差分方法II，作业4

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1. Consider the following scheme for $q_t + aq_x = 0$:

$$Q_i^{n+1} = Q_i^n - \frac{\lambda}{2}[2-(1-\lambda)\phi_{i-1/2}^n]\Delta^- Q_i^n - \frac{\lambda(1-\lambda)}{2}\phi_{i+1/2}^n \Delta^+ Q_i^n,$$

where $\lambda = a\frac{k}{h}$,

$$\phi_{i+1/2}^n := \phi(\theta_{i+1/2}^n) \quad \text{and} \quad \theta_{i+1/2}^n := \frac{\Delta^- Q_i^n}{\Delta^+ Q_i^n}.$$

- Find the truncation error.
- Show that the scheme is second-order if $\phi(1) = 1$.
- Show that the scheme is Beam-Warming if $\phi(\theta) = \theta$.

2. In the MUSCL scheme, if the following conditions hold

$$\min(Q_{i-1}^n, Q_i^n) \le Q_{i-1/2}^{+,n+1/2} \le \max(Q_{i-1}^n, Q_i^n),$$
$$\min(Q_i^n, Q_{i+1}^n) \le Q_{i+1/2}^{-,n+1/2} \le \max(Q_i^n, Q_{i+1}^n),$$

and the flux is monotone, then the MUSCL scheme is TVD-stable.

3. Consider the modified s-stage SSPRK for $q_t = -\mathcal{L}(q)$:

$$\begin{cases} Q^{(i)} = v_i Q^n + \sum_{j=1}^{s}[\alpha_{ij}Q^{(j)} + k\beta_{ij}\mathcal{L}(Q^{(j)})] \quad i = 1, \cdots, s+1, \\ Q^{n+1} = Q^{(s+1)}. \end{cases}$$

Define $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^{(s+1)\times(s+1)}$ as

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1s} & 0 \\ \alpha_{21} & \cdots & \alpha_{2s} & 0 \\ \vdots & & \vdots & \vdots \\ \alpha_{s+1,1} & \cdots & \alpha_{s+1,s} & 0 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_{11} & \cdots & \beta_{1s} & 0 \\ \beta_{21} & \cdots & \beta_{2s} & 0 \\ \vdots & & \vdots & \vdots \\ \beta_{s+1,1} & \cdots & \beta_{s+1,s} & 0 \end{pmatrix}.$$

Show that the above SSPRK can be recast into the s-stage Runge-Kutta method:

$$\begin{cases} Q^{(i)} = Q^n + k\sum_{j=1}^{s} a_{ij}\mathcal{L}(Q^{(j)}) \quad i = 1, \cdots, s, \\ Q^{n+1} = Q^n + k\sum_{i=1}^{s} b_i\mathcal{L}(Q^{(i)}), \end{cases}$$

under the following condition:

$$(\boldsymbol{I} - \boldsymbol{\alpha})^{-1}\boldsymbol{\beta} = \begin{pmatrix} A & 0 \\ b^T & 0 \end{pmatrix}.$$

Here, $A = (a_{ij}) \in \mathbb{R}^{s\times s}$, $b = (b_1, \cdots, b_s)^T \in \mathbb{R}^s$.

4. Consider the WENO reconstruction

$$v_{i+1/2} = \sum_{r=0}^{m-1} \omega_r v^{(r)}_{i+1/2},$$

where the weights are chosen as

$$\omega_r = \frac{\alpha_r}{\sum_{s=0}^{m-1}\alpha_s}, \quad \alpha_r = \frac{d_r}{(\epsilon + \beta_r)^2}.$$

Let the reconstruction polynomial on the stencil $S_r(i)$ be denoted by $p_r(x)$. Define

$$\beta_r = \sum_{l=1}^{m-1}\int_{I_i} h^{2l-1}\left(\frac{\partial^l p_r(x)}{\partial x^l}\right) \mathrm{d}x \qquad r = 0, \cdots, m-1.$$

- Show that when $m = 2$,

$$\beta_0 = (\bar{v}_{i+1} - \bar{v}_i)^2, \quad \beta_1 = (\bar{v}_i - \bar{v}_{i-1})^2.$$

- Verify the condition that $\beta_r = D(1 + \mathcal{O}(h^{m-1}))$ when $m = 2$, where D is a nonzero quantity independent of r (but may depend on h).
- Find the expression of β_0, β_1 and β_2 for $m = 3$.

5. Consider the elliptic problem in non-divergence form:

$$\boldsymbol{A}(x) : D^2u(x) = f(x), \quad \boldsymbol{A}(x) \geq 0.$$

If there exists $\Lambda_0 \geq \lambda_0 > 0$, such that $\lambda_0\boldsymbol{I} \leq A(x) \leq \Lambda_0\boldsymbol{I}$ for all $x \in \Omega$, show that the non-divergence form is uniformly elliptic.