2024春, 差分方法II，作业3

交作业时间：2024/04/07

1. Let (η, ψ) be an entropy pair. Suppose the following cell entropy condition holds:
$$\frac{\eta(Q_i^{n+1}) - \eta(Q_i^n)}{k} + \frac{\Psi_{i+1/2}^n - \Psi_{i-1/2}^n}{h} \le 0,$$
where $\Psi_{i-1/2}^n = \Psi(Q_{i-1}^n, Q_i^n)$ for some numerical entropy flux function $\Psi(\cdot,\cdot)$ that is consistent with ψ. Show that the limiting weak solution q satisfies the entropy condition. (Hint: mimicking the proof of the Lax-Wendroff Theorem.)

2. For scalar conservation law $q_t + f(q)_x = 0$, show that the Godunov flux (by solving the Riemann problem) has the compact form
$$\mathcal{F}(Q_{i-1}, Q_i) = \begin{cases} \min\limits_{Q_{i-1} \le q \le Q_i} f(q) & \text{if } Q_{i-1} \le Q_i, \\ \max\limits_{Q_i \le q \le Q_{i-1}} f(q) & \text{if } Q_{i-1} \ge Q_i. \end{cases}$$

3. For the scalar conservation law, show that LxF (Lax-Friedrichs), LLF (Local Lax-Friedrichs), and Engquist-Osher fluxes are all E-fluxes.

4. For a monotone flux we have that $\mathcal{F}(\uparrow, \downarrow)$. Show that a monotone flux is an E-flux.

5. Consider a conservative scheme with a Lipschitz-continuous numerical flux. If the scheme is TVB (Total Variation Bounded), show that there exists a constant $\tilde{R}$ and k_0, such that
$$\mathrm{TV}_T(Q) \le \tilde{R}, \qquad \forall nk \le T,\ k \le k_0. \tag{1}$$
where
$$\mathrm{TV}_T(Q) := \sum_i \sum_n [k|Q_{i+1}^n - Q_i^n| + h|Q_i^{n+1} - Q_i^n|].$$
Does (1) imply TVB?

6. Show that LxF (Lax-Friedrichs), Engquist-Osher, and Godunov fluxes are entropy stable.