差分方法II, 作业5

交作业时间：2023/06/02

1. For every symmetric positive semi definite (SPSD) matrix A, show that
$$(\det A)^{1/d} = \frac{1}{d} \inf \{\operatorname{tr}(AB) \mid B \text{ is SPD and } \det B = 1\}.$$

2. Show that the function $A \mapsto (\det A)^{1/d}$ is concave on SPSD matrices.

3. Let $\mathbf{S}_+ := \{A \in \mathbf{S}, A \geq 0\}$ and $\mathbf{S}_1 = \{B \in \mathbf{S}_+, \operatorname{tr}(B) = 1\}$, where $\mathbf{S}$ denotes the set of $d \times d$ symmetric real matrices. Define the Bellman operator
$$H(A, f) := \sup_{B \in \mathbf{S}_1} \left(-B : A + f \sqrt[d]{\det B} \right) \quad \forall A \in \mathbf{S}, f \in [0, \infty),$$
and the Monge-Ampère operator
$$M(A, f) := \left(\frac{f}{d} \right)^d - \det A \quad \forall A \in \mathbf{S}, f \in [0, \infty).$$
   (a) Let $f \in [0, \infty)$ and $A \in \mathbf{S}$. Show that $H(A, f) = 0$ holds if and only if $M(A, f) = 0$ and $A \in \mathbf{S}_+$.
   (b) Let $A \in \mathbf{S}_+$, $f \in [0, \infty)$ and let λ be the smallest eigenvalue of A. Show that the function
$$\Phi_{A,f} : [-f, \infty) \to [-\lambda, \infty), \delta \mapsto H(A, f + \delta)$$
   is continuous, strictly monotonically increasing and bijective.

4. Let v_h be a piecewise linear function on the mesh $\mathcal{T}_h$. For any face $F = T^+ \cap T^-$ with $T^\pm \in \mathcal{T}_h$, define the jump
$$[\![v_h]\!]|_F := -n_F^+ \cdot \nabla v_h|_{T^+} - n_F^- \cdot \nabla v_h|_{T^-}.$$
Show that v_h is convex if and only if
$$[\![v_h]\!]|_F \geq 0 \qquad \text{for all faces } F.$$

5. Let $p(x) = \frac{1}{2}(x^2 + y^2)$. A mesh $\mathcal{T}_h$ is called *Delaunay* if the sum of the angles opposite to any edge is less than or equal to π. Show that $I_h p = \Gamma(I_h p)$ if and only if $\mathcal{T}_h$ is Delaunay, where I_h is the interpolant to the continuous piecewise linear space.