差分方法II，作业2

交作业时间：2023/03/31

Finite Difference Schemes and Partial Differential Equations:

- Chapter 9: 9.1.3, Example 9.2.2 (show the uniformly diagonalizable)
- Chapter 10: 10.1.10, 10.1.13, 10.3.4

Finite Volume Methods for Hyperbolic Problems:

- Chapter 8: 8.3

Supplementary Questions:

1. Let q, w be piecewise smooth weak solutions of scalar conservation law $q_t + f(q)_x = 0$, where f is convex. Assume that all the discontinuities are shocks. Let the nonoverlapping $I_k(t) := [x^k(t), x^{k+1}(t)]$ on which $q(x,t) - w(x,t)$ has sign $(-1)^k$. Show that for any k,

$$(-1)^k \left[f(w) - f(q) + (q - w)\frac{\mathrm{d}x}{\mathrm{d}t} \right] \Bigg|_{x^k}^{x^{k+1}} \leq 0.$$

2. Let (η, ψ) be an entropy pair. Suppose the following cell entropy condition holds:

$$\frac{\eta(Q_i^{n+1}) - \eta(Q_i^n)}{k} + \frac{\Psi_{i+1/2}^n - \Psi_{i-1/2}^n}{h} \leq 0,$$

   where $\Psi_{i-1/2}^n = \Psi(Q_{i-1}^n, Q_i^n)$ for some numerical entropy flux function $\Psi(\cdot, \cdot)$ that is consistent with ψ. Show that the limiting weak solution q satisfies the entropy condition. (Hint: mimicking the proof of the Lax-Wendroff Theorem.)

3. Let $\boldsymbol{f}(\boldsymbol{z})$ be a smooth vector-valued function. Show that

$$\nabla_{\boldsymbol{z}} \boldsymbol{f} \text{ is symmetric} \quad \iff \quad \boldsymbol{f}^T = \nabla_{\boldsymbol{z}} r \quad \text{for some scalar-valued function } r(\boldsymbol{z}).$$