差分方法II, 作业4

交作业时间：2023/05/19

1. Given a symmetric matrix $\boldsymbol{A}$ ($\boldsymbol{A} \neq 0$), show that $\gamma = \frac{\mathrm{tr}\boldsymbol{A}}{\|\boldsymbol{A}\|_F^2}$ is the minimizer of
$$\min_{\tau \in \mathbb{R}} \|\tau \boldsymbol{A} - \boldsymbol{I}\|_F^2.$$
Here $\|\cdot\|_F$ represents the Frobenius norm. Find the minimum.

2. Consider the following problem: find $u : \bar{\Omega} \to \mathbb{R}$ such that
$$F(x, u(x), \nabla u(x), D^2 u(x)) = 0 \quad \text{in } \Omega, \qquad u(x) = g(x) \quad \text{on } \partial\Omega,$$
where F is elliptic. If further F is strictly decreasing in the r variable or uniformly elliptic and Lipschitz in p, show that the problem cannot have more than one classical solution.

3. Let A and B be two nonempty compact subsets of $\mathbb{R}^d$ for $d \geq 1$. Then the following inequality holds:
$$|A + B|^{1/d} \geq |A|^{1/d} + |B|^{1/d},$$
where the Minkowski sum is defined by
$$A + B := \{v + w \in \mathbb{R}^d : \ v \in A \text{ and } w \in B\}.$$

4. Consider a function $u(x, y) = x^2 + y^2 + x^2 y^2$ and discretize the Monge-Ampère operator using 9-point stencil, i.e.
$$\mathcal{G}_\theta = \left\{ \{(1,0)^T, (0,1)^T\}, \{(1,1)^T, (-1,1)^T\} \right\}.$$
Show that
$$\min_{\{w_1, w_2\} \in \mathcal{G}_\theta} (\Delta_{w_1} u(0,0))^+ (\Delta_{w_2} u(0,0))^+ = 4,$$
and
$$\min_{w \in \{w_1, w_2\} \in \mathcal{G}_\theta} (\Delta_w u(0,0))^+ \max_{w \in \{w_1, w_2\} \in \mathcal{G}_\theta} (\Delta_w u(0,0))^+ = 4 + 2h^2.$$

5. Let u_ε be a sequence of the uniformly bounded functions on $\bar{\Omega}$, where ε can be understood as a discretization parameter. Define

$$\bar{u}(x) = \lim_{y \to x} \sup_{\varepsilon \downarrow 0} u_\varepsilon(y).$$

Let $\varphi \in C^2(\bar{\Omega})$ and assume that $\bar{u} - \varphi$ has a strictly local maximum at $x_0 \in \bar{\Omega}$ with $\bar{u}(x_0) = \varphi(x_0)$. Show that there are sequences $\{\varepsilon_n\}_{n=1}^{\infty} \subset \mathbb{R}^+$ and $\{y_n\}_{n=1}^{\infty} \subset \bar{\Omega}$, such that

$$\varepsilon_n \downarrow 0, \quad y_n \to x_0, \quad u_{\varepsilon_n}(y_n) \to \bar{u}(x_0)$$

and the sequence of functions $u_{\varepsilon_n} - \varphi$ attains its maximum at y_n.

6. Let $\boldsymbol{w} \in \mathbb{R}^2$ be such that $|\boldsymbol{w}| = 1$. Then, for every $\epsilon > 0$, there exists $\boldsymbol{v} \in \mathbb{Q}^2$ such that $|\boldsymbol{v}| = 1$ and

$$|\boldsymbol{w} - \boldsymbol{v}| \leq \epsilon.$$

Moreover, if $\boldsymbol{v} = (\frac{p_1}{q_1}, \frac{p_2}{q_2})^T$ with $p_1, p_2 \in \mathbb{Z}$ and $q_1, q_2 \in \mathbb{N}$, then we have that

$$0 < q_i \leq \frac{64}{\epsilon^2}.$$