On properties of a class of strong limits for supercritical superprocesses

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Abstract

Suppose that $X = \{X_t, t \geq 0; \mathbb{P}_\mu\}$ is a supercritical superprocess in a locally compact separable metric space E. Let ϕ_0 be a positive eigenfunction corresponding to the first eigenvalue λ_0 of the generator of the mean semigroup of X. Then $M_t := e^{-\lambda_0 t}\langle \phi_0, X_t\rangle$ is a positive martingale. Let M_∞ be the limit of M_t. It is known that M_∞ is non-degenerate iff the $L \log L$ condition is satisfied. When the $L \log L$ condition may not be satisfied, we recently proved in (arXiv:1708.04422) that there exist a non-negative function γ_t on $[0, \infty)$ and a non-degenerate random variable W such that for any finite nonzero Borel measure μ on E,

$$\lim_{t\to\infty} \gamma_t \langle \phi_0, X_t\rangle = W, \qquad \text{a.s.-}\mathbb{P}_\mu.$$

In this paper, we mainly investigate properties of W. We prove that W has strictly positive density on $(0, \infty)$. We also investigate the small value probability and tail probability problems of W.

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1 Background and our model

Consider a supercritical Galton-Watson process $\{Z_n, n \geq 0\}$ with offspring distribution $\{p_n : n \geq 0\}$. In 1968, Seneta [20] proved that there exists a sequence of positive numbers $\{c_n, n \geq 1\}$ such that $c_n Z_n$ converges in distribution to a non-degenerate random variable W; then Heyde [9] strengthened this convergence to almost sure convergence. Since then, the problem of finding $\{c_n, n \geq 1\}$ such that $c_n Z_n$ converges to a non-degenerate limit is called the Seneta-Heyde norming problem, $\{c_n, n \geq 1\}$ are called the norming constants.

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Harris [7] proved that, when $\{p_n : n \geq 0\}$ has a finite second moment, the distribution of W, restricted to $(0, \infty)$, is absolutely continuous; then Stigum [22] extended this to the case when $\{p_n : n \geq 0\}$ satisfies the $L \log L$ condition. Finally, Athreya [1] proved that the same conclusion holds for all supercritical Galton-Watson processes. As for other properties of W, [14] discussed the small value probability problem for W, i.e., the rate at which the probability $P(0 < W \leq r)$ tends to 0 as $r \to 0$; [2] studied the tail probability problem for W, i.e., the rate at which $P(W > r)$ tends to 0 as $r \to \infty$, under the assumption that there exists $N > 0$ such that $p_n = 0$ for all $n \geq N$.

For supercritical multitype Galton-Watson process, Jones [10] studied the corresponding small value probability problem and tail probability problem. Hering [8] established the corresponding results for supercritical branching Markov processes. In the recent paper [18], we studied the Seneta-Heyde type limit problem for supercritical superprocesses: Suppose $\{X_t, t \geq 0; \mathbb{P}_\mu\}$ is a supercritical superprocess on E, we proved that, under certain conditions, there exist a non-negative function γ_t on $[0, \infty)$ and a non-degenerate random variable W such that for all finite Borel measure μ on E,

$$\lim_{t \to \infty} \gamma_t \langle \phi_0, X_t \rangle = W, \qquad \text{a.s.-}\mathbb{P}_\mu,$$

where ϕ_0 is a positive eigenfunction of the infinitesimal generator of the mean semigroup of X corresponding to the first eigenvalue λ_0. The main goals of this paper are to study the absolute continuity of W, when restricted to $(0, \infty)$, and to study the small value probability problem and tail probability problem for W.

1.1 Superprocesses

We first introduce the setup of this paper. Suppose that E is a locally compact separable metric space, and ∂ is a separate point not contained in E. We will use E_∂ to denote $E \cup \{\partial\}$. Suppose that m is a σ-finite Borel measure on E with full support. We will use $\mathcal{B}(E)$ ($\mathcal{B}^+(E)$) to denote the family of (non-negative) Borel functions on E, $\mathcal{B}_b(E)$ ($\mathcal{B}_b^+(E)$) to denote the family of (non-negative) bounded Borel functions on E, and $C(E)$ to denote the family of continuous functions on E. We assume that $\xi = \{\Omega^0, \mathcal{H}, \mathcal{H}_t, \xi_t, \Pi_x, \zeta\}$ is a Hunt process on E, where $\{\mathcal{H}_t : t \geq 0\}$ is the minimal filtration of ξ satisfying the usual conditions and $\zeta := \inf\{t > 0 : \xi_t = \partial\}$ is the lifetime of ξ. We will use $\{P_t : t \geq 0\}$ to denote the semigroup of ξ.

The superprocess $X = \{X_t : t \geq 0\}$ we are going to work with is determined by two parameters: a spatial motion $\xi = \{\xi_t, \Pi_x\}$ on E which is a Hunt process, and a branching mechanism φ of the form

$$\varphi(x, s) = -\alpha(x)s + \beta(x)s^2 + \int_{(0,+\infty)} (e^{-sr} - 1 + sr) n(x, \mathrm{d}r), \quad x \in E,\ s \geq 0, \tag{1.1}$$

where $\alpha \in \mathcal{B}_b(E)$, $\beta \in \mathcal{B}_b^+(E)$ and n is a kernel from E to $(0,\infty)$ satisfying

$$\sup_{x\in E}\int_{(0,+\infty)}(r\wedge r^2)n(x,\mathrm{d}r)<\infty. \tag{1.2}$$

It follows from the above assumptions that there exists $M>0$ such that

$$|\alpha(x)|+\beta(x)+\int_{(0,+\infty)}(r\wedge r^2)n(x,\mathrm{d}r)\le M. \tag{1.3}$$

Let $\mathcal{M}_F(E)$ be the space of finite measures on E, equipped with the topology of weak convergence. The superprocess X with spatial motion ξ and branching mechanism φ is a Markov process taking values in $\mathcal{M}_F(E)$. The existence of such superprocesses is well-known, see [12] or [6], for instance. For any $\mu \in \mathcal{M}_F(E)$, we denote the law of X with initial configuration μ by $\mathbb{P}_\mu$. As usual, $\langle f,\mu\rangle := \int_E f(x)\mu(dx)$ and $\|\mu\| := \langle 1,\mu\rangle$. Throughout this paper, a real-valued function $u(t,x)$ on $[0,\infty)\times E_\partial$ is said to be locally bounded if, for any $t>0$, $\sup_{s\in[0,t],x\in E_\partial}|u(s,x)|<\infty$. Any function f on E is automatically extended to E_∂ by setting $f(\partial)=0$. According to [12, Theorem 5.12], there is a Hunt process $X = \{\Omega,\mathcal{G},\mathcal{G}_t,X_t,\mathbb{P}_\mu\}$ taking values in $\mathcal{M}_F(E)$ such that for every $f\in\mathcal{B}_b^+(E)$ and $\mu\in\mathcal{M}_F(E)$,

$$-\log\mathbb{P}_\mu\left(e^{-\langle f,X_t\rangle}\right)=\langle V_tf,\mu\rangle, \tag{1.4}$$

where $V_tf(x)$ is the unique locally bounded non-negative solution to the equation

$$V_tf(x)+\Pi_x\int_0^t\varphi(\xi_s,V_{t-s}f(\xi_s))\mathrm{d}s=\Pi_xf(\xi_t),\quad x\in E_\partial, \tag{1.5}$$

where we use the convention that $\varphi(\partial,r)=0$ for all $r\ge 0$. Since $f(\partial)=0$, we have $V_tf(\partial)=0$ for any $t\ge0$. In this paper, the superprocess we deal with is always this Hunt realization.

For any $f\in\mathcal{B}_b(E)$ and $(t,x)\in(0,\infty)\times E$, we define

$$T_tf(x):=\Pi_x\left[e^{\int_0^t\alpha(\xi_s)\mathrm{d}s}f(\xi_t)\right]. \tag{1.6}$$

It is well known that $T_tf(x)=\mathbb{P}_{\delta_x}\langle f,X_t\rangle$ for every $x\in E$.

We will always assume that there exists a family of continuous and strictly positive functions $\{p(t,x,y):t>0\}$ on $E\times E$ such that for any $t>0$ and non-negative function f on E, $P_tf(x)=\int_E p(t,x,y)f(y)m(\mathrm{d}y)$. Define

$$a_t(x):=\int_E p(t,x,y)^2\,m(\mathrm{d}y),\qquad \hat{a}_t(x):=\int_E p(t,y,x)^2\,m(\mathrm{d}y).$$

Our first assumption is

Assumption 1 **(i)** *For any $t>0$, $\int_E p(t,x,y)\,m(\mathrm{d}x)\le 1$.*

(ii) *For any $t > 0$, we have*

$$\int_E a_t(x)\, m(\mathrm{d}x) = \int_E \hat{a}_t(x)\, m(\mathrm{d}x) = \int_E \int_E p(t,x,y)^2\, m(\mathrm{d}y)\, m(\mathrm{d}x) < \infty. \tag{1.7}$$

Moreover, the functions $x \to a_t(x)$ and $x \to \hat{a}_t(x)$ are continuous on E.

Note that, in Assumption 1(i), the integration is with respect to the first space variable. This implies that the dual semigroup $\{\widehat{P}_t : t \geq 0\}$ of $\{P_t : t \geq 0\}$ with respect to m is sub-Markovian. By Hölder's inequality, we have

$$p(t+s,x,y) = \int_E p(t,x,z)p(s,z,y)\, m(\mathrm{d}z) \leq (a_t(x))^{1/2}(\hat{a}_s(y))^{1/2}. \tag{1.8}$$

$\{P_t : t \geq 0\}$ and $\{\widehat{P}_t : t \geq 0\}$ are both strongly continuous contraction semigroups on $L^2(E,m)$, see [16] for a proof. We will use $\langle \cdot, \cdot \rangle_m$ to denote the inner product in $L^2(E,m)$. Since $p(t,x,y)$ is continuous in (x,y), by (1.8), Assumption 1(ii) and the dominated convergence theorem, we have that, for any $f \in L^2(E,m)$, $P_t f$ and $\widehat{P}_t f$ are continuous.

It follows from Assumption 1(ii) that, for each $t > 0$, $\{P_t\}$ and $\{\widehat{P}_t\}$ are compact operators on $L^2(E,m)$. Let $\widetilde{L}$ and $\widehat{\widetilde{L}}$ be the infinitesimal generators of the semigroups $\{P_t\}$ and $\{\widehat{P}_t\}$ in $L^2(E,m)$ respectively. Define $\widetilde{\lambda}_0 := \sup \Re(\sigma(\widetilde{L})) = \sup \Re(\sigma(\widehat{\widetilde{L}}))$, where $\Re$ stand for the real part of a complex number. By Jentzsch's theorem ([19, Theorem V.6.6]), $\widetilde{\lambda}_0$ is an eigenvalue of multiplicity 1 for both $\widetilde{L}$ and $\widehat{\widetilde{L}}$. Let $\widetilde{\phi}_0$ and $\widetilde{\psi}_0$ be, respectively, eigenfunctions of $\widetilde{L}$ and $\widehat{\widetilde{L}}$ corresponding to $\widetilde{\lambda}_0$. $\widetilde{\phi}_0$ and $\widetilde{\psi}_0$ can be chosen be strictly positive m-almost everywhere with $\|\widetilde{\phi}_0\|_2 = 1$ and $\langle \widetilde{\phi}_0, \widetilde{\psi}_0 \rangle_m = 1$. Thus for m-almost every $x \in E$,

$$e^{\widetilde{\lambda}_0}\widetilde{\phi}_0(x) = P_1\widetilde{\phi}_0(x), \qquad e^{\widetilde{\lambda}_0}\widetilde{\psi}_0(x) = \widehat{P}_1\widetilde{\psi}_0(x).$$

Hence, by the continuity of $P_1\widetilde{\phi}_0$ and $\widehat{P}_1\widetilde{\psi}_0$, $\widetilde{\phi}_0$ and $\widetilde{\psi}_0$ can be chosen to be continuous and strictly positive everywhere on E.

Our second assumption is that $\{P_t : t \geq 0\}$ and $\{\widehat{P}_t : t \geq 0\}$ are intrinsically ultracontractive.

Assumption 2 **(i)** *$\widetilde{\phi}_0$ is bounded.*

(ii) *The semigroups $\{P_t, t \geq 0\}$ and $\{\widehat{P}_t : t \geq 0\}$ are intrinsically ultracontractive, that is, there exists $c_t > 0$ such that*

$$p(t,x,y) \leq c_t \widetilde{\phi}_0(x)\widetilde{\psi}_0(y). \tag{1.9}$$

In [16], we have given many examples of Hunt processes satisfying Assumptions 1–2. For example, if E be a bounded Lipschitz domain, and ξ is the subprocess of a diffusion process whose generator is a uniformly elliptic second order differential operator, then ξ satisfies Assumptions 1–2, see [4].

By using the boundedness of α and assumptions on ξ, we have proved in [15, Lemma 2.1] that the semigroup $\{T_t\}$ has a continuous and strictly positive density $q(t,x,y)$ with respect to the measure m, that is, for any $f \in \mathcal{B}_b(E)$,

$$T_t f(x) = \int_E q(t,x,y) f(y)\, m(\mathrm{d}y).$$

and, for any $t > 0$, $q(t,x,y)$ is continuous in (x,y), and

$$e^{-Mt} p(t,x,y) \le q(t,x,y) \le e^{Mt} p(t,x,y), \quad (t,x,y) \in (0,\infty) \times E \times E, \tag{1.10}$$

Let $\{\widehat{T}_t, t > 0\}$ be the dual semigroup of $\{T_t, t > 0\}$ in $L^2(E,m)$, that is, for any $f, g \in L^2(E,m)$,

$$\widehat{T}_t f(x) = \int_E q(t,y,x) f(y)\, m(dy).$$

It follows from Assumption 1(ii) and (1.10) that

$$\int_E \int_E q^2(t,x,y)\, m(x)\, m(dy) \le e^{2Mt} \int_E \int_E p^2(t,x,y)\, m(x)\, m(dy) < \infty.$$

Thus using the same analysis as that used before Assumption 2 we can get the following conclusion: for any $t > 0$, T_t and $\widehat{T}_t$ are compact operators on $L^2(E,m)$. Let L and $\widehat{L}$ be the infinitesimal generators of $\{T_t\}$ and $\{\widehat{T}_t\}$ in $L^2(E,m)$ respectively. Define $\lambda_0 := \sup \Re(\sigma(L)) = \sup \Re(\sigma(\widehat{L}))$. λ_0 is an eigenvalue of multiplicity one for both L and $\widehat{L}$. Let ϕ_0 and ψ_0 be, respectively, eigenfunctions of L and $\widehat{L}$ corresponding to λ_0. ϕ_0 and ψ_0 can be chosen to be continuous and strictly positive everywhere with $\|\phi_0\|_2 = 1$, $\langle \phi_0, \psi_0 \rangle_m = 1$.

Using Assumption 2, the boundedness of α and an argument similar to that used in the proof of [4, Theorem 3.4], one can show the following:

(i) ϕ_0 is bounded.

(ii) The semigroup $\{T_t, t \ge 0\}$ and $\{\widehat{T}_t, t > 0\}$ are intrinsically ultracontractive, that is, there exists $c_t > 0$ such that

$$q(t,x,y) \le c_t \phi_0(x) \psi_0(y). \tag{1.11}$$

Define $q_t(x) := \mathbb{P}_{\delta_x}(\|X_t\| = 0)$ and $\mathcal{E} := \{\exists t > 0, \|X_t\| = 0\}$. Note that $q_t(x)$ is non-decreasing in t. Thus the limit

$$q(x) := \lim_{t\to\infty} q_t(x) = \mathbb{P}_{\delta_x}(\mathcal{E})$$

exist. We call $q(x)$ the extinction probability of the superprocess. Let $v(x) := -\log q(x)$. It follows from the branching property of X that $\mathbb{P}_\mu(\mathcal{E}) = e^{-\langle v, \mu \rangle}$. The main interest of this paper is on supercritical superprocesses, so we assume that

Assumption 3 $\lambda_0 > 0$.

We also assume that

Assumption 4 *There exists $t_0 > 0$ such that*

$$\inf_{x\in E} q_{t_0}(x) > 0. \tag{1.12}$$

Assumption 4 guarantees that $\|v\|_\infty \le \sup_{x\in E}(-\log q_{t_0}(x)) < \infty$, thus v is a bounded function. In [15, Section 2.2], we gave a sufficient condition for Assumption 4. In particular, if $\inf_{x\in E}\beta(x) > 0$, then Assumption 4 holds. Under Assumptions 1–4, we have proven in [18, Lemma 3.1] that $q(x) < 1$, for all $x \in E$, which is a reflection of supercriticality.

1.2 Main results

Define

$$M_t := e^{-\lambda_0 t}\langle \phi_0, X_t\rangle, \quad t \ge 0.$$

It follows from the Markov property that, for every $\mu \in \mathcal{M}_F(E)$, $\{M_t, t \ge 0\}$ is a non-negative $\mathbb{P}_\mu$-martingale. Thus $\{M_t, t \ge 0\}$ has a $\mathbb{P}_\mu$-a.s. finite limit denoted as M_∞. According to [13], M_∞ is non-degenerate if and only if the $L\log L$ condition holds. When M_∞ is a non-degenerate random variable, X_t grows exponentially and the growth rate is $e^{\lambda_0 t}$. When M_∞ is a degenerate random variable, $e^{\lambda_0 t}$ is no longer the growth rate of X_t. In [18], we proved that, when the $L\log L$ condition may not be satisfied, the growth rate of X_t is $e^{\lambda_0 t}L(t)$, where $L(t)$ is a slowly varying function after some transform. Now we state the main results of [18].

In [18], we proved that there exists a family of non-negative functions $\{\eta_t(x), t \ge 0\}$, satisfying $0 \le \eta_t(x) \le v(x)$, such that

$$\eta_t(x) = V_s(\eta_{t+s})(x), \qquad t, s \ge 0, x \in E.$$

Furthermore, η_0 is not identically 0, is also not identically equal to v. Let $\gamma_t = \langle \eta_t, \psi_0\rangle_m$, then

$$\lim_{t\to\infty} \frac{\gamma_t}{\gamma_{t+s}} = e^{\lambda_0 s}, \quad \forall s \ge 0,$$

and the following assertions are valid.

Theorem 1.1 [18, Theorem 1.2]. *There exists a non-degenerate random variable W such that for all $\mu \in \mathcal{M}_F(E)$,*

$$\lim_{t\to\infty} \gamma_t\langle \phi_0, X_t\rangle = W, \qquad a.s.\text{-}\mathbb{P}_\mu$$

and

$$\mathbb{P}_\mu(W = 0) = e^{-\langle v,\mu\rangle}, \qquad \mathbb{P}_\mu(W < \infty) = 1.$$

Define a new measure $n^{\phi_0}(x, \mathrm{d}r)$ by

$$\int_0^\infty f(r) n^{\phi_0}(x, \mathrm{d}r) = \int_0^\infty f(r\phi_0(x)) n(x, \mathrm{d}r).$$

Let $l(x) := \int_1^\infty r \ln r \, n^{\phi_0}(x, \mathrm{d}r)$. Necessary and sufficient conditions for M_∞ to be non-degenerate are as follows:

Theorem 1.2 [18, Theorem1.3]. *The following are equivalent:*

(1) *for some $\mu \in \mathcal{M}_F(E)$, M_∞ is a non-degenerate random variable under $\mathbb{P}_\mu$;*

(2) *for every non-zero $\mu \in \mathcal{M}_F(E)$, M_∞ is a non-degenerate random variable under $\mathbb{P}_\mu$;*

(3) $l_0 := \lim_{t\to\infty} e^{\lambda_0 t}\gamma_t < \infty$*;*

(4) *($L \log L$ **criterion:**)* $\int_E \psi_0(x) l(x) m(\mathrm{d}x) < \infty$;

(5) *for some non-zero $\mu \in \mathcal{M}_F(E)$, $\mathbb{P}_\mu W < \infty$;*

(6) *for every non-zero $\mu \in \mathcal{M}_F(E)$, $\mathbb{P}_\mu W < \infty$.*

It follows from [18, Remark 1.1] that, $\int_E \psi_0(x) l(x) m(\mathrm{d}x) < \infty$ if and only if

$$\int_E \phi_0(x)\psi_0(x) m(\mathrm{d}x) \int_1^\infty (r \ln r) n(x, \mathrm{d}r) < \infty. \tag{1.13}$$

The main purpose of this paper is to further study properties of W: including whether W has a density function, the small value probability problem and the tail problem for W. The main results of this paper are as follows.

Theorem 1.3 *For any non-zero $\mu \in \mathcal{M}_F(E)$, under $\mathbb{P}_\mu$, the restriction of the random variable W on $(0, \infty)$ has a strictly positive density.*

In Subsection 3.1, we will introduce another semigroup $\{T_t^*\}$ with largest eigenvalue $\lambda_0^* < 0$. Define

$$\epsilon_0 := \frac{-\lambda_0^*}{\lambda_0}.$$

Let

$$L(t) = e^{-\lambda_0 t}\gamma_t, \tag{1.14}$$

then

$$\lim_{t\to\infty} \frac{L(t+s)}{L(t)} = 1.$$

Define

$$\widetilde{L}(\theta) := L(\log\theta/\lambda_0), \quad \theta \geq 1, \tag{1.15}$$

then $\widetilde{L}$ is a slowly varying function at ∞.

Theorem 1.4 *For any non-zero* $\mu \in \mathcal{M}_F(E)$,

$$\lim_{r\to 0} r^{-\epsilon_0}\mathbb{P}_\mu(0 < W \le r) = e^{-\langle v,\mu\rangle}A(\psi(1))\langle v\phi_0^*,\,\mu\rangle/\Gamma(\epsilon_0+1),$$

where $\Gamma(\cdot)$ *is the usual* Γ *function,* ϕ_0^* *is an egenfunction of* T_t^* *corresponding to the eigenvalue* $e^{\lambda_0^* t}$*, the operator* A *is defined in* (3.16)*. Furthermore*

$$\lim_{r\to\infty} r\widetilde{L}(r)^{-1}\mathbb{P}_\mu(W > r) = 0.$$

Remark 1.5 For a Galton-Watson process, the small value probability problem of W can be divided into two cases: the Schröder case and Böttcher case, see [10, 14]. Suppose $\{Z_n, n \ge 0\}$ is a Galton-Watson process with offspring distribution $\{p_n, n \ge 0\}$. Let q be its extinction probability, $f(s)$ be the probability generating function of $\{p_n, n \ge 0\}$, and $m > 1$ be the mean of the offspring distribution. Let $\gamma = f'(q)$.

(1) If $p_0 + p_1 > 0$, then $F(s) := \lim_{n\to\infty} \gamma^{-n}(f_n(s) - q)$ exists, and F satisfies the Schröder equation: $F(f(s)) = \gamma f(s)$. Let $\epsilon = -\log\gamma/\log m$, then

$$P(W \le r) \asymp r^{-\epsilon}.$$

(2) If $p_0+p_1 = 0$, then $\lambda = \min\{n : p_n > 0\} \ge 2$. In this case $G(s) := \lim_{n\to\infty} -\lambda^{-n}\log f_n(s)$ exists, and the function $\overline{G} = e^{-G}$ satisfies the Böttcher equation $\overline{G}(f) = \overline{G}^\lambda$. Let $\beta = \log\lambda/\log m$, then one can obtain

$$-\log P(W \le r) \asymp r^{-\beta/(1-\beta)}.$$

For the branching Markov process in [8] and the superprocess in this paper, the small value probability problem of the strong limit W has only one case, the Schröder case. In fact, for the branching Markov process $\{Z_t, t \ge 0\}$ in [8], when the extinction probability is 0, one can show that $\lim_{t\to\infty} e^{-\lambda_0^* t}\overline{F}_t f$ exists, where $\overline{F}_t f := P_{\delta_x}(e^{\langle \log f, Z_t\rangle})$.

Suppose that there exists $N > 0$ such that the offspring distribution $\{p_n\}$ of the Galton-Watson process satisfies $p_n = 0$ for all $n \ge N$, then [2] obtained the rate at which the tail probability of W tends to 0. For results on the rate at which tail probability of W tends to 0 for multitype Galton-Watson processes, see [10]. For superprocesses, under some condition, the rate at which tail probability of W tends to 0 is determined by the skeleton process (a branching Markov process) of X. When the branching mechanisms $n(x, \mathrm{d}r)$ is not 0, the offspring distribution $\{p_n\}$ (see (2.12) and (2.13)) of the skeleton process of X does not satisfy this condition, thus we could not get the rate at which the tail probability $\mathbb{P}_\mu(W > r)$ tends to 0 as $r \to \infty$. We only obtain a weaker result.

In Section 2, we will prove that W is a compound Poisson random variable of the form $W = \sum_{n=1}^{N} Y_n$, where N is a Poisson random variable, $Y_j, j \geq 1$ is a sequence of independent and identically distributed random variables independent of N. We will also prove that the distribution of Y_1 is the distribution of the corresponding strong limit of some branching Markov process. In Section 3, we will analyze and estimate the Laplace transform and characteristic function of Y_1, and show that Y_1 restricted to $(0, \infty)$ has a density function, thus proving 1.3. By using the Tauberian theorem, we can prove Theorem 1.4.

2 Compound Poisson random variable and branching Markov process

2.1 Compound Poisson random variable

The Laplace exponent of W is defined as

$$\Phi(\theta, x) := -\log \mathbb{P}_{\delta_x} \exp\{-\theta W\}. \tag{2.1}$$

Using the Markov property and the branching property, we have shown in [18, (5.3)] that

$$\Phi(\theta, x) = V_t(\Phi(\theta e^{-\lambda_0 t}, \cdot))(x). \tag{2.2}$$

Lemma 2.1 *For any $x \in E$, there exists a finite measure $\pi(x, dr)$ on $(0, \infty)$ such that $\pi(x, (0, \infty)) = v(x)$, and*

$$\Phi(\theta, x) = \int_{(0,\infty)} \left(1 - e^{-\theta r}\right) \pi(x, dr).$$

Proof: Since

$$\mathbb{P}_\mu\left[e^{-\theta W}\right] = \left(\mathbb{P}_{\mu/n}\left[e^{-\theta W}\right]\right)^n,$$

the distribution of the random variable W under $\mathbb{P}_\mu$ is infinitely divisible. Since W is non-negative, there exist a non-negative function $a(x)$ and a σ-finite measure $\pi(x, dr)$ satisfying the condition

$$\int_0^\infty (1 \wedge r) \pi(x, dr) < \infty$$

such that

$$\Phi(\theta, x) = a(x)\theta + \int_{(0,\infty)} \left(1 - e^{-\theta r}\right) \pi(x, dr). \tag{2.3}$$

It follows from [18, Theorem 1.2] that

$$\Phi(\infty, x) = -\log \mathbb{P}_{\delta_x}(W = 0) = v(x).$$

From this one gets $a(x) = 0$ and

$$\pi(x, (0, \infty)) = v(x).$$

The assertion of the lemma follows immediately. $\square$

It follows from Lemma 2.1 that, under $\mathbb{P}_\mu$, W is a compound Poisson random variable, that is,

$$W = \sum_{n=1}^{N} Y_n,$$

where N is a Poisson random variable with parameter $\langle v, \mu\rangle$, $\{Y_j, j \geq 1\}$ is a sequence of independent and identically distributed random variables with common distribution $\frac{\int_E \pi(x,\mathrm{d}y)\,\mu(dx)}{\langle v,\mu\rangle}$ and independent of N. From now on, we assume that Y is a random variable with distribution $\frac{\int_E \pi(x,\mathrm{d}y)\,\mu(dx)}{\langle v,\mu\rangle}$.

Lemma 2.2 *For any non-zero $\mu \in \mathcal{M}_F(E)$, under $\mathbb{P}_\mu$, the random variable W restricted to $(0, \infty)$ has a density if and only if the random variable Y has a density. Furthermore, if the density of Y is $g_\mu(y)$, then for any $0 < a < b$,*

$$\mathbb{P}_\mu(W \in (a, b)) = \int_a^b f_\mu(y)\, dy,$$

where

$$f_\mu(y) = \sum_{k=1}^{\infty} g_\mu^{*k}(y) \frac{\langle v, \mu\rangle^k}{k!} e^{-\langle v,\mu\rangle}. \tag{2.4}$$

Remark 2.3 *If for every $x \in E$, Y has a density function $g(x, y)$ under $\mathbb{P}_{\delta_x}$, then*

$$\pi(x, dy) = v(x)g(x, y)dy.$$

Thus for every $\mu \in \mathcal{M}_F(E)$, Y has a density function under $\mathbb{P}_\mu$:

$$g_\mu(y) = \frac{\int_E v(x)g(x, y)\,\mu(dx)}{\langle v, \mu\rangle}, \quad y > 0.$$

It follows from Lemma 2.2, it suffices to prove that the random variable Y has a density function. For this we will analyze the Laplace transform and characteristic function of Y. Define

$$\psi(\theta, x) := \frac{v(x) - \Phi(\theta, x)}{v(x)} = v(x)^{-1} \int_{(0,\infty)} e^{-\theta r} \pi(x, dr), \quad \theta \geq 0.$$

Thus

$$\mathbb{P}_\mu\Big(e^{-\theta Y}\Big) = \frac{\langle v\psi(\theta, \cdot), \mu\rangle}{\langle v, \mu\rangle}, \quad \theta \geq 0.$$

Note that $\psi(\theta, x)$ is the Laplace transform of the distribution $v(x)^{-1}\pi(x, dr)$. For any $x \in E$, $\theta \geq 0$, $\psi(\theta, x) \in [0, 1]$.

Similarly, we define

$$\psi(i\theta, x) := \mathbb{P}_{\delta_x}\left(e^{-i\theta Y}\right) = v(x)^{-1} \int_{(0,\infty)} e^{-i\theta r}\pi(x, dr), \quad \theta \in \mathbb{R}.$$

For any $a > 0$, let $\mathcal{D}_a := \{f \in \mathcal{B}(E) : 0 \leq f(x) \leq a, x \in E\}$. Define an operator $\overline{V}_t : \mathcal{D}_1 \to \mathcal{D}_1$ by

$$\overline{V}_t f(x) := \frac{v(x) - V_t(v(1-f))(x)}{v(x)}, \qquad f \in \mathcal{D}_1. \tag{2.5}$$

It follows from (2.2) that, for all $\theta \geq 0$,

$$\psi(\theta, x) = \overline{V}_t(\psi(\theta e^{-\lambda_0 t}, \cdot))(x). \tag{2.6}$$

Obviously, we can extend the definition of $\overline{V}_t$ to the space of all complex-valued functions on E with sup norm less than or equal to 1.

In the next subsection, we will show that $\overline{V}_t$ is the Laplace functional of some branching Markov process. A skeleton decomposition of superprocesses was established under some conditions in [3, 5]. When the conditions of [3, 5] are satisfied, the branching Markov process we are going to introduce below is just the skeleton process of the (ξ, φ)-superprocess. [5] dealt with the skeleton decomposition of super-diffusions, while [3] dealt with the skeleton decomposition of superprocesses with a symmetric spatial motion. [3, 5] can not completely cover the superprocesses dealt with in this paper. In this paper we do not use the skeleton decomposition of superprocesses. We start with that W is a compound Poisson random variable, and introduce the corresponding branching Markov process.

2.2 Branching Markov processes

Define

$$N_t := \frac{v(\xi_t)}{v(\xi_0)} \exp\Big\{ - \int_0^t \frac{\varphi(\xi_s, v(\xi_s))}{v(\xi_s)}\, ds\Big\}.$$

Lemma 2.4 *Under Π_x, $\{N_t : t \geq 0\}$ is a non-negative martingale with respect to the filtration $\{\mathcal{H}_t, t \geq 0\}$, and $\Pi_x(N_t) = 1$.*

Proof: It follows from the Markov property and the branching property that, for any $t > 0$, $v(x) = V_t v(x)$. Thus

$$v(x) + \Pi_x \int_0^t \varphi(\xi_s, v(\xi_s))\mathrm{d}s = \Pi_x v(\xi_t). \tag{2.7}$$

It is easy to see that

$$|\varphi(x, z)| \leq 2M(z + z^2), \quad z \geq 0,$$

Thus

$$\frac{|\varphi(x, v(x))|}{v(x)} \le 2M(1 + v(x)) \le 2M(1 + \|v\|_\infty). \tag{2.8}$$

Hence it follows from the Feynman-Kac formula that

$$v(x) = \Pi_x\Big[\exp\Big\{ - \int_0^t \frac{\varphi(\xi_s, v(\xi_s))}{v(\xi_s)}\, ds\Big\} v(\xi_t)\Big]. \tag{2.9}$$

It follows immediately from the Markov property and (2.9) that

$$\begin{aligned}\Pi_x(N_{t+s}|\mathcal{H}_t) =& v(\xi_0)^{-1} \exp\Big\{ - \int_0^t \frac{\varphi(\xi_s, v(\xi_s))}{v(\xi_s)}\, ds\Big\} \\ &\times \Pi_{\xi_t}\Big[v(\xi_s) \exp\Big\{ - \int_0^s \frac{\varphi(\xi_s, v(\xi_s))}{v(\xi_s)}\, ds\Big\} = N_t.\end{aligned}$$

Thus $\{N_t : t \ge 0\}$ is a non-negative martingale. □

We use the martingale $\{N_t\}$ to define a new probability measure $\overline{\Pi}_x$:

$$\frac{d\overline{\Pi}_x}{d\Pi_x}\Big|_{\mathcal{H}_t} = N_t, \qquad t \ge 0.$$

Proposition 2.5 *For* $f \in \mathcal{D}_1$,

$$\overline{V}_t f(x) = \overline{\Pi}_x \int_0^t \varphi^*(\xi_s, \overline{V}_{t-s} f(\xi_s))\mathrm{d}s + \overline{\Pi}_x f(\xi_t), \tag{2.10}$$

where

$$\varphi^*(x, \lambda) := \frac{\varphi(x, v(x)(1-\lambda)) - \varphi(x, v(x))(1-\lambda)}{v(x)}, \quad \lambda \in [0, 1].$$

Proof: It follows from (1.5) that for $f \in \mathcal{D}_1$,

$$V_t(v(1-f))(x) + \Pi_x \int_0^t \varphi(\xi_s, V_{t-s}(v(1-f))(\xi_s))\mathrm{d}s = \Pi_x v(\xi_t)(1 - f(\xi_t)).$$

Thus by (2.7), we have

$$v(x)\overline{V}_t f(x) + \Pi_x \int_0^t \varphi(\xi_s, v(\xi_s)) - \varphi(\xi_s, v(\xi_s)(1 - \overline{V}_{t-s} f(\xi_s)))\mathrm{d}s = \Pi_x v(\xi_t) f(\xi_t).$$

Hence

$$\begin{aligned}v(x)\overline{V}_t f(x) =& \Pi_x \int_0^t v(\xi_s)\varphi^*(\xi_s, \overline{V}_{t-s} f(\xi_s))\mathrm{d}s \\ &- \Pi_x \int_0^t \frac{\varphi(\xi_s, v(\xi_s))}{v(\xi_s)} v(\xi_s)\overline{V}_{t-s} f(\xi_s)\mathrm{d}s + \Pi_x v(\xi_t) f(\xi_t).\end{aligned}$$

It follows from the Feynman-Kac formula that

$$v(x)\overline{V}_tf(x)=\Pi_x\int_0^t e^{-\int_0^s\frac{\varphi(\xi_u,v(\xi_u))}{v(\xi_u)}\,du}v(\xi_s)\varphi^*(\xi_s,\overline{V}_{t-s}f(\xi_s))\mathrm{d}s+\Pi_x\Big[e^{-\int_0^t\frac{\varphi(\xi_s,v(\xi_s))}{v(\xi_s)}\,ds}v(\xi_t)f(\xi_t)\Big],$$

from which (2.10) follows immediately □

With the preparation above, we now introduce a branching Markov process corresponding to $\overline{V}_t$. By the definition of $\varphi^*(x,\lambda)$, we have

$$\begin{aligned}\varphi^*(x,\lambda)=&\frac{\varphi(x,v(x)(1-\lambda))-\varphi(x,v(x))(1-\lambda)}{v(x)}\\=&\beta(x)v(x)(\lambda^2-\lambda)+v(x)^{-1}\int_0^\infty\Big((e^{r\lambda v(x)}-1+\lambda)e^{-rv(x)}-\lambda\Big)n(x,dr)\\=&\beta(x)v(x)\lambda^2+\sum_{n=2}^\infty\int_0^\infty\frac{v(x)^{n-1}(r\lambda)^n}{n!}e^{-rv(x)}\,n(x,dr)\\&-\lambda\Big(\beta(x)v(x)+v(x)^{-1}\int_0^\infty(e^{rv(x)}-1-rv(x))e^{-rv(x)}\,n(x,dr)\Big).\end{aligned}$$

Thus we have

$$\varphi^*(x,\lambda)=b(x)\Big(\sum_{n=2}^\infty\lambda^np_n(x)-\lambda\Big),\tag{2.11}$$

where

$$b(x)=\beta(x)v(x)+v(x)^{-1}\int_0^\infty(e^{rv(x)}-1-rv(x))e^{-rv(x)}\,n(x,dr);$$

$$p_2(x)=\frac{v(x)}{b(x)}\Big(\beta(x)+\frac{1}{2}\int_0^\infty r^2e^{-v(x)r}n(x,dr)\Big);\tag{2.12}$$

$$p_n(x)=\frac{v^{n-1}(x)}{n!b(x)}\int_0^\infty r^ne^{-v(x)r}n(x,dr),\quad n>2.\tag{2.13}$$

It is easy to verify that $\sum_{n=2}^\infty p_n(x)=1$ and $b(x)$ is a bounded non-negative function. In fact,

$$\begin{aligned}b(x)\le&\beta(x)v(x)+v(x)^{-1}\int_0^\infty((rv(x))\wedge(rv(x))^2)\,n(x,dr)\\\le&\beta(x)v(x)+v(x)\int_0^1r^2\,n(x,dr)+\int_1^\infty r\,n(x,dr)\le M\|v\|_\infty+M.\end{aligned}$$

It is also easy to see that $b(x)>0$.

Consider a branching Markov process $\{Z_t,t\ge0;P_\nu\}$ with spatial motion $\{\xi_t,t\ge0;\overline{\Pi}_x\}$, branching rate function $b(x)$ and spatially dependent offspring distribution $\{p_n(x):n\ge2\}$. Then for any $g\in\mathcal{B}_b^+(E)$,

$$P_{\delta_x}(e^{-\langle g,Z_t\rangle})=\overline{V}_t(e^{-g})(x),$$

and

$$Q_t g(x) := P_{\delta_x}(\langle g, Z_t\rangle) = \overline{\Pi}_x\left(\exp\Big\{\int_0^t \frac{\partial}{\partial\lambda}\varphi^*(\xi_s,1)\,ds\Big\}g(\xi_t)\right) = v(x)^{-1}T_t(vg)(x), \quad (2.14)$$

where the last equality follows from the definitions of φ^* and $\overline{\Pi}_x$. Hence the first eigenvalue of the infinitesimal generator of the semigroup $\{Q_t\}$ is λ_0, and $v(x)^{-1}\phi_0(x)$ is the corresponding eigenfunction. It follows from the boundedness of v that $\{Q_t\}$ is also intrinsically untracontractive, and thus condition (M) in [8] holds. Hence it follows from [8, Proposition 3.6] that there exist a non-negative function $\overline{\gamma}_t$ and a non-degenerate random variable W^Z such that

$$\overline{\gamma}_t\langle v^{-1}\phi_0, Z_t\rangle \to W^Z, \quad P_\nu\text{-a.s.},$$

and the Laplace transform of W^Z, defined by

$$\psi^Z(\theta,x) := P_{\delta_x}\left(e^{-\theta W^Z}\right), \quad \theta\in\mathbb{R},$$

is a solution of (2.6). We already know that the Laplace transform $\psi(\theta,x)$ of Y is also a solution of (2.6), thus it follows from [8, Proposition 3.8] that there exists $a\in(0,\infty)$ such that $(Y,\mathbb{P}_{\delta_x})$ and (aW^Z, P_{δ_x}) have the same distribution. Since $p_0(x)=p_1(x)=0$, the extinction probability of Z is 0. Using the assertions about W^Z in [8, Propositions 5.1, 5.10 and 5.11], one can deduce the corresponding properties of $(Y,\mathbb{P}_{\delta_x})$, and thus obtaining the proofs of Theorem 1.3 and Theorem 1.4.

In Theorem 1.4, the semigroup $\{T_t^*\}$ (see Subsection 3.1), especially the first eigenvalue λ_0^* of its infinitesimal generator and its corresponding eigenfunction, play very important roles. Theorem 1.4 contains another important operator A, which is determined by the limit of $e^{-\lambda_0^* t}\overline{V}_t f(x)$, see (3.16). The semigroup $\{\delta\bar{F}_t(0), t\geq 0\}$ in [8] coincides with the semigroup $\{T_t^*\}$ of this paper, and the operator Q there coincides with our operator A, but [8] did not give explicit expressions for these two quantities. For completeness, we do not quote the conclusions of [8] directly. In Subsection 3.1, we will give the definitions of $\{T_t^*\}$ and A. In Subsections 3.2 and 3.3, we will give the proofs of Theorem 1.3 and Theorem 1.4. The main ideas are similar to that of [8].

3 Proofs of Main Results

3.1 Estimates on the operator $\overline{V}_t$

In this subsection, we will give some estimates on the operator $\overline{V}_t$. We then use these estimates and (2.6) to obtain some estimates on the Laplace transform $\psi(\theta,x)$. In the proof below, C stands for a constant whose value might change from one appearance to another.

We first list some estimates from [18] that we will use in this paper.

(1) **Estimates on the semigroup $\{T_t\}$:** It follows from [11, Theorem 2.7] that, under Assumptions 1–2, for any $\delta > 0$, there exist constants $\gamma = \gamma(\delta) > 0$ and $c = c(\delta) > 0$ such that for all $(t, x, y) \in [\delta, \infty) \times E \times E$, we have

$$\left| e^{-\lambda_0 t} q(t, x, y) - \phi_0(x)\psi_0(y) \right| \le c e^{-\gamma t} \phi_0(x)\psi_0(y). \tag{3.1}$$

Take t large enough so that $ce^{-\gamma t} < \frac{1}{2}$, then

$$e^{-\lambda_0 t} q(t, x, y) \ge \frac{1}{2} \phi_0(x)\psi_0(y).$$

Since $q(t, x, \cdot) \in L^1(E, m)$, we have $\psi_0 \in L^1(E, m)$. Thus for any $f \in \mathcal{B}_b^+(E)$, we have $\langle f, \psi_0 \rangle_m < \infty$. Consequently, for any $f \in \mathcal{B}_b^+(E)(t, x) \in [\delta, \infty) \times E$, we have

$$\left| e^{-\lambda_0 t} T_t f(x) - \langle f, \psi_0 \rangle_m \phi_0(x) \right| \le c e^{-\gamma t} \langle |f|, \psi_0 \rangle_m \phi_0(x) \tag{3.2}$$

and

$$(1 - ce^{-\gamma t}) \langle |f|, \psi_0 \rangle_m \phi_0(x) \le e^{-\lambda_0 t} T_t |f|(x) \le (1 + c) \langle |f|, \psi_0 \rangle_m \phi_0(x). \tag{3.3}$$

(2) v **and** ϕ_0 **are comparable:** It follows from [18, Lemma 4.4] that

$$v(x) = V_1(v)(x) \ge C T_1(v)(x) \ge C \phi_0(x). \tag{3.4}$$

Furthermore,

$$v(x) = V_1 v(x) \le T_1 v(x) \le C \phi_0(x). \tag{3.5}$$

(3) It follows from [18, Proposition 5.3 and Lemma 4.6] that

$$\gamma_t = \langle \Phi(e^{-\lambda_0 t}), \psi_0 \rangle_m,$$

$$\Phi(e^{-\lambda_0 t}, x) = (1 + h_t(x)) e^{-\lambda_0 t} L(t) \phi_0(x), \tag{3.6}$$

where $\lim_{t \to \infty} \|h_t\|_\infty = 0$.

Define a semigroup $P_t^{\varphi'}$ by

$$P_t^{\varphi'} f(x) := \Pi_x \Big(f(\xi_t) e^{-\int_0^t \partial_\lambda \varphi(\xi_s, v(\xi_s))\, ds} \Big). \tag{3.7}$$

It follows from (1.3) and the boundedness of v that $\partial_\lambda \varphi(x, v(x))$ is bounded. By using the same argument as in the paragraph above (1.11), one can show that the semigroup $(P_t^{\varphi'})$ is also intrinsically ultracontractive. Let λ_0^* be the largest (simple) eigenvalue of the infinitesimal generator of $(P_t^{\varphi'})$, let $\overline{\phi}_0$ and $\overline{\psi}_0$ be, respectively, eigenfunctions of the infinitesimal generators of $(P_t^{\varphi'})$ and its dual semigroup corresponding to λ_0^*. $\overline{\phi}_0$ and $\overline{\psi}_0$ can be chosen to be strictly positive continuous functions on E and satisfy $\|\overline{\phi}_0\|_2 = 1$, $\langle \overline{\phi}_0, \overline{\psi}_0 \rangle_m = 1$. Furthermore, $\overline{\phi}_0$ is a bounded function, and $\overline{\psi}_0 \in L^1(E, m)$.

It follows from [11, Theorem 2.7] that, under Assumptions 1–2, for any $\delta > 0$, there exist constants $\gamma = \gamma(\delta) > 0$ and $c = c(\delta) > 0$ such that for all $(t, x, y) \in [\delta, \infty) \times E \times E$, we have

$$\left| e^{-\lambda_0^* t} P_t^{\varphi'} f(x) - \langle f, \overline{\psi}_0 \rangle_m \overline{\phi}_0(x) \right| \le c e^{-\gamma t} \langle |f|, \overline{\psi}_0 \rangle_m \overline{\phi}_0(x). \tag{3.8}$$

Based on this, we define another semigroup

$$T_t^* f(x) := v(x)^{-1} P_t^{\varphi'}(vf)(x) = v(x)^{-1} \Pi_x \Big((vf)(\xi_t) e^{-\int_0^t \partial_\lambda \varphi(\xi_s, v(\xi_s))\, ds} \Big) = \overline{\Pi}_x \big(f(\xi_t) e^{-\int_0^t b(\xi_s)\, ds} \big).$$

Let $\phi_0^*(x) := v(x)^{-1} \overline{\phi}_0(x)$ and $\psi_0^*(x) := v(x) \overline{\psi}_0(x)$. It follows from (3.8) that

$$\left| e^{-\lambda_0^* t} T_t^* f(x) - \langle f, \psi_0^* \rangle_m \phi_0^*(x) \right| \le c e^{-\gamma t} \langle |f|, \psi_0^* \rangle_m \phi_0^*(x). \tag{3.9}$$

Note that $\partial_\lambda \varphi(x, \lambda) \ge -\alpha(x)$. Hence

$$\overline{\phi}_0(x) = e^{\lambda_0^*} P_1^{\varphi'}(\overline{\phi}_0)(x) \le C T_1(\overline{\phi}_0)(x) \le C \phi_0(x).$$

Using (3.4) we see that $\|\phi_0^*\|_\infty < \infty$. It is also easy to see that $\psi_0^* \in L^1(E, m)$.

It can be shown that the semigroup $\{T_t^*, t \ge 0\}$ defined above coincides with the semigroup $\{\delta \bar{F}_t(0)\}$ defined in [8], where $\{\delta \bar{F}_t(0)\}$ is defined via a Fréchet derivative.

Lemma 3.1

$$\lambda_0^* < 0.$$

Proof: It follows from $V_t v(x) = v(x)$ and [17, Lemma 4.1] that

$$\mathbb{P}_{\delta_x} \Big(\langle f, X_t \rangle e^{-\langle v, X_t \rangle} \Big) = \Pi_x \Big(f(\xi_t) e^{-\int_0^t \partial_z \varphi(\xi_s, v(\xi_s))\, ds} \Big) e^{-v(x)}.$$

Thus,

$$T_t^* f(x) = v(x)^{-1} e^{v(x)} \mathbb{P}_{\delta_x} \Big(\langle vf, X_t \rangle e^{-\langle v, X_t \rangle} \Big). \tag{3.10}$$

By [18, Lemma 3.2], we have

$$\mathbb{P}_{\delta_x} (\lim_{t \to \infty} \langle v, X_t \rangle = 0) = 1 - \mathbb{P}_{\delta_x} (\lim_{t \to \infty} \langle v, X_t \rangle = \infty) = e^{-v(x)}.$$

Hence by the dominated convergence theorem, we have

$$\lim_{t \to \infty} T_t^* 1(x) = 0.$$

Combining with (3.9), we immediately get $\lambda_0^* < 0$. □

Lemma 3.2 *For any $a \in [0, 1)$, there exists a constant $c(a) > 0$ such that for $t \ge 1$,*

$$\overline{V}_t f(x) \le c(a) e^{\lambda_0^* (1-a) t} \phi_0^*(x)^{1-a}, \quad \forall f \in \mathcal{D}_a.$$

Proof: Note that

$$\sup_{0\le\lambda\le a}\frac{\varphi^*(x,\lambda)}{\lambda}=\sup_{0\le\lambda\le a}b(x)(\sum_{n=2}^{\infty}p_n(x)\lambda^{n-1}-1)\le b(x)(a-1).$$

Since $\varphi^*(x,\lambda)\le 0$ for all $\lambda\in[0,1]$, it follows from (2.10) that for any $f\in\mathcal{D}_1$,

$$\overline{V}_t f(x)\le\overline{\Pi}_x f(x)\le\|f\|_\infty. \tag{3.11}$$

Thus for $f\in\mathcal{D}_a$,

$$\varphi^*(x,\overline{V}_{t-s}f(x))\le b(x)(a-1)\overline{V}_{t-s}f(x). \tag{3.12}$$

It follows from (2.10) and the Feynman-Kac formula that, for $t\ge 1$,

$$\begin{aligned}\overline{V}_t f(x)=&\overline{\Pi}_x\int_0^t e^{-(1-a)\int_0^s b(\xi_u)\,du}\Big[\varphi^*(\xi_s,\overline{V}_{t-s}f(\xi_s))+(1-a)b(\xi_s)\overline{V}_{t-s}f(\xi_s)\Big]\mathrm{d}s\\&+\overline{\Pi}_x\Big[e^{-(1-a)\int_0^t b(\xi_s)\,ds}f(\xi_t)\Big]\\\le&a\overline{\Pi}_x\Big[e^{-(1-a)\int_0^t b(\xi_s)\,ds}\Big]\le a\Big[\overline{\Pi}_x\Big[e^{-\int_0^t b(\xi_s)\,ds}\Big]\Big]^{1-a}\\=&a\left[T_t^*1(x)\right]^{1-a}\le a(1+c)e^{\lambda_0^*(1-a)t}\langle 1,\psi_0^*\rangle_m^{1-a}\phi_0^*(x)^{1-a},\end{aligned}$$

where the last inequality follows from (3.9). □

Lemma 3.3 *For any $f\in\mathcal{D}_1$,*

$$T_t^*f(x)\le\overline{V}_t f(x)\le(1+\|f\|_\infty e^{\|b\|_\infty t})T_t^*f(x).$$

Proof: It follows from (3.11) that

$$\overline{V}_t f(x)\le\overline{\Pi}_x f(x)\le e^{\|b\|_\infty t}T_t^*f(x). \tag{3.13}$$

Using (2.10) and the Feynman-Kac formula, we can get

$$\overline{V}_t f(x)=\int_0^t T_s^*[\varphi_0^*(\cdot,\overline{V}_{t-s}f)](x)\mathrm{d}s+T_t^*(f)(x), \tag{3.14}$$

where $\varphi_0^*(x,\lambda)=\varphi^*(x,\lambda)+b(x)\lambda\ge 0$. Hence $\overline{V}_t f(x)\ge T_t^*f(x)$. Note that

$$\varphi_0^*(x,\lambda)\le b(x)\lambda^2,\qquad 0\le\lambda\le 1. \tag{3.15}$$

Combining (3.11) and (3.13), we get

$$\varphi_0^*(x,\overline{V}_{t-s}f(x))\le b(x)\overline{V}_{t-s}f(x)^2\le\|f\|_\infty\|b\|_\infty e^{\|b\|_\infty(t-s)}T_{t-s}^*f(x).$$

Hence,

$$\int_0^t T_s^*[\varphi_0^*(\cdot,\overline{V}_{t-s}f)](x)\mathrm{d}s\le\|f\|_\infty\int_0^t\|b\|_\infty e^{\|b\|_\infty(t-s)}\mathrm{d}sT_t^*f(x)\le\|f\|_\infty e^{\|b\|_\infty t}T_t^*f(x).$$

Summarizing the above, we get

$$T_t^* f(x) \le \overline{V}_t f(x) \le (1 + \|f\|_\infty e^{\|b\|_\infty t}) T_t^* f(x).$$

□

For any $f \in \mathcal{D}_1$, define

$$A(f) := \int_0^\infty e^{-\lambda_0^* s} \langle \varphi_0^*(\cdot, \overline{V}_s f), \psi_0^* \rangle_m \mathrm{d}s + \langle f, \psi_0^* \rangle_m. \tag{3.16}$$

Lemma 3.4 *For any $a \in [0, 1)$ and $f \in \mathcal{D}_a$,*

$$\sup_{t>0} e^{-\lambda_0^* t} \|\overline{V}_t f\|_\infty < \infty.$$

Furthermore,

$$\lim_{t\to\infty} e^{-\lambda_0^* t} \overline{V}_t f(x) = A(f)\phi_0^*(x),$$

where $A(f)$is defined in (3.16).

Proof: Note that $\phi_0^*(x)$ is bounded. It follows from Lemma 3.2 that there exists $s_0 > 1$ such that

$$\overline{V}_{s_0} f(x) \le 1/4, \quad \forall f \in \mathcal{D}_a.$$

Using this and Lemma 3.2 we obtain that, for any $s > s_0 + 1$,

$$\overline{V}_s f(x) = \overline{V}_{s-s_0}(\overline{V}_{s_0} f)(x) \le \overline{V}_{s-s_0}(1/4)(x) \le C e^{3\lambda_0^* s/4} \phi_0^*(x)^{3/4}. \tag{3.17}$$

It follows from (3.14) that for any $t > s_0 + 1$,

$$\begin{aligned} e^{-\lambda_0^* t} \overline{V}_t f(x) = & \int_0^t e^{-\lambda_0^* s} e^{-\lambda_0^*(t-s)} T_{t-s}^* [\varphi_0^*(\cdot, \overline{V}_s f)](x) \mathrm{d}s + e^{-\lambda_0^* t} T_t^*(f)(x) \\ = & \Big(\int_0^{s_0+1} + \int_{s_0+1}^t \Big) e^{-\lambda_0^* s} e^{-\lambda_0^*(t-s)} T_{t-s}^* [\varphi_0^*(\cdot, \overline{V}_s f)](x) \mathrm{d}s + e^{-\lambda_0^* t} T_t^*(f)(x) \\ =: & J_1(t,x) + J_2(t,x) + J_3(t,x). \end{aligned} \tag{3.18}$$

For J_3, using (3.9) we can easily get $J_3(t,x) \le C \langle f, \psi_0^* \rangle_m \phi_0^*(x)$ and

$$\lim_{t\to\infty} J_3(t,x) = \langle f, \psi_0^* \rangle_m \phi_0^*(x).$$

For $J_2(t,x)$, we can use (3.17) and (3.15) to get that, for any $t > s > s_0 + 1$,

$$e^{-\lambda_0^*(t-s)} |T_{t-s}^* [\varphi_0^*(\cdot, \overline{V}_s f)](x)| \le C e^{3\lambda_0^* s/2} e^{-\lambda_0^*(t-s)} T_{t-s}^* [(\phi_0^*)^{3/2}](x) \le C e^{3\lambda_0^* s/2} \phi_0^*(x).$$

Hence,

$$|J_2(t,x)| \le C \int_{s_0+1}^t e^{\lambda_0^* s/2} \mathrm{d}s \phi_0^*(x) \le C \phi_0^*(x).$$

It follows from the dominated convergence theorem that

$$\lim_{t\to\infty} J_2(t,x) = \int_{s_0+1}^{\infty} e^{-\lambda_0^* s}\langle \varphi_0^*(\cdot, \overline{V}_s f), \psi_0^*\rangle_m \mathrm{d}s\phi_0^*(x).$$

Finally, we deal with $J_1(t,x)$. Since $\overline{V}_s f(x) \le 1$, we have $\varphi_0^*(x, \overline{V}_s f(x)) \le \|b\|_\infty$. Thus for $t-s > t-s_0 > 1$, we have

$$e^{-\lambda_0^*(t-s)}|T_{t-s}^*[\varphi_0^*(\cdot, \overline{V}_s f)](x)| \le Ce^{-\lambda_0^*(t-s)}T_{t-s}^* 1(x) \le C\phi_0^*(x).$$

Hence

$$J_1(t,x) < C\phi_0^*(x),$$

and it follows from the dominated convergence theorem that

$$\lim_{t\to\infty} J_1(t,x) = \int_0^{s_0+1} e^{-\lambda_0^* s}\langle \varphi_0^*(\cdot, \overline{V}_s f), \psi_0^*\rangle_m \mathrm{d}s\phi_0^*(x).$$

Summarizing, we get the conclusion of the lemma. □

3.2 Proof of Theorem 1.3

It follows from Lemma 2.2 that to prove Theorem 1.3, it suffices to show that Y has a density $g_\mu(y)$ and that, for any $y > 0$, $g_\mu(y) > 0$. In this subsection, we will show that Y has a density function by analyzing the properties of the characteristic function of Y. By (2.6), we have

$$\psi(i\theta, x) = \overline{V}_t(\psi(i\theta e^{-\lambda_0 t}, \cdot))(x), \quad \theta \in \mathbb{R}. \tag{3.19}$$

For simplicity, for any $\theta \in \mathbb{R}$, we write $\psi(i\theta, \cdot)$ as $\psi(i\theta)$; similarly, for any $\theta > 0$, we write $\psi(\theta, \cdot)$ as $\psi(\theta)$.

Lemma 3.5 *For any bounded closed interval I not containing 0, we have*

$$\sup_{\theta\in I} \|\psi(i\theta)\|_\infty < 1.$$

Proof: It is easy to see that

$$\begin{aligned} &|\|\psi(i\theta)\|_\infty - \|\psi(i(\theta+\epsilon))\|_\infty| \le \|\psi(i\theta) - \psi(i(\theta+\epsilon))\|_\infty \\ &= \|\overline{V}_1(\psi(i\theta e^{-\lambda_0})) - \overline{V}_1(\psi(i(\theta+\epsilon)e^{-\lambda_0}))\|_\infty. \end{aligned}$$

It is well known that, for any $|x_j| \le 1$, $|y_j| \le 1$, it holds that

$$|\prod_{j=1}^n x_j - \prod_{j=1}^n y_j| \le \sum_{j=1}^n |x_j - y_j|.$$

For any complex-valued function f on E with sup norm less or equal to 1, we have

$$\overline{V}_t f(x) = P_{\delta_x} \prod_{u \in \mathcal{L}_t} f(\xi_t(u)),$$

where $\mathcal{L}_t$ is the collection of particles of the branching Markov process Z which are alive at time t, $\xi_t(u)$ stands for the position of particle u at time t. Thus,

$$\begin{aligned}
&|\overline{V}_1(\psi(i\theta e^{-\lambda_0}))(x) - \overline{V}_1(\psi(i(\theta+\epsilon)e^{-\lambda_0}))(x)| \\
\le& P_{\delta_x}\langle|\psi(i\theta e^{-\lambda_0}) - \psi(i(\theta+\epsilon)e^{-\lambda_0})|, Z_1\rangle \\
=& v(x)^{-1} T_1(v|\psi(i\theta e^{-\lambda_0})) - \psi(i(\theta+\epsilon)e^{-\lambda_0}))|)(x) \\
\le& C\langle|\psi(i\theta e^{-\lambda_0})) - \psi(i(\theta+\epsilon)e^{-\lambda_0}))|, \psi_0\rangle_m \to 0, \quad \epsilon \to 0.
\end{aligned}$$

The equality above is due to (2.14), the last inequality is due to (3.13) and (3.4), and the last limit is due to the continuity of the characteristic function and the dominated convergence theorem. Thus $\|\psi(i\theta)\|_\infty$ is continuous in θ. Now, we only need to show that, for any $\theta \neq 0$, $\|\psi(i\theta)\|_\infty < 1$.

We use contradiction. Suppose that for all $\theta \in \mathbb{R}$ and $x \in E$, $|\psi(i\theta, x)| = 1$. Then by the uniqueness of characteristic functions, there exists a positive-valued function $c(x)$ such that $P_{\delta_x}(Y = c(x)) = 1$, that is, $\psi(\theta, x) = e^{-\theta c(x)}$. Using (2.6) (with θ replaced by $\theta e^{\lambda_0 t}$),

$$\exp\{-\theta e^{\lambda_0 t} c(x)\} = P_{\delta_x}(e^{-\theta\langle c, Z_t\rangle}),$$

that is, the Laplace transform of the random variable $\langle c, Z_t\rangle$ is the same as $e^{\lambda_0 t}c(x)$, thus $P_{\delta_x}(\langle c, Z_t\rangle = e^{\lambda_0 t}c(x)) = 1$. By the definition of branching Markov processes, we know that $\langle c, Z_t\rangle$ can not be concentrated at one point, a contradiction! Thus there exist $\theta_0 \in \mathbb{R}$ and $x_0 \in E$ such that $|\psi(i\theta_0, x_0)| < 1$. Hence there exists $\delta = \delta(x_0) > 0$, such that $|\psi(i\theta, x_0)| < 1$ for all $|\theta| \in (0, \delta)$. Since $x \to \psi(i\theta, x)$ is a continuous function, for all $|\theta| \in (0, \delta)$, we have

$$m(y \in E : |\psi(i\theta, y)| < 1) > 0.$$

For any complex-valued function f on E with sup norm less or equal to 1,

$$|\overline{V}_t f(x)| = |P_{\delta_x} \prod_{u \in \mathcal{L}_t} f(\xi_t(u))| \le P_{\delta_x} \prod_{u \in \mathcal{L}_t} |f|(\xi_t(u)) = \overline{V}_t|f|(x).$$

Thus by (3.13), we have

$$1 - |\psi(i\theta e^{\lambda_0 t}, x)| \ge 1 - \overline{V}_t(|\psi(i\theta)|)(x) \ge \overline{\Pi}_x(1 - |\psi(i\theta, \xi_t)|).$$

Note that

$$\frac{\varphi(x, v(x))}{v(x)} \le -\alpha(x) + \beta(x)v(x) + \frac{1}{2}v(x)\int_0^1 r^2 n(x, dr) + \int_1^\infty r n(x, dr) \le -\alpha(x) + M(\|v\|_\infty + 1).$$

Suppose that c and γ are the constants from (3.3). For t large enough, we have $1-ce^{-\gamma t} > 0$, hence by the definition of $\overline{\Pi}_x$,

$$\begin{aligned}\overline{\Pi}_x(1-|\psi(i\theta,\xi_t)|) &\geq v(x)^{-1}e^{-M(\|v\|_\infty+1)t}T_t(v(1-|\psi(i\theta)|))(x)\\ &\geq e^{-M(\|v\|_\infty+1)t}(1-ce^{-\gamma t})e^{\lambda_0 t}\langle v(1-|\psi(i\theta)|),\psi_0\rangle_m\frac{\phi_0(x)}{v(x)},\end{aligned}$$

where the last inequality is due to (3.3).

It follows from (3.5) that, for $|\theta|\in(0,\delta)$ and t sufficiently large, $\|\psi(i\theta e^{\lambda_0 t})\|_\infty < 1$. Thus for all $\theta\neq 0$, $\|\psi(i\theta)\|_\infty<1$.

Summarizing, we get the conclusion of the lemma. □

Lemma 3.6 *For any $\delta\in(0,-\frac{\lambda_0^*}{\lambda_0})$, there exists a constant $C>0$, such that for $|\theta|$ sufficiently large,*

$$\|\psi(i\theta)\|_\infty \leq C|\theta|^{-\delta}.$$

Proof: For any $\delta\in(0,-\frac{\lambda_0^*}{\lambda_0})$, there exists $\epsilon\in(0,1)$ such that

$$(1+\epsilon)e^{\lambda_0^*} \leq e^{-\lambda_0\delta}. \tag{3.20}$$

It follows from Lemma 3.2 and Lemma 3.5 that there exists $j\geq 1$ such that for all $k\geq j$,

$$\sup_{\theta\in[1,e^{\lambda_0}]}\|\overline{V}_k(|\psi(i\theta)|)\|_\infty \leq \epsilon e^{-\|b\|_\infty}. \tag{3.21}$$

Thus by 3.3 we get that, for all $\theta\in[1,e^{\lambda_0}]$ and $n\geq 1$,

$$\begin{aligned}|\overline{V}_{n+j}(\psi(i\theta))(x)| &\leq \overline{V}_{n+j}(|\psi(i\theta)|)(x) = \overline{V}_1\overline{V}_{n+j-1}(|\psi(i\theta)|)(x)\\ &\leq (1+\epsilon)T_1^*(\overline{V}_{n+j-1}(|\psi(i\theta)|))(x).\end{aligned}$$

By iteration and (3.9) we get that, for $\theta\in[1,e^{\lambda_0}]$, we can use (3.20) to get that

$$\begin{aligned}|\overline{V}_{n+j}(\psi(i\theta))(x)| &\leq (1+\epsilon)^n T_n^*(\overline{V}_j(|\psi(i\theta)|))(x)\\ &\leq (1+\epsilon)^n T_n^*(1)(x) \leq (1+c)(1+\epsilon)^n e^{\lambda_0^* n}\langle 1,\psi_0^*\rangle_m\|\phi_0^*\|_\infty\\ &\leq (1+c)\langle 1,\psi_0^*\rangle_m\|\phi_0^*\|_\infty e^{\lambda_0\delta(j+1)}e^{-\lambda_0\delta(n+j)}\theta^{-\delta}.\end{aligned}$$

Since $\psi(i\theta e^{\lambda_0(n+j)})(x) = \overline{V}_{n+j}(\psi(i\theta))(x)$, we have

$$\|\psi(i\theta)\|_\infty \leq C|\theta|^{-\delta}, \qquad \theta \geq e^{\lambda_0 j}.$$

Using $\psi(-i\theta)(x) = \overline{\psi(i\theta)(x)}$, we get

$$\|\psi(i\theta)\|_\infty \leq C|\theta|^{-\delta}, \qquad \theta \leq -e^{\lambda_0 j}.$$

Summarizing, we get the conclusion of the lemma. □

Proposition 3.7 *For any non-zero $\mu \in \mathcal{M}_F(E)$, under $\mathbb{P}_\mu$, Y is an absolutely continuous random variable, that is, it has a density function $g_\mu(y)$.*

Proof: By Remark 2.3, it suffices to prove that the conclusion holds when $\mu = \delta_x$. It follows from $\psi(\theta, x) = \overline{V}_t(\psi(\theta e^{-\lambda_0 t}))(x)$ that

$$Y =^d e^{-\lambda_0 t} \sum_{u \in \mathcal{L}_t} Y^u,$$

where $\mathcal{L}_t$ is the collection of particles of the branching Markov process Z which are alive at time t. Given Z_t, $\{Y^u, u \in \mathcal{L}_t\}$ is a family of independent random varaibles with $Y^u =^d (Y, \mathbb{P}_{\delta_{\xi_t(u)}})$.

Take $\delta \in (0, -\frac{\lambda_0^*}{\lambda_0})$ and $K > 0$ such that $K\delta > 1$. For any Lebesgue null set $B \subset (0, \infty)$, we have

$$\mathbb{P}_{\delta_x}(Y \in B) \le P_{\delta_x}(\|Z_t\| \le K) + \sum_{n=K+1}^{\infty} P_{\delta_x}(\|Z_t\| = n, e^{-\lambda_0 t} \sum_{u \in \mathcal{L}_t} Y^u \in B).$$

Given Z_t and $\|Z_t\| = n > K$, for $|\theta|$ sufficiently large, we have

$$\left| P_{\delta_x}\left(e^{i\theta \sum_{u \in \mathcal{L}_t} Y^u} | Z_t\right) \right| \mathbf{1}_{\|Z_t\|=n} \le C^n |\theta|^{-\delta n},$$

implying that the characteristic function of $\sum_{u \in \mathcal{L}_t} Y^u$ is L^1 integrable. Thus $\sum_{u \in \mathcal{L}_t} Y^u$ has a density function, and hence

$$P_{\delta_x}(\|Z_t\| = n, e^{-\lambda_0 t} \sum_{u \in \mathcal{L}_t} Y^u \in B) = 0.$$

Summarizing the above, we have

$$\mathbb{P}_{\delta_x}(Y \in B) \le P_{\delta_x}(\|Z_t\| \le K).$$

Letting $t \to \infty$, we immediately get $\mathbb{P}_{\delta_x}(Y \in B) = 0$, that is, the distribution of Y is absolutely continuous with respect to the Lebesgue measure, and thus has a density function. □

Proposition 3.8 *For any non-zero $\mu \in \mathcal{M}_F(E)$, under $\mathbb{P}_\mu$, the density function of Y is strictly positive on $(0, \infty)$.*

Proof: Note that $\{Y, \mathbb{P}_{\delta_x}\}$ and $\{aW^Z, P_{\delta_x}\}$ have the same distribution, where $a > 0$ is a constant. By Remark 2.3, it suffices to show that, under P_{δ_x}, the density function of W^Z is strictly positive on $(0, \infty)$.

It has been proven in [8, Proposition 5.6] that, for branching Markov processes satisfying certain conditions, the density function of W^Z is strictly positive on $(0, \infty)$. For the branching

Markov process $\{Z_t\}$ of this paper, we can use the same argument to show that the same conclusion holds. We omit the details. $\square$

Proof of Theorem 1.3: Combining Lemma 2.2, Proposition 3.7 and Proposition 3.8, we immediately get that, under $\mathbb{P}_\mu$, the distribution of W is absolutely continuous on $(0,\infty)$ with density function f_μ satisfying that, for all $y>0$,

$$f_\mu(y) \geq g_\mu(y)\langle v,\mu\rangle e^{-\langle v,\mu\rangle} > 0.$$

$\square$

3.3 Proof of Theorem 1.4

Recall that

$$\epsilon_0 = \frac{-\lambda_0^*}{\lambda_0}.$$

Proof of Theorem 1.4 First, we deal with the small value probability problem. It follows from Lemma 3.4 that

$$e^{-\lambda_0^* t}\psi(e^{\lambda_0 t}, x) = e^{-\lambda_0^* t}\overline{V}_t(\psi(1))(x) \to A(\psi(1))\phi_0^*(x),$$

that is,

$$\lim_{\theta\to\infty}\theta^{\epsilon_0}\psi(\theta,x) = A(\psi(1))\phi_0^*(x).$$

Simple calculations give that, as $\theta\to\infty$,

$$\begin{aligned}\mathbb{P}_\mu(e^{-\theta W}|W>0) =& \frac{1}{1-e^{-\langle v,\,\mu\rangle}}\left(e^{-\langle\Phi(\theta),\mu\rangle} - e^{-\langle v,\,\mu\rangle}\right)\\ =& \frac{1}{e^{\langle v,\,\mu\rangle}-1}\left(e^{\langle\psi(\theta)v,\mu\rangle}-1\right)\\ \sim& \frac{1}{e^{\langle v,\,\mu\rangle}-1}\langle\psi(\theta)v,\,\mu\rangle.\end{aligned}$$

Summarizing the above, we get

$$\lim_{\theta\to\infty}\theta^{\epsilon_0}\mathbb{P}_\mu(e^{-\theta W}|W>0) = \frac{1}{e^{\langle v,\,\mu\rangle}-1}\lim_{\theta\to\infty}\theta^{\epsilon_0}\langle\psi(\theta)v,\,\mu\rangle = \frac{1}{e^{\langle v,\,\mu\rangle}-1}A(\psi(1))\langle v\phi_0^*,\,\mu\rangle.$$

It follows from the Tauberian theorem that

$$\lim_{r\to0} r^{-\epsilon_0}\mathbb{P}_\mu(W\leq r|W>0) = \frac{1}{e^{\langle v,\,\mu\rangle}-1}A(\psi(1))\langle v\phi_0^*,\,\mu\rangle/\Gamma(\epsilon_0+1).$$

Thus

$$\lim_{r\to0} r^{-\epsilon_0}\mathbb{P}_\mu(0<W\leq r) = e^{-\langle v,\,\mu\rangle}A(\psi(1))\langle v\phi_0^*,\,\mu\rangle/\Gamma(\epsilon_0+1).$$

Now we deal with the tail probability problem. Let

$$G(s) := \int_0^s \mathbb{P}_\mu(W>r)\,dr.$$

Then the Laplace transform of G is

$$\begin{aligned}\int_0^\infty e^{-\theta r}\, dG(r) =& \int_0^\infty e^{-\theta r}\mathbb{P}_\mu(W>r)\, dr\\ =&\theta^{-1}\Big(1-\theta\int_0^\infty e^{-\theta r}\mathbb{P}_\mu(W\le r)\, dr\Big)\\ =&\theta^{-1}\Big(1-\mathbb{P}_\mu(e^{-\theta W})\Big)\\ =&\theta^{-1}(1-e^{-\langle\Phi(\theta),\mu\rangle}).\end{aligned}$$

It follows from (3.6) that

$$\lim_{\theta\to 0}\theta^{-1}\widetilde{L}(\theta^{-1})^{-1}\Phi(\theta,x)=\lim_{t\to\infty}e^{\lambda_0 t}L(t)^{-1}\Phi(e^{-\lambda_0 t},x)=\phi_0(x),$$

where $L(t)$ are $\widetilde{L}$ are defined in (1.14) and (1.15). Hence,

$$\lim_{\theta\to 0}\widetilde{L}(\theta^{-1})^{-1}\int_0^\infty e^{-\theta r}\, dG(r)=\lim_{\theta\to 0}\theta^{-1}\widetilde{L}(\theta^{-1})^{-1}\langle\Phi(\theta),\mu\rangle=\langle\phi_0,\mu\rangle.$$

It follows from the Tauberian theorem that

$$\lim_{r\to\infty}\widetilde{L}(r)^{-1}G(r)=\langle\phi_0,\mu\rangle.$$

Therefore, by [21], we have

$$\lim_{r\to\infty}r\widetilde{L}(r)^{-1}\mathbb{P}_\mu(W>r)=0.$$

□

References

[1] Athreya, K. B.: On the absolute continuity of the limit random variable in the supercritical Galton-Watson branching process. *Proc. Amer. Math. Soc.* **30** (1971), 563–565.

[2] Biggins, J. D. and Bingham, N. H.: Large deviations in the supercritical branching process. *Adv. Appl. Probab.* **25** (1993), 757–772.

[3] Chen, Z.-Q., Ren, Y.-X., Yang, T.: Skeleton decomposition and law of large numbers for supercritical superprocesses. *Acta Appl. Math.*, to appear, 2018. Arxiv: 1709.00847.

[4] Davies, E. B. and Simon, B.: Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.* **59** (1984), 335–395.

[5] Eckhoff, M., Kyprianou, A. E. and Winkel, M.: Spines, skeletons and the strong law of large numbers for superdiffusions. *Ann. Probab.* **43** (2015), 2594C-2659.

[6] Dynkin, E. B.: Superprocesses and partial differential equations. *Ann. Probab.* **21** (1993), 1185–1262.

[7] Harris, T. E.: *The Theory of Branching Processes.* Springer, Berlin, 1963.

[8] Hering, H.: The non-degenerate limit for supercritical branching diffusions. *Duke Math. J.* **45** (1978), 561–599.

[9] Heyde, C. C.: Extension of a result of Seneta for the supercritical branching process. *Ann. Math. Statist.* **41**(1970), 739–742.

[10] Jones, O.D.: Large deviations for supercritial multitype branching processes. *J. Appl. Prob.* **41** (2004),703–720.

[11] Kim, P. and Song, R.: Intrinsic ultracontractivity of non-symmetric diffusion semigroups in bounded domains. *Tohoku Math. J.* **60** (2008), 527–547.

[12] Li, Z.: *Measure-Valued Branching Markov Processes.* Springer, Heidelberg, 2011.

[13] Liu, R.-L., Ren, Y.-X. and Song, R.: $L \log L$criterion for a class of superdiffusions. *J. Appl. Probab.* **46** (2009), 479–496.

[14] Mörters, P. and Ortgiese, M.: Small value probabilities via the branching tree heuristic. *Bernoulli* **14** (2008), 277–299.

[15] Ren, Y.-X., Song, R. and Zhang, R.: Limit theorems for some critical superprocesses. *Illinois J. Math.* **59** (2015), 235–276.

[16] Ren, Y.-X., Song, R. and Zhang, R.: Central limit theorems for supercritical branching non-symmetric Markov processes. *Ann. Probab.* **45** (2017), 564–623.

[17] Ren, Y.-X., Song, R. and Zhang, R.: Williams decomposition for superprocesses. *Elect. J. Probab.*, **23** (2018), No. 23, 1–33.

[18] Ren, Y.-X., Song, R. and Zhang, R.: Supercritical superprocesses: proper normalization and non-degenerate strong limit. *Sci. China Math.* (to appear), 2018. ArXiv: 1708.04422

[19] Schaefer, H. H.: *Banach Lattices and Positive Operators.* Springer, New York, 1974

[20] Seneta, E.: On recent theorems concerning the supercritical Galton-Watson process. *Ann. Math. Statist.* **39**(1968), 2098–2102.

[21] Seneta, E.: A Tauberian theorem of E. Landau and W. Feller. *Ann. Prob.* **1** (1973), 1057–1058.

[22] Stigum, B. P.: A theorem on the Galton-Watson process. *Ann. Math. Stat.* **37** (1966), 695–698.

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