差分方法II，作业3

交作业时间：2022/04/18

Finite Volume Methods for Hyperbolic Problems:

- Chapter 8: 8.3
- Chapter 12: 12.3

Supplementary Questions:

1. Let q, w be piecewise smooth weak solutions of scalar conservation law $q_t + f(q)_x = 0$, where f is convex. Assume that all the discontinuities are shocks. Let the nonoverlapping $I_k(t) := [x^k(t), x^{k+1}(t)]$ on which $q(x,t) - w(x,t)$ has sign $(-1)^k$. Show that for any k,

$$(-1)^k \left[f(w) - f(q) + (q-w)\frac{\mathrm{d}x}{\mathrm{d}t} \right] \Bigg|_{x^k}^{x^{k+1}} \leq 0.$$

2. For the scalar conservation law, show that Engquist-Osher flux is monotone when the CFL condition holds.

3. For the scalar conservation law, show that LxF (Lax-Friedrichs), LLF (Local Lax-Friedrichs), and Engquist-Osher fluxes are all E-fluxes.

4. For a monotone flux we have that $\mathcal{F}(\uparrow, \downarrow)$. Show that a monotone flux is an E-flux.

5. Let $\boldsymbol{f}(\boldsymbol{z})$ be a smooth vector-valued function. Show that

$$\nabla_{\boldsymbol{z}} \boldsymbol{f} \text{ is symmetric} \quad \Longleftrightarrow \quad \boldsymbol{f}^T = \nabla_{\boldsymbol{z}} r \quad \text{for some scalar-valued function } r(\boldsymbol{z}).$$

6. Consider a conservative scheme with a Lipschitz-continuous numerical flux. If the scheme is TVB (Total Variation Bounded), then there exists a constant $\tilde{R}$ and k_0, such that

$$\mathrm{TV}_T(Q) \leq \tilde{R}, \qquad \forall nk \leq T,\ k \leq k_0. \tag{1}$$

where

$$\mathrm{TV}_T(Q) := \sum_i \sum_n [k|Q_{i+1}^n - Q_i^n| + h|Q_i^{n+1} - Q_i^n|].$$

Does (1) imply TVB?