差分方法II，作业4

交作业时间：2021/05/31

1. For every symmetric positive semi definite (SPSD) matrix A, show that
$$(\det A)^{1/d} = \frac{1}{d} \inf \{\mathrm{tr}(AB) \mid B \text{ is SPD and } \det B = 1\}.$$

2. Show that the function $A \mapsto (\det A)^{1/d}$ is concave on SPSD matrices.

3. Let $\mathbf{S}_+ := \{A \in \mathbf{S}, A \geq 0\}$ and $\mathbf{S}_1 = \{B \in \mathbf{S}_+, \mathrm{tr}(B) = 1\}$, where $\mathbf{S}$ denotes the set of $d \times d$ symmetric real matrices. Define the Bellman operator
$$H(A, f) := \sup_{B \in \mathbf{S}_1} \left(-B : A + f \sqrt[d]{\det B}\right) \quad \forall A \in \mathbf{S}, f \in [0, \infty),$$
and the Monge-Ampère operator
$$M(A, f) := \left(\frac{f}{d}\right)^d - \det A \quad \forall A \in \mathbf{S}, f \in [0, \infty).$$
   (a) Let $f \in [0, \infty)$ and $A \in \mathbf{S}$. Show that $H(A, f) = 0$ holds if and only if $M(A, f) = 0$ and $A \in \mathbf{S}_+$.
   (b) Let $A \in \mathbf{S}_+$, $f \in [0, \infty)$ and let λ be the smallest eigenvalue of A. Show that the function
$$\Phi_{A,f} : [-f, \infty) \to [-\lambda, \infty), \delta \mapsto H(A, f + \delta)$$
   is continuous, strictly monotonically increasing and bijective.

4. Consider the LBR for Monge-Ampère:
$$\mathrm{MA}^{\mathrm{LBR}}_{h,S} u(x) := \min_{\substack{(e,f,g) \in S^3 \\ \text{superbase}}} \gamma(\nabla^{2,+}_{e,h} u(x), \nabla^{2,+}_{f,h} u(x), \nabla^{2,+}_{g,h} u(x)),$$
where $\gamma : \mathbb{R}^3_+ \to \mathbb{R}$ is defined by
$$\gamma(a, b, c) = \begin{cases} bc & \text{if } a \geq b + c, \\ ac & \text{if } b \geq a + c, \\ ab & \text{if } c \geq a + b, \\ \frac{1}{2}(ab + bc + ca) - \frac{1}{4}(a^2 + b^2 + c^2) & \text{otherwise.} \end{cases}$$

(a) Show that $\mathrm{MA}_{h,S}^{\mathrm{LBR}}$ is monotone and satisfies the discrete maximal principle (DCP).

(b) If replacing min by $\min^{\delta}$ (regularized min), show that $\mathrm{MA}_{h,S,\delta}^{\mathrm{LBR}}$ is also monotone and satisfies DCP.

5. Let u_ε be the solution of two-scale method for MA, $I_h u$ be the interpolation of exact solution. Give the convergence rate estimate of

$$\max_{\Omega}(I_h u - u_\varepsilon).$$