差分方法II，作业5

交作业时间：2021/06/14

1. Let v_h be a piecewise linear function on the mesh $\mathcal{T}_h$. For any face $F = T^+ \cap T^-$ with $T^\pm \in \mathcal{T}_h$, define the jump
$$[\![v_h]\!]|_F := -n_F^+ \cdot \nabla v_h|_{T^+} - n_F^- \cdot \nabla v_h|_{T^-}.$$
Show that v_h is convex if and only if
$$[\![v_h]\!]|_F \geq 0 \qquad \text{for all faces } F.$$

2. Let $p(x) = \frac{1}{2}(x^2 + y^2)$. A mesh $\mathcal{T}_h$ is called *Delaunay* if the sum of the angles opposite to any edge is less than or equal to π. Show that $I_h p = \Gamma(I_h p)$ if and only if $\mathcal{T}_h$ is Delaunay, where I_h is the interpolant to the continuous piecewise linear space.

3. For ODE $y' = f(t, y)$, consider the linear multi-step method
$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \ldots + a_0 y_n = b_k f_{n+k} + b_{k-1} f_{n+k-1} + \ldots + b_0 f_n.$$
Show that the multistep is of order p

$\iff$ (i) $\sum_{i=0}^k a_i = 0$ and $\sum_{i=0}^k a_i i^q = q \sum_{i=0}^k b_i i^{q-1}, q = 1, \ldots, p$;
$\iff$ (ii) $\rho(\mathrm{e}^\tau) - \tau\sigma(\mathrm{e}^\tau) = \mathcal{O}(\tau^{p+1})$ as $\tau \to 0$;
$\iff$ (ii) $\frac{\rho(\zeta)}{\log(\zeta)} - \sigma(\zeta) = \mathcal{O}((\zeta - 1)^p)$ as $\zeta \to 1$.

Here, $\rho(\zeta)$ and $\sigma(\zeta)$ are the generating functions of $\{a_i\}_{i=0}^k$ and $\{b_i\}_{i=0}^k$, respectively.

4. Assume $u(t)$ is continuous on $[0, T]$ and $u(0) = 0$. Show that
$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \partial_\xi^\alpha u(\xi) \mathrm{d}\xi.$$
Notice that the Caputo and Riemann-Liouville fractional derivatives coincide due to $u(0) = 0$.

5. Consider the two-parameter Milttag-Leffler function
$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)} \quad z \in \mathbb{C}.$$

For $\lambda > 0$, $\alpha > 0$, and $t > 0$, show that

$$\begin{aligned} {}^{C}\partial_t^{\alpha} E_{\alpha,1}(-\lambda t^{\alpha}) &= -\lambda E_{\alpha,1}(-\lambda t^{\alpha}), \\ E_{\alpha,1}(-\lambda t^{\alpha}) &= 1 - \lambda t^{\alpha} E_{\alpha,1+\alpha}(-\lambda t^{\alpha}). \end{aligned}$$