高等数学(B). 作业15. 参考答案

习题6.8. 1. (3). $x+z-\varepsilon\sin z=y$. $(0<\varepsilon<1)$.

解. 两边对 x 求导有 $$1+\frac{\partial z}{\partial x}-\varepsilon\cdot\cos z\cdot\frac{\partial z}{\partial x}=0.$$

则 $$\frac{\partial z}{\partial x}=\frac{1}{\varepsilon\cos z-1}$$

两边对 y 求导有 $$\frac{\partial z}{\partial y}-\varepsilon\cos z\cdot\frac{\partial z}{\partial y}=1$$

则 $$\frac{\partial z}{\partial y}=\frac{1}{1-\varepsilon\cos z}$$

(4). $z^x=y^z$.

解: 两边对 x 求导,(将 z^x 看做 $e^{x\cdot\ln z}$) 有

$$z^x\cdot\left(\ln z+\frac{x}{z}\cdot\frac{\partial z}{\partial x}\right)=y^z\cdot\frac{\partial z}{\partial x}\cdot\ln y=z^x\cdot\frac{\partial z}{\partial x}\cdot\ln y.$$

$(y^z=z^x)$.

则 $$\frac{\partial z}{\partial x}=\frac{z\cdot\ln z}{z\cdot\ln y-x}.$$

两边对 y 求导有(将 y^z 看做 $e^{z\ln y}$).

$$x\cdot z^{x-1}\cdot\frac{\partial z}{\partial y}=y^z\cdot\left(\frac{\partial z}{\partial y}\cdot\ln y+\frac{z}{y}\right)=z^x\cdot\left(\frac{\partial z}{\partial y}\cdot\ln y+\frac{z}{y}\right)$$

则 $$\frac{\partial z}{\partial y}=\frac{z^2}{xy-zy\cdot\ln y}$$

3. 设 $z+\cos xy=e^z$. 求 $\frac{\partial z}{\partial x}$ 及 $\frac{\partial^2 z}{\partial x^2}$.

解: 对等式两边对 x 求导有:

$$\frac{\partial z}{\partial x}+(-\sin xy)\cdot y=e^z\cdot\frac{\partial z}{\partial x}. \qquad (1)$$

$$\frac{\partial z}{\partial x}=\frac{\sin(xy)\cdot y}{1-e^z}.$$

(1) 式两边对 x 求导有:

$$\frac{\partial^2 z}{\partial x^2}-\cos(xy)\cdot y^2=e^z\cdot\left(\frac{\partial z}{\partial x}\right)^2+e^z\cdot\frac{\partial^2 z}{\partial x^2}.$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{e^z\cdot\left(\frac{\partial z}{\partial x}\right)^2+y^2\cos(xy)}{1-e^z}$$

将 $\frac{\partial z}{\partial x}=\frac{\sin(xy)\cdot y}{1-e^z}$ 代入有: $$\frac{\partial^2 z}{\partial x^2}=\frac{y^2}{(1-e^z)^3}\cdot\left((1-e^z)^2\cos(xy)+e^z\sin^2(xy)\right)$$

5. 解：(1). ∵ $F(x,y,z)=0$

则 $F_x\cdot dx + F_y\cdot dy + F_z\cdot dz = 0$.

$dz = -\frac{F_x}{F_z}dx - \frac{F_y}{F_z}dy$. $(F_z \neq 0)$.

(2) ∵ $F(x^2+y^2+z^2,\ xy-z^2)=0$.

有. $F_1'(d(x^2+y^2+z^2)) + F_2'\cdot(d(xy-z^2)) = 0$

即. $F_1'(2x\,dx+2y\cdot dy+2z\,dz) + F_2'\cdot(x\,dy+y\cdot dx-2z\cdot dz)=0$.

化简后有：$dz = \frac{(2xF_1'+yF_2')dx+(2yF_1'+xF_2')\cdot dy}{2z(F_2'-F_1')}$.

8. 解：

$$\begin{cases} xu + yv = 0 \\ uv - xy = 5 \end{cases} \quad (1).$$

对(1)中式x求导有：
$$\begin{cases} u + x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} = 0 \\ u\cdot\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} - y = 0. \end{cases} \quad (2).$$

将 $x=1$. $y=1$, $u=v=2$ 代入有
$$\begin{cases} \frac{\partial u}{\partial x}\Big|_{(1,1)} = -\frac{5}{4} \\ \frac{\partial v}{\partial x}\Big|_{(1,1)} = \frac{3}{4}. \end{cases} \quad (3).$$

对(1)中式y求导有：
$$\begin{cases} x\cdot\frac{\partial u}{\partial y} + v + y\cdot\frac{\partial v}{\partial y} = 0. \\ \frac{\partial u}{\partial y}\cdot v + u\cdot\frac{\partial v}{\partial y} - x = 0. \end{cases} \quad (4).$$

将 $x=1$. $y=1$. $u=v=2$ 代入有：
$$\begin{cases} \frac{\partial u}{\partial y} = -\frac{3}{4} \\ \frac{\partial v}{\partial y} = \frac{5}{4}. \end{cases}$$

对(2)中 x对导有.
$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\cdot\frac{\partial^2 v}{\partial x^2} = 0 \\ u\cdot\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + v\frac{\partial^2 u}{\partial x^2} = 0 \end{cases}$$

将 $x=1$, $y=1$, $v=u=2$, $\frac{\partial u}{\partial x}\Big|_{(1,1)} = -\frac{5}{4}$, $\frac{\partial v}{\partial x}\Big|_{(1,1)} = \frac{3}{4}$ 代入有 $\frac{\partial^2 u}{\partial x^2} = \frac{55}{32}$.

对(2)中 y对导有
$$\begin{cases} \frac{\partial u}{\partial y} + x\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial v}{\partial x} + y\frac{\partial^2 v}{\partial x\partial y} = 0 \\ \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} + u\cdot\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial x} + v\cdot\frac{\partial^2 u}{\partial x\partial y} = 1 \end{cases}$$

P3.

将 $x=1, y=1$, $u=v=2$, $\left.\frac{\partial u}{\partial y}\right|_{(1,1)}=-\frac{3}{4}$, $\left.\frac{\partial v}{\partial y}\right|_{(1,1)}=\frac{3}{4}$ 代入有 $\frac{\partial^2 u}{\partial x \partial y}=\frac{25}{32}$.

11. 证明:

$$\frac{\partial u}{\partial \xi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta}.$$

$$\frac{\partial v}{\partial \xi}=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \xi}.$$

$$\frac{\partial v}{\partial \eta}=\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \eta}.$$

$$\frac{D(u,v)}{D(\xi,\eta)}=\begin{vmatrix}\frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta}\\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta}\end{vmatrix}=\begin{vmatrix}\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta}\\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \xi} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \eta}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}\cdot\begin{vmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}\end{vmatrix}=\frac{D(u,v)}{D(x,y)}\cdot\frac{D(x,y)}{D(\xi,\eta)}.$$

习题6.9.

1. (2) $z=2xy-5x^2-2y^2+4x+4y-1$

$$\begin{cases}\frac{\partial z}{\partial x}=2y-10x+4=0\\ \frac{\partial z}{\partial y}=2x-4y+4=0\end{cases}$$ 联立解得 $\begin{cases}x=\frac{2}{3}\\ y=\frac{4}{3}.\end{cases}$

又 $A=\frac{\partial^2 z}{\partial x^2}=-10$

$B=\frac{\partial^2 z}{\partial x\partial y}=2$

$C=\frac{\partial^2 z}{\partial y^2}=-4.$

$B^2<AC$ 且 $A<0$, $\Rightarrow$ $(\frac{2}{3},\frac{4}{3})$ 是极大值点.

极大值为 $\left.z\right|_{(\frac{2}{3},\frac{4}{3})}=3.$

(5). $z = x^3y^2(6-x-y).$ $(x>0, y>0).$

$$\begin{cases} \frac{\partial z}{\partial x} = y^2(3x^2(6-x-y)-x^3) = x^2y^2(3(6-x-y)-x)=0 \\ \frac{\partial z}{\partial y} = x^3(2y(6-x-y)-y^2) = x^3y(2(6-x-y)-y)=0 \end{cases} \quad (x>0, y>0)$$

联立解得 $\begin{cases} x=3 \\ y=2. \end{cases}$

$$A = \frac{\partial^2 z}{\partial x^2}\Big|_{(3,2)} = y^2(2x\cdot(18-4x-3y)-4x^2)\Big|_{(3,2)} = -144.$$

$$B = \frac{\partial^2 z}{\partial x\partial y}\Big|_{(3,2)} = x^2(2y\cdot(18-4x-3y)-3y^2)\Big|_{(3,2)} = -108.$$

$$C = \frac{\partial^2 z}{\partial y^2}\Big|_{(3,2)} = x^3((12-2x-3y)-3y)\Big|_{(3,2)} = -162.$$

$B^2 < AC$ 且 $A<0$, $\Rightarrow$ $(3,2)$是极大值点.

极大值 $z|_{(3,2)} = 108.$

2.(2). $z = 3x+4y$, 当 $x^2+y^2=1$ 时.

解. 利用 Lagrange 乘子法. 构造 $g(x,y) = 3x+4y+\lambda(x^2+y^2-1)$.

则 $\begin{cases} g'_x = 3+2\lambda x & (1) \\ g'_y = 4+2\lambda y & (2) \\ g'_\lambda = x^2+y^2-1. & (3) \end{cases}$

令 $(1)=0$ 有 $x = -\frac{3}{2\lambda}$ (4)

令 $(2)=0$ 有 $y = -\frac{2}{\lambda}.$ (5).

令 $(3)=0$ 且将(4).(5)代入有 $\lambda = \pm\frac{5}{2}.$

当 $\lambda = \frac{5}{2}$ 时. $x = -\frac{3}{5}.$ $y = -\frac{4}{5}.$ $z|_{(-\frac{3}{5},-\frac{4}{5})} = -5.$

当 $\lambda = -\frac{5}{2}$ 时, $x = \frac{3}{5}.$ $y = \frac{4}{5}.$ $z|_{(\frac{3}{5},\frac{4}{5})} = 5.$

比较可得. 条件最小值点为$(-\frac{3}{5},-\frac{4}{5})$, 条件最小值为 -5.

条件最大值点为$(\frac{3}{5},\frac{4}{5})$, 条件最大值为 5.

6. 解. 点(x,y,z)距离原点的距离平方为$f(x,y,z)=x^2+y^2+z^2$.

条件为 $\begin{cases} x^2+y^2+\frac{z^2}{4}=1 \\ x+y+z=0. \end{cases}$

构造函数 $L(x,y,z,\lambda,\mu)=x^2+y^2+z^2+\lambda(x^2+y^2+\frac{z^2}{4}-1)+\mu(x+y+z)$.

则 $\begin{cases} L'_x=2x+2\lambda x+\mu & (1) \\ L'_y=2y+2\lambda y+\mu. & (2) \\ L'_z=2z+\frac{1}{2}\lambda z+\mu. & (3) \\ L'_\lambda=x^2+y^2+\frac{z^2}{4}-1 & (4) \\ L'_\mu=x+y+z & (5). \end{cases}$

令(1)=0且(3)=0并联立有 $2x(1+\lambda)=2z+\frac{1}{2}\lambda z$.

令(2)=0且(3)=0并联立有 $2y(1+\lambda)=2z+\frac{1}{2}\lambda z$.

1° 若$\lambda=-1$，则有$z=0$. 联立(4),(5)可解得

(x,y,z)为$(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0)$或$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)$.

此时$f(x,y,z)=1$. 距离原点为1.

2° 若$\lambda\neq-1$，则有$x=y$. 联立(4),(5)=0可得有

$(x,y,z)=(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{2}{\sqrt{3}})$或$(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{2}{\sqrt{3}})$.

此时$f(x,y,z)=2$，距离原点距离为$\sqrt{2}$.

则最小距离为1，最大距离为$\sqrt{2}$.

9. 解. 由椭圆对称性，仅考虑$x>0$, $y>0$时情况.(即第一象限).

在椭圆方程中对x求导有

$$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0.$$

则切线斜率 $y'=-\frac{b^2x}{a^2y}$.

对于第一象限中点 (x_0, y_0)，切线方程为 $y-y_0=-\frac{b^2x_0}{a^2y_0}(x-x_0)$.

则其在 y 轴截距为 $y_1=y_0+\frac{b^2x_0^2}{a^2y_0}$.

其在 x 轴截距为 $x_1=x_0+\frac{a^2y_0^2}{b^2x_0}$

则三角形面积 $f(x,y)=\frac{1}{2}(x_0+\frac{a^2y^2}{b^2x})\cdot(y+\frac{b^2x}{a^2y})$

$=\frac{1}{2}\cdot\frac{a^2b^2}{xy}$.

条件为 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

考虑函数 $L(x,y,\lambda)=\frac{1}{2}\frac{a^2b^2}{xy}+\lambda(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)$.

则
$$\begin{cases} L_x'=-\frac{1}{2}\cdot\frac{a^2b^2}{x^2y}+\frac{2\lambda x}{a^2} & (1) \\ L_y'=-\frac{1}{2}\frac{a^2b^2}{xy^2}+\frac{2\lambda y}{b^2} & (2) \\ L_\lambda'=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 & (3). \end{cases}$$

令 (1),(2)=0 并联立可得 $\frac{\lambda x}{a^2y}=\frac{\lambda y}{b^2x}$.

又 ∵ $\lambda\neq 0$ $(x,y>0)$.

∴ 有 $y=\frac{b}{a}x$.

代入可解得 $(x,y)=(\frac{a}{\sqrt2},\frac{b}{\sqrt2})$.

又 ∵ $x\to 0$ 时, $f(x,y)\to+\infty$. 故该点为最小值点.

由对称性可知最小值点为 $(\frac{a}{\sqrt2},\frac{b}{\sqrt2})$, $(-\frac{a}{\sqrt2},\frac{b}{\sqrt2})$, $(\frac{a}{\sqrt2},-\frac{b}{\sqrt2})$, $(-\frac{a}{\sqrt2},-\frac{b}{\sqrt2})$.

10. 解: $f(x,y)=\frac{1}{2}(x^n+y^n)$. 条件 $x+y=A$.

考虑函数 $L(x,y,\lambda)=\frac{1}{2}(x^n+y^n)+\lambda(x+y-A)$.

则
$$\begin{cases} L_x'=\frac{1}{2}\cdot nx^{n-1}+\lambda & (1) \\ L_y'=\frac{1}{2}ny^{n-1}+\lambda & (2) \\ L_\lambda'=x+y-A. & (3) \end{cases}$$

令 (1).(2)=0 并联立有 $x^{n-1}=y^{n-1}$. (4).

P7.

联立 (3), (4). 有 $x=y=\frac{A}{2}$.

此时. $f(x,y)=(\frac{A}{2})^n$.

又当 $x=0$, $y=A$ 时 $f(x,y)=\frac{1}{2}A^n \geq (\frac{A}{2})^n$.

$\therefore$ $(\frac{A}{2},\frac{A}{2})$ 为 $f(x,y)$ 的最小值点. 最小值为 $(\frac{A}{2})^n$.

即 $f(x,y)=\frac{1}{2}(x^n+y^n) \geq (\frac{A}{2})^n = (\frac{x+y}{2})^n$.