2024春, 差分方法II，作业4

交作业时间：2024/04/19

1. Show that the optimal SSPERK(3,3) is

$$\begin{aligned} Q^{(1)} &= Q^n + k\mathcal{L}(Q^n) \\ Q^{(2)} &= \frac{3}{4}Q^n + \frac{1}{4}Q^{(1)} + \frac{k}{4}\mathcal{L}(Q^{(1)}) \\ Q^{n+1} &= \frac{1}{3}Q^n + \frac{2}{3}Q^{(2)} + \frac{2k}{3}\mathcal{L}(Q^{(2)}). \end{aligned}$$

   That is, no 3-state third-order explicit Shu-Osher form has better SSP coefficient than the above scheme.

2. Consider the modified s-stage SSPRK for $q_t = -\mathcal{L}(q)$:

$$\begin{cases} Q^{(i)} = v_i Q^n + \sum_{j=1}^{s}[\alpha_{ij}Q^{(j)} + k\beta_{ij}\mathcal{L}(Q^{(j)})] \quad i = 1, \cdots, s+1, \\ Q^{n+1} = Q^{(s+1)}. \end{cases}$$

   Define $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^{(s+1)\times(s+1)}$ as

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1s} & 0 \\ \alpha_{21} & \cdots & \alpha_{2s} & 0 \\ \vdots & & \vdots & \vdots \\ \alpha_{s+1,1} & \cdots & \alpha_{s+1,s} & 0 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_{11} & \cdots & \beta_{1s} & 0 \\ \beta_{21} & \cdots & \beta_{2s} & 0 \\ \vdots & & \vdots & \vdots \\ \beta_{s+1,1} & \cdots & \beta_{s+1,s} & 0 \end{pmatrix}.$$

   Show that the above SSPRK can be recast into the s-stage Runge-Kutta method:

$$\begin{cases} Q^{(i)} = Q^n + k\sum_{j=1}^{s} a_{ij}\mathcal{L}(Q^{(j)}) \quad i = 1, \cdots, s, \\ Q^{n+1} = Q^n + k\sum_{i=1}^{s} b_i\mathcal{L}(Q^{(i)}), \end{cases}$$

   under the following condition:

$$(\boldsymbol{I} - \boldsymbol{\alpha})^{-1}\boldsymbol{\beta} = \begin{pmatrix} A & 0 \\ b^T & 0 \end{pmatrix}.$$

   Here, $A = (a_{ij}) \in \mathbb{R}^{s\times s}$, $b = (b_1, \cdots, b_s)^T \in \mathbb{R}^s$.

3. Consider the WENO reconstruction

$$v_{i+1/2} = \sum_{r=0}^{m-1} \omega_r v_{i+1/2}^{(r)},$$

where the weights are chosen as

$$\omega_r = \frac{\alpha_r}{\sum_{s=0}^{m-1} \alpha_s}, \quad \alpha_r = \frac{d_r}{(\epsilon + \beta_r)^2}.$$

Here, d_r are the coefficients such that $\sum_{r=0}^{m-1} d_r v_{i+1/2}^{(r)}$ coincides with the reconstruction using all the $(2m-1)$ cell averages. Show that

(a) For $m = 2$, $d_0 = \frac{2}{3}, d_1 = \frac{1}{3}$;
(b) For $m = 3$, $d_0 = \frac{3}{10}$, $d_1 = \frac{3}{5}$, $d_2 = \frac{1}{10}$;
(c) d_r is away positive.

4. In the WENO reconstruction, let the reconstruction polynomial on the stencil $S_r(i)$ be denoted by $p_r(x)$. Define

$$\beta_r = \sum_{l=1}^{m-1} \int_{I_i} h^{2l-1} \left(\frac{\partial^l p_r(x)}{\partial x^l} \right) \mathrm{d}x \qquad r = 0, \cdots, m-1.$$

(a) Show that when $m = 2$,

$$\beta_0 = (\bar{v}_{i+1} - \bar{v}_i)^2, \quad \beta_1 = (\bar{v}_i - \bar{v}_{i-1})^2.$$

(b) Verify the condition that

$$\beta_r = D(1 + \mathcal{O}(h^{m-1})) \tag{1}$$

when $m = 2$, where D is a nonzero quantity independent of r (but may depend on h).

(c) Find the expression of β_0, β_1 and β_2 for $m = 3$ and verify the condition (1).

5. Consider the elliptic problem in non-divergence form:

$$\boldsymbol{A}(x) : D^2 u(x) = f(x), \quad \boldsymbol{A}(x) \geq 0.$$

If there exists $\Lambda_0 \geq \lambda_0 > 0$, such that $\lambda_0 \boldsymbol{I} \leq A(x) \leq \Lambda_0 \boldsymbol{I}$ for all $x \in \Omega$, show that the non-divergence form is uniformly elliptic.

6. Consider the following problem: find $u : \bar{\Omega} \to \mathbb{R}$ such that

$$F(x, u(x), \nabla u(x), D^2 u(x)) = 0 \quad \text{in } \Omega, \qquad u(x) = g(x) \quad \text{on } \partial\Omega,$$

where F is elliptic. If further F is strictly decreasing in the r variable or uniformly elliptic and Lipschitz in p, show that the problem cannot have more than one classical solution.