

## 差分方法II, 作业3

交作业时间: 2021/05/12

1. Consider the following problem: find  $u : \bar{\Omega} \rightarrow \mathbb{R}$  such that

$$F(x, u(x), \nabla u(x), D^2 u(x)) = 0 \quad \text{in } \Omega, \quad u(x) = g(x) \quad \text{on } \partial\Omega,$$

where  $F$  is elliptic. If further  $F$  is strictly decreasing in the  $r$  variable or uniformly elliptic, show that the problem cannot have more than one classical solution.

2. Let  $A$  and  $B$  be two nonempty compact subset of  $\mathbb{R}^d$  for  $d \geq 1$ . Then the following inequality holds:

$$|A + B|^{1/d} \geq |A|^{1/d} + |B|^{1/d},$$

where the Minkowski sum is defined by

$$A + B := \{v + w \in \mathbb{R}^d : v \in A \text{ and } w \in B\}.$$

3. Consider a function  $u(x, y) = x^2 + y^2 + x^2 y^2$  and discretize the Monge-Ampère operator using 9-point stencil, i.e.

$$\mathcal{G}_\theta = \{\{(1, 0)^T, (0, 1)^T\}, \{(1, 1)^T, (-1, 1)^T\}\}.$$

Show that

$$\min_{\{w_1, w_2\} \in \mathcal{G}_\theta} (\Delta_{w_1} u(0, 0))^+ (\Delta_{w_2} u(0, 0))^+ = 4,$$

and

$$\min_{w \in \{w_1, w_2\} \in \mathcal{G}_\theta} (\Delta_w u(0, 0))^+ \max_{w \in \{w_1, w_2\} \in \mathcal{G}_\theta} (\Delta_w u(0, 0))^+ = 4 + 2h^2.$$

4. Consider the 17-point stencil for the regularized discretized Monge-Ampère operator  $\text{MA}_{h, \theta, \delta}^{\text{WS}}$ . Find its Jacobian.
5. Let  $u_\varepsilon$  be a sequence of the uniformly bounded functions on  $\bar{\Omega}$ , where  $\varepsilon$  can be understood as a discretization parameter. Define

$$\bar{u}(x) = \lim_{y \rightarrow x} \sup_{\varepsilon \downarrow 0} u_\varepsilon(y).$$

Let  $\varphi \in C^2(\bar{\Omega})$  and assume that  $\bar{u} - \varphi$  has a strictly local maximum at  $x_0 \in \bar{\Omega}$  with  $\bar{u}(x_0) = \varphi(x_0)$ . Show that there are sequences  $\{\varepsilon_n\}_{n=1}^\infty \subset \mathbb{R}^+$  and  $\{y_n\}_{n=1}^\infty \subset \bar{\Omega}$ , such that

$$\varepsilon_n \downarrow 0, \quad y_n \rightarrow x_0, \quad u_{\varepsilon_n}(y_n) \rightarrow \bar{u}(x_0)$$

and the sequence of functions  $u_{\varepsilon_n} - \varphi$  attains its maximum at  $y_n$ .

6. Show the existence of the wide stencil scheme by constructing the discrete Perron iteration.