2024春, 差分方法II，作业5

交作业时间：2024/05/08

1. Let A and B be two nonempty compact subsets of $\mathbb{R}^d$ for $d \geq 1$. Then the following inequality holds:

$$|A+B|^{1/d} \geq |A|^{1/d} + |B|^{1/d},$$

where the Minkowski sum is defined by

$$A + B := \{v + w \in \mathbb{R}^d : \ v \in A \text{ and } w \in B\}.$$

2. Consider a function $u(x, y) = x^2 + y^2 + x^2y^2$ and discretize the Monge-Ampère operator using 9-point stencil, i.e.

$$\mathcal{G}_\theta = \left\{\{(1,0)^T, (0,1)^T\}, \{(1,1)^T, (-1,1)^T\}\right\}.$$

Show that

$$\min_{\{w_1,w_2\}\in\mathcal{G}_\theta} (\Delta_{w_1} u(0,0))^+ (\Delta_{w_2} u(0,0))^+ = 4,$$

and

$$\min_{w\in\{w_1,w_2\}\in\mathcal{G}_\theta} (\Delta_w u(0,0))^+ \max_{w\in\{w_1,w_2\}\in\mathcal{G}_\theta} (\Delta_w u(0,0))^+ = 4 + 2h^2.$$

3. Let u_ε be a sequence of the uniformly bounded functions on $\bar{\Omega}$, where ε can be understood as a discretization parameter. Define

$$\bar{u}(x) = \lim_{y\to x} \sup_{\varepsilon \downarrow 0} u_\varepsilon(y).$$

Let $\varphi \in C^2(\bar{\Omega})$ and assume that $\bar{u} - \varphi$ has a strictly local maximum at $x_0 \in \bar{\Omega}$ with $\bar{u}(x_0) = \varphi(x_0)$. Show that there are sequences $\{\varepsilon_n\}_{n=1}^\infty \subset \mathbb{R}^+$ and $\{y_n\}_{n=1}^\infty \subset \bar{\Omega}$, such that

$$\varepsilon_n \downarrow 0, \quad y_n \to x_0, \quad u_{\varepsilon_n}(y_n) \to \bar{u}(x_0)$$

and the sequence of functions $u_{\varepsilon_n} - \varphi$ attains its maximum at y_n.

4. Let $\boldsymbol{w} \in \mathbb{R}^2$ be such that $|\boldsymbol{w}| = 1$. Then, for every $\epsilon > 0$, there exists $\boldsymbol{v} \in \mathbb{Q}^2$ such that $|\boldsymbol{v}| = 1$ and

$$|\boldsymbol{w} - \boldsymbol{v}| \leq \epsilon.$$

   Moreover, if $\boldsymbol{v} = (\frac{p_1}{q_1}, \frac{p_2}{q_2})^T$ with $p_1, p_2 \in \mathbb{Z}$ and $q_1, q_2 \in \mathbb{N}$, then we have that

$$0 < q_i \leq \frac{64}{\epsilon^2}.$$

   Is this estimate about the order of ϵ sharp (i.e., whether there can exist $\alpha < 2$ and a constant C independent of ϵ, such that $q_i \leq C\epsilon^{-\alpha}$)?

5. Show the discrete comparison principle and L^∞ stability of the regularized wide stencil FD for Monge-Ampère $MA^{\mathrm{WS}}_{h,\theta,\delta}$.

6. For every symmetric positive semi definite (SPSD) matrix A, show that

$$(\det A)^{1/d} = \frac{1}{d} \inf \left\{ \operatorname{tr}(AB) \mid B \text{ is SPD and } \det B = 1 \right\}.$$

7. Show that the function $A \mapsto (\det A)^{1/d}$ is concave on SPSD matrices.