Tail probability of maximal displacement in critical and subcritical branching stable processes

Haojie Hou[*] **Yiyang Jiang** **Yan-Xia Ren**[†] and **Renming Song**[‡]

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Abstract

In this paper, we study critical and subcritical branching α-stable processes, $\alpha \in (0,2)$. We obtain the exact asymptotic behaviors of the tails of the maximal positions of all subcritical branching α-stable processes with positive jumps. In the case of subcritical branching spectrally negative α-stable processes, we obtain the exact asymptotic behaviors of the tails of the maximal positions under the assumption that the offspring distributions satisfy the $L\log L$ condition. For critical branching α-stable processes, we obtain the exact asymptotic behaviors of the tails under the assumption that the offspring distributions belong to the domain of attraction of a γ-distribution, $\gamma \in (1,2]$.

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1 Introduction and main results

1.1 Introduction

A branching Lévy process is a continuous-time Markov process defined as follows. At time 0, there is a particle at $x \in \mathbb{R}$ and it moves according to a Lévy process $(\xi_t, \mathbf{P}_x)$. After an exponential time with parameter 1, independent of the motion, it dies and produces k offspring with probability p_k, $k \geqslant 0$. The offspring move independently according to ξ from the place where they are born and obey the same branching mechanism as their parent. This procedure goes on. For $t \geqslant 0$, let N_t be the collection of particles alive at time t. For $u \in N_t$, we use $X_u(t)$ to denote the position of the particle u at time t. The point process $(Z_t)_{t\geqslant 0}$ defined by

$$Z_t := \sum_{u\in N_t} \delta_{X_u(t)}, \qquad t \geqslant 0,$$

is called a branching Lévy process. We will denote the law of $(Z_t)_{t\geqslant 0}$ by $\mathbb{P}_x$ and write $\mathbb{P}$ for $\mathbb{P}_0$ for simplicity. When ξ is a Brownian motion, $(Z_t)_{t\geqslant 0}$ is called a branching Brownian motion. When ξ is an α-stable process, $\alpha \in (0,2)$, $(Z_t)_{t\geqslant 0}$ is called a branching α-stable process.

Let $m := \sum_{k=0}^{\infty} kp_k$ be the mean of the offspring distribution. When $m > 1(= 1, < 1)$, we say that the branching Lévy process $(Z_t)_{t\geqslant 0}$ is supercritical (critical, subcritical). It is well known that in the critical and subcritical cases, $(Z_t)_{t\geqslant 0}$ will die out in finite time with probability 1. Thus in this case we can define the maximal position of $(Z_t)_{t\geqslant 0}$ by

$$M := \sup_{t>0} \sup_{u\in N_t} X_u(t).$$

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When $m = 1$ and ξ is a standard Brownian motion, Sawyer and Fleischman [17] proved that, under the assumption $\sum_{k=0}^{\infty} k^3 p_k < \infty$,

$$\lim_{x\to+\infty} x^2 \mathbb{P}(M \geqslant x) = \frac{6}{\sigma^2},$$

where σ^2 is the variance of the offspring distribution. Profeta [13] extended the result above to some critical branching spectrally negative Lévy processes under the same third moment condition on the offspring distribution. When $m = 1$, ξ is an α-stable process with positive jumps and $\sum_{k=0}^{\infty} k^3 p_k < \infty$, Lalley and Shao [10], and Profeta [12] proved that

$$\lim_{x\to+\infty} x^{\alpha/2} \mathbb{P}(M \geqslant x) = c_1(\alpha),$$

where $c_1(\alpha)$ is an explicit positive constant. For critical branching spectrally negative α-stable processes with $\alpha \in (1, 2)$, Profeta [13, Corollary 1.3] proved that, when $\sum_{k=0}^{\infty} k^3 p_k < \infty$,

$$0 < \liminf_{x\to\infty} x^{\alpha} \mathbb{P}(M \geqslant x) \leqslant \limsup_{x\to\infty} x^{\alpha} \mathbb{P}(M \geqslant x) < \infty. \tag{1}$$

When $m < 1$ and ξ is a standard Brownian motion, Sawyer and Fleischman [17] proved that, under the assumption $\sum_{k=0}^{\infty} k^3 p_k < \infty$, there exists a function $c(\cdot)$ bounded between two positive constants such that

$$\lim_{x\to+\infty} \frac{\mathbb{P}(M \geqslant x)}{(1-m)c(x)e^{-\sqrt{2(1-m)}x}} = 1.$$

For subcritical branching α-stable processes with positive jumps, Profeta [12] proved that, under the assumption $\sum_{k=0}^{\infty} k^3 p_k < \infty$,

$$\lim_{x\to+\infty} x^{\alpha} \mathbb{P}(M \geqslant x) = \frac{c_2(\alpha)}{1-m}$$

for some explicit constant $c_2(\alpha)$. For subcritical branching spectrally negative processes, Profeta [13, Theorem 1] proved that, under the assumption $\sum_{k=0}^{\infty} k^3 p_k < \infty$,

$$\lim_{x\to\infty} e^{c_4 x} \mathbb{P}(M \geqslant x) = c_3$$

for some explicit positive constants c_3, c_4.

For results on the asymptotic behaviors of the tails of maximal positions of (sub)critical branching random walks, see [9, 11].

The purpose of this paper is to establish the exact asymptotic behaviors of the tails of the maximal positions of (sub)critical branching α-stable processes under minimal conditions on the offspring distributions. More precisely, we will obtain (i) the exact asymptotic behaviors of the tails of the maximal positions of subcritical branching α-stable processes with positive jumps without any extra assumption on the offspring distributions; (ii) the exact asymptotic behaviors of the tail of the maximal positions of subcritical branching spectrally negative α-stable processes under the assumption that the offspring distributions satisfy the $L \log L$ condition; (iii) the exact asymptotic behaviors of the tails of critical branching α-stable processes under the assumption that the offspring distributions belong to the domain of attraction of a γ-distribution, $\gamma \in (1, 2]$.

1.2 Main results

Before we state our main results, we recall some useful facts about stable processes. In this paper, we always assume that the spatial motion ξ is a (strictly) α-stable process, $\alpha \in (0, 2)$. For basic information on α-stable process, see, for instance, [1, Chap. VIII]. We assume that the Lévy measure of ξ is given by

$$v_\alpha(\mathrm{d}x) := c_+ x^{-(1+\alpha)} \mathbf{1}_{(0,\infty)}(x)\mathrm{d}x + c_- |x|^{-(1+\alpha)} \mathbf{1}_{(-\infty,0)}(x)\mathrm{d}x,$$

where c_+ and c_- are non-negative numbers with at least one of them being positive. Let $\Psi(\theta) := -\ln \mathbf{E}(e^{\mathrm{i}\theta\xi_1})$ be the characteristic exponent of ξ. It is known (see, for instance [8, Chapter 1.2.6]) that,

$$\Psi(\theta) = \begin{cases} c_*|\theta|^{\alpha}\left(1 - \mathrm{i}\beta \tan\frac{\pi\alpha}{2} \operatorname{sgn}\theta\right), & \text{for } \alpha \in (0,1)\cup(1,2) \text{ and } \beta \in [-1,1], \\ c_*|\theta| + \mathrm{i}\theta\eta, & \text{for } \alpha = 1 \end{cases}$$

where $\eta \in \mathbb{R}$, $c_* := -(c_+ + c_-)\,\Gamma(-\alpha)\cos(\pi\alpha/2)$, and $\beta = (c_+ - c_-)\,/\,(c_+ + c_-)$ if $\alpha \in (0,1) \cup (1,2)$. Here $\operatorname{sgn}\theta = 1_{(\theta>0)} - 1_{(\theta<0)}$. Note that, for (strictly) 1-stable process, $c_+ = c_-$ (see, for example, [16, Theorem 14.7 (v)]). It follows from [15, Example 1.1, Lemma 2.1 and the last sentence in Subsection 1.1] (see also [1, Proposition VIII.4]) that, when $c_+ > 0$,

$$\lim_{x\to\infty} x^\alpha \mathbf{P}_0(\xi_1 \geqslant x) = \nu_\alpha((1,\infty)) = \frac{c_+}{\alpha}. \tag{2}$$

When $c_+ = 0$ and $\alpha \in (0,1)$, $-\xi$ is a subordinator and $M = 0$ almost surely. A spectrally negative 1-stable process reduces to the drift $-\eta t$. When $\eta \geqslant 0$, $M = 0$. Thus in the case of branching spectrally negative α-stable processes, exclude the above cases and assume either $\alpha \in (1,2)$, or $\alpha = 1$ and $\eta < 0$. In this case, it is easy to see that for any $\lambda > 0$,

$$\mathbf{E}_0(e^{\lambda\xi_1}) = \exp\{C_1(\alpha)\lambda^\alpha\}, \quad \text{where} \quad C_1(\alpha) = -\Psi(-\mathrm{i}). \tag{3}$$

Define $\tau_y = \inf\{t > 0, \xi_t \geqslant y\}$. For any $x < y$, combining (3) with the fact that $\mathbf{P}_x\left(\xi_{\tau_y} = y\right) = 1$, we have that for any $\lambda > 0$,

$$\mathbf{E}_x\left(e^{-\lambda\tau_y}\right) = \exp\left\{-C_1(\alpha)^{-1/\alpha}\lambda^{1/\alpha}(y-x)\right\}. \tag{4}$$

In the critical case, we will need the following assumption on the offspring distribution:

(H) The offspring distribution $\{p_k : k \geqslant 0\}$ belongs to the domain of attraction of a γ-stable, $\gamma \in (1,2]$, distribution. More precisely, either there exist $\gamma \in (1,2)$ and $\kappa_\gamma \in (0,\infty)$ such that

$$\lim_{n\to\infty} n^\gamma \sum_{k=n}^{\infty} p_k = \kappa_\gamma,$$

or that (corresponding $\gamma = 2$)

$$\sum_{k=0}^{\infty} k^2 p_k < \infty.$$

Recall that σ^2 is the variance of the offspring distribution. We define

$$C_2(\gamma) := \begin{cases} \frac{\Gamma(2-\gamma)}{\gamma-1}\kappa_\gamma & \text{for } \gamma \in (1,2); \\ \frac{1}{2}\sigma^2 & \text{for } \gamma = 2. \end{cases} \tag{5}$$

Our first main result is on (sub)critical branching stable processes with positive jumps.

Theorem 1. *Suppose that $c_+ > 0$.*

i) If $m < 1$, then

$$\lim_{x\to+\infty} x^\alpha \mathbb{P}(M \geqslant x) = \frac{c_+}{(1-m)\alpha}.$$

ii) If $m = 1$ and **(H)** *holds, then*

$$\lim_{x\to+\infty} x^{\alpha/\gamma} \mathbb{P}(M \geqslant x) = \left(\frac{c_+}{\alpha C_2(\gamma)}\right)^{\frac{1}{\gamma}},$$

where $C_2(\gamma)$ is defined in (5).

Remark 1. Theorem 1 (ii) is also contained in Theorem 3 of the recent preprint [14]. Since there are some differences between the proofs of Theorem 1 (ii) and [14, Theorem 3], we include it here for completeness. The proof of Theorem 1 is an adaptation of that of the corresponding result in [12] below. **(H)** only changes the behaviors of Φ_0 and Φ_R, defined in (7). In [12, (1.5)], the third moment condition on the offspring distribution is used to estimate Φ_R directly while in the γ-stable branching case, we have to do a more careful analysis.

Our second main result is on the case for critical branching spectrally negative α-stable process with $\alpha \in (1,2)$, and it generalizes and refines (1).

Theorem 2. *Suppose $c_+ = 0$ and $\alpha \in (1,2)$ or $\alpha = 1$ and $\eta < 0$. If $m = 1$ and* **(H)** *holds, then there exists a constant $C_3(\alpha,\beta,\gamma) \in (0,\infty)$ such that*

$$\lim_{x\to\infty} x^{\frac{\alpha}{\gamma-1}} \mathbb{P}(M \geqslant x) = C_3(\alpha,\beta,\gamma).$$

Remark 2. The constant $C_3(\alpha,\beta,\gamma)$ has a probabilistic representation via a super α-stable process, see the proof of Proposition 2. In the proof of Proposition 2, we use the fact that a superprocess is an appropriate scaling limit of branching Markov processes.

Our last main result is on subcritical branching spectrally negative α-stable process with $\alpha \in (1,2)$.

Theorem 3. *Suppose that $c_+ = 0$, and that either $\alpha \in (1,2)$ or $\alpha = 1$ and $\eta < 0$. If $m < 1$ and $\sum_{k=0}^{\infty} k(\log k)p_k < \infty$, then there exists a positive constant $C_4(\alpha)$ such that*

$$\lim_{x\to\infty} e^{((1-m)/C_1(\alpha))^{1/\alpha}x} \mathbb{P}(M \geqslant x) = C_4(\alpha).$$

Remark 3. Our proof of Theorem 3 can be easily adapted to a widely class of subcritical branching spectrally negative Lévy processes.

2 An integral equation for $\mathbb{P}(M \geqslant x)$

Let

$$u(x) := \mathbb{P}(M \geqslant x) \quad \text{and} \quad S_t := \sup_{s\in[0,t]} \xi_s$$

be the tail probability of M and the supremum of ξ up to time t. Let $\mathbf{e}$ be an exponential random variable with parameter 1 independent of ξ. Define

$$G(x) := \sum_{k=0}^{\infty} p_k(1-x)^k - 1 + mx, \quad x \in [0,1].$$

It is easy to see that G is a non-negative function in $[0,1]$.

Proposition 1. *The function u is a solution of the integral equation:*

$$u(x) = \mathbf{P}_0(S_{\mathbf{e}} \geqslant x) + \mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}} u(x-\xi_{\mathbf{e}})\right] - \Phi_0(x) - \Phi_R(x), \tag{6}$$

where

$$\Phi_0(x) = (1-m)\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}} u(x-\xi_{\mathbf{e}})\right] \quad \textit{and} \quad \Phi_R(x) = \mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}} G(u(x-\xi_{\mathbf{e}}))\right]. \tag{7}$$

(i) If $m = 1$ and **(H)** *holds, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\Phi_R(x) \geqslant (1-\varepsilon)C_2(\gamma)\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}} u^{\gamma}(x-\xi_{\mathbf{e}})\right] - \frac{(1-\varepsilon)C_2(\gamma)}{\delta}\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}} u^{\gamma+1}(x-\xi_{\mathbf{e}})\right] \tag{8}$$

and

$$\Phi_R(x) \leqslant (1+\varepsilon)C_2(\gamma)\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}} u^{\gamma}(x-\xi_{\mathbf{e}})\right] + \frac{1}{\delta^{\gamma+1}}\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}} u^{\gamma+1}(x-\xi_{\mathbf{e}})\right], \tag{9}$$

where $C_2(\gamma)$ is defined in (5).

(ii) If $m < 1$ and $c_+ > 0$, then

$$\lim_{x\to\infty} \frac{\Phi_R(x)}{\Phi_0(x)} = 0. \tag{10}$$

Proof. Applying the Markov property at the first branching time, we get

$$\mathbb{P}(M < x) = p_0 \mathbf{P}(S_{\mathbf{e}} < x) + \sum_{n=1}^{+\infty} p_n \mathbb{P}\left(S_{\mathbf{e}} < x, \xi_{\mathbf{e}} + M^{(1)} < x, \ldots, \xi_{\mathbf{e}} + M^{(n)} < x\right),$$

where $\left(M^{(n)}\right)_{n\in\mathbb{N}}$ are independent copies of M, which are also independent of $(\xi_{\mathbf{e}}, S_{\mathbf{e}})$. Hence,

$$\begin{aligned}1-u(x) &= p_0\mathbf{P}_0\left(S_{\mathbf{e}}<x\right)+\sum_{n=1}^{+\infty}p_n\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}\left(1-u\left(x-\xi_{\mathbf{e}}\right)\right)^n\right]\\ &= p_0\mathbf{P}_0\left(S_{\mathbf{e}}<x\right)-m\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u\left(x-\xi_{\mathbf{e}}\right)\right]\\ &\quad+\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}\left(\sum_{n=1}^{+\infty}p_n\left(1-u\left(x-\xi_{\mathbf{e}}\right)\right)^n+mu\left(x-\xi_{\mathbf{e}}\right)\right)\right].\end{aligned}$$

Therefore,

$$\begin{aligned}u(x)=&\mathbf{P}_0\left(S_{\mathbf{e}}\geqslant x\right)+\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u\left(x-\xi_{\mathbf{e}}\right)\right]-(1-m)\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u\left(x-\xi_{\mathbf{e}}\right)\right]\\ &-\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}\left(\sum_{n=1}^{+\infty}p_n\left(1-u\left(x-\xi_{\mathbf{e}}\right)\right)^n+mu\left(x-\xi_{\mathbf{e}}\right)-(1-p_0)\right)\right]\\ =&\mathbf{P}_0\left(S_{\mathbf{e}}\geqslant x\right)+\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u\left(x-\xi_{\mathbf{e}}\right)\right]-\Phi_0(x)-\Phi_R(x),\end{aligned}$$

where Φ_0 and Φ_R are given in (7).

We prove (i) first. When $m=1$ and **(H)** holds, from [4, Lemma 3.1] (for $\gamma\in(1,2)$) and L'Hopital's rule (for $\gamma=2$), we get that

$$\lim_{u\downarrow 0}\frac{G(u)}{u^\gamma}=C_2(\gamma). \tag{11}$$

Therefore, for any $\varepsilon>0$, there exists $\delta>0$ such that for all $u\leqslant\delta$,

$$(1-\varepsilon)C_2(\gamma)\leqslant\frac{G(u)}{u^\gamma}\leqslant(1+\varepsilon)C_2(\gamma). \tag{12}$$

Plugging (12) into the definition of Φ_R in (7), we get

$$\begin{aligned}\Phi_R(x)&\geqslant\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}G(u\left(x-\xi_{\mathbf{e}}\right))1_{\{u(x-\xi_{\mathbf{e}})<\delta\}}\right]\\ &\geqslant(1-\varepsilon)C_2(\gamma)\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^\gamma\left(x-\xi_{\mathbf{e}}\right)1_{\{u(x-\xi_{\mathbf{e}})<\delta\}}\right]\\ &\geqslant(1-\varepsilon)C_2(\gamma)\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^\gamma\left(x-\xi_{\mathbf{e}}\right)\right]-\frac{(1-\varepsilon)C_2(\gamma)}{\delta}\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^{\gamma+1}\left(x-\xi_{\mathbf{e}}\right)\right],\end{aligned}$$

where in the last inequality we used $1_{\{u<\delta\}}=1-1_{\{u\geqslant\delta\}}$ and $1_{\{u\geqslant\delta\}}\leqslant u/\delta$. Thus (8) is valid. On the other hand, since $G(x)\leqslant 1$ for $x\in[0,1]$, we have

$$\begin{aligned}\Phi_R(x)&=\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}G(u\left(x-\xi_{\mathbf{e}}\right))1_{\{u(x-\xi_{\mathbf{e}})<\delta\}}\right]+\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}G(u\left(x-\xi_{\mathbf{e}}\right))1_{\{u(x-\xi_{\mathbf{e}})\geqslant\delta\}}\right]\\ &\leqslant(1+\varepsilon)C_2(\gamma)\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^\gamma\left(x-\xi_{\mathbf{e}}\right)\right]+\frac{1}{\delta^{\gamma+1}}\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^{\gamma+1}\left(x-\xi_{\mathbf{e}}\right)\right],\end{aligned} \tag{13}$$

which implies (9).

We now consider the case $m<1$ and prove (ii) under the assumption $c_+>0$. Note that for any $\varepsilon>0$, there exists $\delta>0$ such that

$$G(u)\leqslant\varepsilon u,\qquad\text{for all } u\leqslant\delta.$$

Similar to (13), using the fact that $G(x)\leqslant 1$, we get that

$$\begin{aligned}\Phi_R(x)&=\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}G(u\left(x-\xi_{\mathbf{e}}\right))1_{\{u(x-\xi_{\mathbf{e}})<\delta\}}\right]+\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}G(u\left(x-\xi_{\mathbf{e}}\right))1_{\{u(x-\xi_{\mathbf{e}})\geqslant\delta\}}\right]\\ &\leqslant\varepsilon\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u\left(x-\xi_{\mathbf{e}}\right)\right]+\frac{1}{\delta^2}\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^2\left(x-\xi_{\mathbf{e}}\right)\right].\end{aligned}$$

Therefore, to prove (10), it suffices to show that

$$\lim_{x\to\infty}\frac{\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^2\left(x-\xi_{\mathbf{e}}\right)\right]}{\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u\left(x-\xi_{\mathbf{e}}\right)\right]}=0. \tag{14}$$

Considering the case where the initial particle does not split before time 1 and using (2), we get

$$u(x)\geqslant e^{-1}\mathbf{P}_0(S_1>x)\geqslant e^{-1}\mathbf{P}_0(\xi_1>x)\geqslant\frac{c_+}{2e\alpha}x^{-\alpha}.$$

when x is large enough. Therefore, when x is large enough,

$$\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u\left(x-\xi_{\mathbf{e}}\right)\right] \geqslant \mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u\left(x-\xi_{\mathbf{e}}\right)1_{\left\{|\xi_{\mathbf{e}}|<\frac{1}{2}x\right\}}\right] \geqslant \frac{c_+}{2e\alpha}\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}\left(x-\xi_{\mathbf{e}}\right)^{-\alpha}1_{\left\{|\xi_{\mathbf{e}}|<\frac{1}{2}x\right\}}\right]$$
$$\geqslant \frac{c_+}{2e\alpha}\left(\frac{3}{2}x\right)^{-\alpha}\mathbf{P}_0\left(S_{\mathbf{e}}<x,|\xi_{\mathbf{e}}|<\frac{1}{2}x\right) \geqslant c_1x^{-\alpha}, \tag{15}$$

for some positive constant c_1, where we used the fact that $\lim_{x\to\infty}\mathbf{P}_0\left(S_{\mathbf{e}}<x,|\xi_{\mathbf{e}}|<\frac{1}{2}x\right)=1$. Now we consider the numerator. Using the fact that $u\leqslant 1$, we get that that for $\delta'>0$ sufficiently small,

$$\begin{aligned}&\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^2\left(x-\xi_{\mathbf{e}}\right)\right]\\&=\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^2\left(x-\xi_{\mathbf{e}}\right)1_{\{\xi_{\mathbf{e}}<(1-\delta')x\}}\right]+\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^2\left(x-\xi_{\mathbf{e}}\right)1_{\{(1-\delta')x<\xi_{\mathbf{e}}<x\}}\right]\\&\leqslant u(\delta'x)\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u\left(x-\xi_{\mathbf{e}}\right)\right]+\mathbf{P}_0\left((1-\delta')x<\xi_{\mathbf{e}}<x\right).\end{aligned} \tag{16}$$

Combining (15) and (16), we obtain that

$$\begin{aligned}&\limsup_{x\to\infty}\frac{\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^2\left(x-\xi_{\mathbf{e}}\right)\right]}{\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u\left(x-\xi_{\mathbf{e}}\right)\right]} \leqslant \lim_{x\to\infty}u(\delta'x)+c_1\limsup_{x\to\infty}x^\alpha\mathbf{P}_0\left((1-\delta')x<\xi_{\mathbf{e}}<x\right)\\&=c_1\limsup_{x\to\infty}x^\alpha\int_0^\infty e^{-z}\mathbf{P}_0\left((1-\delta')x<\xi_z<x\right)\mathrm{d}z.\end{aligned}$$

From [15, Lemma 2.2], when x is large enough,

$$x^\alpha\mathbf{P}_0\left((1-\delta')x<\xi_z<x\right)\leqslant x^\alpha\mathbf{P}_0\left(|\xi_z|>(1-\delta')x\right)\leqslant c_2z, \quad \text{for all } z>0$$

for some positive constant c_2. Therefore, combining the dominated convergence theorem and (2), we get that

$$\limsup_{x\to\infty}\frac{\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u^2\left(x-\xi_{\mathbf{e}}\right)\right]}{\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u\left(x-\xi_{\mathbf{e}}\right)\right]} \leqslant \frac{c_2c_+}{\alpha}\int_0^\infty ze^{-z}\mathrm{d}z\frac{\delta'}{(1-\delta')^{1+\alpha}}\xrightarrow{\delta'\downarrow 0}0,$$

which implies (14). ∎

3 Proof of Theorem 1

3.1 The case $0<\alpha<1$

Define

$$\eta_\alpha(\lambda):=\Gamma(1-\alpha)\lambda^{\alpha-1}$$

and $\xi_{\mathbf{e}}^+=\max(0,\xi_{\mathbf{e}})$. It follows from [12, (2.1)] that

$$\lim_{\lambda\downarrow 0}\frac{1-\mathbf{E}_0\left[e^{-\lambda S_{\mathbf{e}}}\right]}{\lambda\cdot\eta_\alpha(\lambda)}=\lim_{\lambda\downarrow 0}\frac{1-\mathbf{E}_0\left[e^{-\lambda\xi_{\mathbf{e}}^+}\right]}{\lambda\cdot\eta_\alpha(\lambda)}=\frac{c_+}{\alpha}. \tag{17}$$

We will use $\mathcal{L}[f]$ to denote the Laplace transform of a positive function f:

$$\mathcal{L}[f](\lambda):=\int_0^{+\infty}e^{-\lambda x}f(x)\mathrm{d}x, \quad \lambda>0.$$

The following lemma is given in [12, Lemma 2.1]

Lemma 1. *Assume that $\alpha<1$ and that $f:[0,\infty)\to[0,\infty)$ is a positive and decreasing function.*
(i) For any $\lambda>0$, it holds that

$$\int_0^\infty e^{-\lambda x}\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}f(x-\xi_{\mathbf{e}})\right]\mathrm{d}x\leqslant\mathbf{E}_0\left[e^{-\lambda S_{\mathbf{e}}}\right]\mathcal{L}[f](\lambda).$$

(ii) For any $\lambda > 0$, it holds that

$$\begin{aligned}&\int_0^\infty e^{-\lambda x}\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}f(x-L_{\mathbf{e}})\right]\mathrm{d}x\\&\geqslant \mathbf{E}_0\left[e^{-\lambda\xi_{\mathbf{e}}^+}\right]\mathcal{L}[f](\lambda)+f(0)\frac{\mathbf{E}_0\left[e^{-\lambda S_{\mathbf{e}}}\right]-\mathbf{E}_0\left[e^{-\lambda\xi_{\mathbf{e}}^+}\right]}{\lambda}-\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}<0\}}\int_0^{-\xi_{\mathbf{e}}}e^{-\lambda z}f(z)\mathrm{d}z\right].\end{aligned}$$

(iii) If in addition that $\lim_{x\to\infty}f(x)=0$, then

$$\lim_{\lambda\downarrow 0}\frac{1}{\eta_\alpha(\lambda)}\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}<0\}}\int_0^{-\xi_{\mathbf{e}}}e^{-\lambda z}f(z)\mathrm{d}z\right]=0.$$

The next lemma can be found in [12, Lemma 2.2].

Lemma 2. *For any $\lambda > 0$, it holds that*

$$\frac{\lambda}{1-\mathbf{E}_0\left[e^{-\lambda S_{\mathbf{e}}}\right]}\mathcal{L}\left[\Phi_0+\Phi_R\right](\lambda)\leqslant 1$$

and

$$\frac{\lambda}{1-\mathbf{E}_0\left[e^{-\lambda\xi_{\mathbf{e}}^+}\right]}\mathcal{L}\left[\Phi_0+\Phi_R\right](\lambda)\geqslant 1-\lambda\mathcal{L}[u](\lambda)-\frac{\lambda}{1-\mathbf{E}_0\left[e^{-\lambda\xi_{\mathbf{e}}^+}\right]}\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}<0\}}\int_0^{-\xi_{\mathbf{e}}}e^{-\lambda z}u(z)\mathrm{d}z\right].$$

Proof of Theorem 1 for $\alpha<1$. Recall that $u(x)=\mathbb{P}(M\geqslant x)$. Using a change of variables and the monotone convergence theorem, we get

$$\lambda\mathcal{L}[u](\lambda)=\int_0^{+\infty}e^{-z}u\left(\frac{z}{\lambda}\right)\mathrm{d}z\underset{\lambda\downarrow 0}{\longrightarrow}0.$$

Combining (17) and Lemma 1 (iii) with $f=u$ and Lemma 2, we get

$$\lim_{\lambda\downarrow 0}\frac{\mathcal{L}\left[\Phi_0+\Phi_R\right](\lambda)}{\eta_\alpha(\lambda)}=\frac{c_+}{\alpha}.\tag{18}$$

We first consider the case $m=1$. In this case, $\Phi_0(x)=0$. Combining (9) and Lemma 1 (i) with $f=u^\gamma$ and $f=u^{\gamma+1}$, we get

$$\mathcal{L}\left[\Phi_R\right](\lambda)\leqslant C_2(\gamma)(1+\varepsilon)\mathcal{L}\left[u^\gamma\right](\lambda)+\frac{1}{\delta^{\gamma+1}}\mathcal{L}\left[u^{\gamma+1}\right](\lambda).\tag{19}$$

Since $\lim_{x\to+\infty}u(x)=0$, for any $\varepsilon_1>0$, there exists $A_1>0$ such that $u(x)\leqslant\varepsilon_1$ for $x\geqslant A_1$. Hence,

$$\begin{aligned}\mathcal{L}\left[u^{\gamma+1}\right](\lambda)&=\int_0^{A_1}e^{-\lambda x}u^{\gamma+1}(x)\mathrm{d}x+\int_{A_1}^{+\infty}e^{-\lambda x}u^{\gamma+1}(x)\mathrm{d}x\\&\leqslant A_1+\varepsilon_1\int_{A_1}^{+\infty}e^{-\lambda x}u^{\gamma}(x)\mathrm{d}x\leqslant A_1+\varepsilon_1\mathcal{L}\left[u^\gamma\right](\lambda),\end{aligned}\tag{20}$$

where in the first inequality we used the fact that $u\leqslant 1$. Thus, combining (18), (19) and (20), we have

$$\begin{aligned}\frac{c_+}{\alpha}=\lim_{\lambda\downarrow 0}\frac{\mathcal{L}\left[\Phi_R\right](\lambda)}{\eta_\alpha(\lambda)}&\leqslant\liminf_{\lambda\downarrow 0}\frac{\left(C_2(\gamma)(1+\varepsilon)+\varepsilon_1/\delta^{\gamma+1}\right)\mathcal{L}\left[u^\gamma\right](\lambda)+A_1/\delta^{\gamma+1}}{\eta_\alpha(\lambda)}\\&=\liminf_{\lambda\downarrow 0}\frac{\left(C_2(\gamma)(1+\varepsilon)+\varepsilon_1/\delta^{\gamma+1}\right)\mathcal{L}\left[u^\gamma\right](\lambda)}{\eta_\alpha(\lambda)}.\end{aligned}$$

Letting $\varepsilon_1\to 0$ first and then $\varepsilon\to 0$, we get that

$$\frac{c_+}{\alpha}\leqslant\liminf_{\lambda\downarrow 0}\frac{C_2(\gamma)\mathcal{L}\left[u^\gamma\right](\lambda)}{\eta_\alpha(\lambda)}.\tag{21}$$

On the other hand, combining (8), (20), Lemma 1 (ii) with $f = u^\gamma$ and Lemma 1 (i) with $f = u^{\gamma+1}$, we see that

$$\begin{aligned}
&\mathcal{L}\left[\Phi_R\right](\lambda) \\
&\geqslant C_2(\gamma)(1-\varepsilon)\left(\mathbf{E}_0\left[e^{-\lambda\xi_{\mathbf{e}}^+}\right]\mathcal{L}[u^\gamma](\lambda) + \frac{\mathbf{E}_0\left[e^{-\lambda S_{\mathbf{e}}}\right] - \mathbf{E}_0\left[e^{-\lambda\xi_{\mathbf{e}}^+}\right]}{\lambda} - \mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}<0\}}\int_0^{-\xi_{\mathbf{e}}} e^{-\lambda z}u^\gamma(z)\mathrm{d}z\right]\right) \\
&- \frac{C_2(\gamma)(1-\varepsilon)}{\delta}\left(A_1 + \varepsilon_1\mathcal{L}\left[u^\gamma\right](\lambda)\right).
\end{aligned}$$

Dividing both sides by $\eta_\alpha(\lambda)$ and using Lemma 1 (iii) with $f = u^\gamma$, we obtain

$$\frac{c_+}{\alpha} \geqslant \limsup_{\lambda\downarrow 0}\frac{C_2(\gamma)(1-\varepsilon)\left(1-\varepsilon_1/\delta\right)\mathcal{L}[u^\gamma](\lambda)}{\eta_\alpha(\lambda)}.$$

Letting $\varepsilon_1 \to 0$ first and then $\varepsilon \to 0$, we conclude that

$$\frac{c_+}{\alpha} \geqslant \limsup_{\lambda\downarrow 0}\frac{C_2(\gamma)\mathcal{L}\left[u^\gamma\right](\lambda)}{\eta_\alpha(\lambda)}. \tag{22}$$

Combining (21) and (22), we conclude that

$$\lim_{\lambda\downarrow 0}\frac{\mathcal{L}\left[u^\gamma\right](\lambda)}{\eta_\alpha(\lambda)} = \frac{c_+}{\alpha C_2(\gamma)}.$$

Hence, by the Tauberian theorem, the above limit is equivalent to

$$\lim_{x\to+\infty}\frac{1}{\eta_\alpha\left(\frac{1}{x}\right)}\int_0^x u^\gamma(z)\mathrm{d}z = \frac{c_+}{\alpha\Gamma(2-\alpha)C_2(\gamma)}.$$

Applying Karamata's monotone density theorem [2, Theorem 1.7.2], we get the desired result for $m = 1$.

Now we deal with the subcritical case $m < 1$. For any $\varepsilon' > 0$, by (10), we see that there exists a constant A' such that for all $x \geqslant A'$,

$$0 \leqslant \Phi_R(x) \leqslant \varepsilon'\Phi_0(x). \tag{23}$$

Similar to (20), using (23), we get that for all $\lambda > 0$,

$$0 \leqslant \mathcal{L}[\Phi_R](\lambda) \leqslant A' + \varepsilon'\int_{A'}^\infty e^{-\lambda x}\Phi_0(x)\mathrm{d}x \leqslant A' + \varepsilon'\mathcal{L}[\Phi_0](\lambda),$$

which together with (18) implies that

$$\limsup_{\lambda\downarrow 0}\frac{\mathcal{L}[\Phi_0](\lambda)}{\eta_\alpha(\lambda)} \leqslant \frac{c_+}{\alpha} \leqslant \liminf_{\lambda\downarrow 0}\frac{(1+\varepsilon')\mathcal{L}[\Phi_0](\lambda) + A'}{\eta_\alpha(\lambda)} = (1+\varepsilon')\liminf_{\lambda\downarrow 0}\frac{\mathcal{L}[\Phi_0](\lambda)}{\eta_\alpha(\lambda)}.$$

Letting $\varepsilon' \downarrow 0$, we get

$$\lim_{\lambda\downarrow 0}\frac{1}{\eta_\alpha(\lambda)}\mathcal{L}[u](\lambda) = \frac{c_+}{\alpha(1-m)}.$$

Hence, by the Tauberian theorem, we have

$$\lim_{x\to+\infty}\frac{1}{\eta_\alpha\left(\frac{1}{x}\right)}\int_0^x u(z)\mathrm{d}z = \frac{c_+}{\alpha\Gamma(2-\alpha)(1-m)}.$$

Applying Karamata's monotone density theorem [2, Theorem 1.7.2], we get the desired result. □

3.2 Proof of Theorem 1 for $1 \leqslant \alpha < 2$

It follows from [12, (3.3)] that for $\alpha \in (1,2)$,

$$\lim_{\lambda \downarrow 0} \frac{\int_0^{+\infty} e^{-\lambda x} x \mathbf{P}_0\left(S_{\mathbf{e}} \geqslant x\right) \mathrm{d}x}{\lambda^{\alpha-2}} = \lim_{\lambda \downarrow 0} \frac{1 - \mathbf{E}_0\left[e^{-\lambda S_{\mathbf{e}}}\right] - \lambda \mathbf{E}_0\left[S_{\mathbf{e}} e^{-\lambda S_{\mathbf{e}}}\right]}{\lambda^2 \cdot \lambda^{\alpha-2}} = \frac{c_+ \Gamma(2-\alpha)}{\alpha}. \tag{24}$$

For $\alpha = 1$, combining (2), [15, Lemma 2.2] and the dominated convergence theorem, (24) remains true for $\alpha = 1$ since $\Gamma(1) = 1$. Moreover, (24) also holds with $S_{\mathbf{e}}$ replaced by $\xi_{\mathbf{e}}^+$.

Lemma 3. *Assume that $\alpha \in [1,2)$ and that $f : [0,\infty) \to [0,\infty)$ is a positive and decreasing function.*
(i) We have the following upper bound

$$\int_0^\infty e^{-\lambda x} x \mathbf{E}_0\left[1_{\{S_{\mathbf{e}} < x\}} f(x - \xi_{\mathbf{e}})\right] \mathrm{d}x \leqslant \mathbf{E}_0\left[e^{-\lambda S_{\mathbf{e}}}\right] \mathcal{L}[x f(x)](\lambda) + \mathbf{E}_0\left[S_{\mathbf{e}} e^{-\lambda S_{\mathbf{e}}}\right] \mathcal{L}[f](\lambda).$$

(ii) We have the following lower bound

$$\begin{aligned}
&\int_0^\infty e^{-\lambda x} x \mathbf{E}_0\left[1_{\{S_{\mathbf{e}} < x\}} f(x - \xi_{\mathbf{e}})\right] \mathrm{d}x \\
&\geqslant f(0) \int_0^{+\infty} e^{-\lambda x} x \left(\mathbf{P}_0\left(\xi_{\mathbf{e}} \geqslant x\right) - \mathbf{P}_0\left(S_{\mathbf{e}} \geqslant x\right)\right) \mathrm{d}x + \mathbf{E}_0\left[e^{-\lambda \xi_{\mathbf{e}}} 1_{\{\xi_{\mathbf{e}} \geqslant 0\}}\right] \mathcal{L}[x f(x)](\lambda) \\
&\quad + \int_0^\infty e^{-\lambda x} x \mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}} < 0\}} f(x - \xi_{\mathbf{e}})\right] \mathrm{d}x.
\end{aligned}$$

Proof. For (i), see [12, Lemma 3.1] for the proof of $\alpha \in (1,2)$ and the proof for $\alpha = 1$ is the same. Now we prove (ii). Combining the inequalities $1_{\{S_{\mathbf{e}} \geqslant x\}} \geqslant 1_{\{\xi_{\mathbf{e}} \geqslant x\}}$ and $f(x) \leqslant f(0)$ for all $x \geqslant 0$, we have

$$\begin{aligned}
&\int_0^\infty e^{-\lambda x} x \mathbf{E}_0\left[1_{\{S_{\mathbf{e}} < x\}} f(x - \xi_{\mathbf{e}})\right] \mathrm{d}x \\
&\geqslant f(0) \int_0^{+\infty} e^{-\lambda x} x \left(\mathbf{P}_0\left(\xi_{\mathbf{e}} \geqslant x\right) - \mathbf{P}_0\left(S_{\mathbf{e}} \geqslant x\right)\right) \mathrm{d}x + \int_0^\infty e^{-\lambda x} x \mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}} < x\}} f(x - \xi_{\mathbf{e}})\right] \mathrm{d}x.
\end{aligned} \tag{25}$$

By Fubini's theorem, we have

$$\begin{aligned}
&\int_0^\infty e^{-\lambda x} x \mathbf{E}_0\left[1_{\{0 \leqslant \xi_{\mathbf{e}} < x\}} f(x - \xi_{\mathbf{e}})\right] \mathrm{d}x = \int_0^\infty e^{-\lambda x} x \int_0^x f(x-y) \mathbf{P}_0(\xi_{\mathbf{e}} \in \mathrm{d}y) \mathrm{d}x \\
&= \int_0^\infty \mathbf{P}_0(\xi_{\mathbf{e}} \in \mathrm{d}y) \int_y^\infty e^{-\lambda x} x f(x-y) \mathrm{d}x \geqslant \int_0^\infty \mathbf{P}_0(\xi_{\mathbf{e}} \in \mathrm{d}y) \int_y^\infty e^{-\lambda x} (x-y) f(x-y) \mathrm{d}x \\
&= \mathbf{E}_0\left[e^{-\lambda \xi_{\mathbf{e}}} 1_{\{\xi_{\mathbf{e}} \geqslant 0\}}\right] \mathcal{L}[x f(x)](\lambda).
\end{aligned} \tag{26}$$

Now (ii) follows from (25) and (26) $\square$

Lemma 4. *For any $\lambda > 0$, it holds that*
(i)

$$\frac{\lambda^2}{1 - \mathbf{E}_0\left[e^{-\lambda S_{\mathbf{e}}}\right] - \lambda \mathbf{E}_0\left[S_{\mathbf{e}} e^{-\lambda S_{\mathbf{e}}}\right]} \mathcal{L}\left[x(\Phi_0(x) + \Phi_R(x))\right](\lambda) \leqslant 1 + \frac{\lambda^2 \mathbf{E}_0\left[S_{\mathbf{e}} e^{-\lambda S_{\mathbf{e}}}\right]}{1 - \mathbf{E}_0\left[e^{-\lambda S_{\mathbf{e}}}\right] - \lambda \mathbf{E}_0\left[S_{\mathbf{e}} e^{-\lambda S_{\mathbf{e}}}\right]} \mathcal{L}[u](\lambda),$$

(ii)

$$\begin{aligned}
&\frac{\lambda^2}{1 - \mathbf{E}_0\left[e^{-\lambda \xi_{\mathbf{e}}^+}\right] - \lambda \mathbf{E}_0\left[\xi_{\mathbf{e}}^+ e^{-\lambda \xi_{\mathbf{e}}^+}\right]} \mathcal{L}\left[x(\Phi_0(x) + \Phi_R(x))\right](\lambda) \\
&\geqslant 1 - \frac{\lambda^2 \mathbf{E}_0\left[1 - e^{-\lambda \xi_{\mathbf{e}}^+}\right] \mathcal{L}[x u(x)](\lambda)}{1 - \mathbf{E}_0\left[e^{-\lambda \xi_{\mathbf{e}}^+}\right] - \lambda \mathbf{E}_0\left[\xi_{\mathbf{e}}^+ e^{-\lambda \xi_{\mathbf{e}}^+}\right]} \\
&\quad - \frac{\lambda^2 \left(\mathbf{P}_0\left(\xi_{\mathbf{e}} \leqslant 0\right) \mathcal{L}(x u(x))(\lambda) - \int_0^\infty e^{-\lambda x} x \mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}} < 0\}} u(x - \xi_{\mathbf{e}})\right] \mathrm{d}x\right)}{1 - \mathbf{E}_0\left[e^{-\lambda \xi_{\mathbf{e}}^+}\right] - \lambda \mathbf{E}_0\left[\xi_{\mathbf{e}}^+ e^{-\lambda \xi_{\mathbf{e}}^+}\right]}.
\end{aligned}$$

Proof. The proof of (i) for $\alpha \in (1,2)$ can be found in [12, Lemma 3.2] and the case $\alpha = 1$ can be treated similarly. Now we prove (ii). Combining (6) and Lemma 3 (ii) with $f = u$, we see that

$$\begin{aligned}
&\mathcal{L}\left[x(\Phi_0(x)+\Phi_R(x))\right](\lambda)+\mathcal{L}(xu(x))(\lambda)\\
&=\int_0^{+\infty} e^{-\lambda x} x\mathbf{P}_0\left(S_{\mathbf{e}} \geqslant x\right)\mathrm{d}x+\int_0^{\infty} e^{-\lambda x} x\mathbf{E}_0\left[1_{\{S_{\mathbf{e}}<x\}}u(x-\xi_{\mathbf{e}})\right]\mathrm{d}x\\
&\geqslant \int_0^{+\infty} e^{-\lambda x} x\mathbf{P}_0\left(\xi_{\mathbf{e}} \geqslant x\right)\mathrm{d}x+\mathbf{E}_0\left[e^{-\lambda\xi_{\mathbf{e}}}1_{\{\xi_{\mathbf{e}}\geqslant 0\}}\right]\mathcal{L}[xu(x)](\lambda)\\
&\qquad+\int_0^{\infty} e^{-\lambda x} x\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}<x\}}f(x-\xi_{\mathbf{e}})\right]\mathrm{d}x.
\end{aligned}$$

From the argument of (24) and the fact that $\xi_{\mathbf{e}} = \xi_{\mathbf{e}}^+$ on the set $\{\xi_{\mathbf{e}} \geqslant x\}$ for $x \geqslant 0$, we get the lower bound

$$\begin{aligned}
&\mathcal{L}\left[x(\Phi_0(x)+\Phi_R(x))\right](\lambda)+\mathcal{L}(xu(x))(\lambda)\\
&\geqslant \int_0^{+\infty} e^{-\lambda x} x\mathbf{P}_0\left(\xi_{\mathbf{e}}^+ \geqslant x\right)\mathrm{d}x-\mathbf{E}_0\left[(1-e^{-\lambda\xi_{\mathbf{e}}^+})1_{\{\xi_{\mathbf{e}}\geqslant 0\}}\right]\mathcal{L}(xu(x))(\lambda)\\
&\quad-\left(\mathbf{P}_0\left(\xi_{\mathbf{e}}\leqslant 0\right)\mathcal{L}(xu(x))(\lambda)-\int_0^{\infty} e^{-\lambda x} x\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}<0\}}u(x-\xi_{\mathbf{e}})\right]\mathrm{d}x\right)\\
&=\frac{1-\mathbf{E}_0\left[e^{-\lambda\xi_{\mathbf{e}}^+}\right]-\lambda\mathbf{E}_0\left[\xi_{\mathbf{e}}^+e^{-\lambda\xi_{\mathbf{e}}^+}\right]}{\lambda^2}-\mathbf{E}_0\left[1-e^{-\lambda\xi_{\mathbf{e}}^+}\right]\mathcal{L}(xu(x))(\lambda)\\
&\quad-\left(\mathbf{P}_0\left(\xi_{\mathbf{e}}\leqslant 0\right)\mathcal{L}(xu(x))(\lambda)-\int_0^{\infty} e^{-\lambda x} x\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}<0\}}u(x-\xi_{\mathbf{e}})\right]\mathrm{d}x\right),
\end{aligned}$$

which implies the desired result. □

In the critical case $m = 1$, we will need the following a priori upper bound of u.

Lemma 5. *If $\alpha \in (1,2)$ and $m = 1$, then there exists a constant $A > 0$ such that,*

$$u(x) \leqslant Ax^{-\alpha/\gamma}, \qquad \textit{for all } x \geqslant 0.$$

Proof. The proof is similar to that of [12, Lemma 3.3]. The main difference is that we have (29) for our branching mechanism. Denote by $\underline{M}^{(t)}$ the maximum of $(Z_s : s \geqslant 0)$ on $[0,t]$ and by $\overline{M}^{(t)}$ the maximum of $(Z_s : s \geqslant 0)$ on $[t,+\infty]$. Since $\alpha \in (1,2)$, we have $\rho := \mathbf{P}_0(\xi_1 \geqslant 0) \in [1-\frac{1}{\alpha}, \frac{1}{\alpha}]$. It follows that

$$\begin{aligned}
u(x)=\mathbb{P}(M\geqslant x)&\leqslant \mathbb{P}(\underline{M}^{(t)}\geqslant x)+\mathbb{P}(\overline{M}^{(t)}\geqslant x)\\
&\leqslant \mathbb{P}(\underline{M}^{(t)}\geqslant x)+\mathbb{P}\left(N_t\geqslant 1\right).
\end{aligned}\tag{27}$$

Define $T_x = \inf\{t \geqslant 0 : \exists u \in N_t, X_u(t) \geqslant x\}$. Then $\{T_x \leqslant t\} = \left\{\underline{M}^{(t)} \geqslant x\right\}$. Applying strong Markov property at T_x, we get

$$\mathbb{E}\left[\sum_{u\in N_t} 1_{\{X_u(t)\geqslant x\}} \mid \underline{M}^{(t)}\geqslant x\right]\geqslant \rho,$$

which implies that

$$\mathbb{P}\left(\underline{M}^{(t)}\geqslant x\right)\leqslant \rho^{-1}\mathbb{E}\left[\sum_{u\in N_t}1_{\{X_u(t)\geqslant x\}}\right]\leqslant \rho^{-1}\mathbb{E}[N_t]\mathbf{P}_0\left(\xi_t\geqslant x\right)=\rho^{-1}\mathbf{P}_0\left(t^{1/\alpha}\xi_1\geqslant x\right),\tag{28}$$

where we used the facts that N_t is independent of the spatial positions, and $\mathbb{E}[N_t] = 1$ for all $t \geqslant 0$. It follows from (12) that the function $\sum_{k=0}^{\infty} p_k(1-x)^k - (1-x)$ is a regularly varying at 0 with index γ. Hence by [18] [Theorem 4] we know that $Q(t) := \mathbb{P}(N_t \geqslant 1)$ satisfies the following equation for some positive constant c:

$$\lim_{t\to\infty}\frac{Q(t)}{t(\gamma-1)cQ(t)^{\gamma}}=1,$$

which implies that

$$\mathbb{P}(N_t \geqslant 1) \sim \left(\frac{1}{c(\gamma-1)t}\right)^{\frac{1}{\gamma-1}}. \tag{29}$$

Taking $t = x^{\alpha(1-\frac{1}{\gamma})}$ in (28) and (29), using (27), we get that there is some constant $A > 0$ such that

$$u(x) \leqslant Ax^{-\alpha/\gamma}.$$

This completes the proof of the Lemma. □

Proof of Theorem 1 for $\alpha \in [1,2)$. We first prove that

$$\lim_{\lambda\downarrow 0} \frac{\mathcal{L}\left[x(\Phi_0(x)+\Phi_R(x))\right](\lambda)}{\lambda^{\alpha-2}} = \frac{c_+\Gamma(2-\alpha)}{\alpha}. \tag{30}$$

For the upper bound, combining (24) and Lemma 4(i), we have

$$\begin{aligned}
&\frac{\alpha}{c_+\Gamma(2-\alpha)} \limsup_{\lambda\downarrow 0} \frac{\mathcal{L}\left[x(\Phi_0(x)+\Phi_R(x))\right](\lambda)}{\lambda^{\alpha-2}} \leqslant 1 + \limsup_{\lambda\downarrow 0} \frac{\lambda^2\mathbf{E}_0\left[S_{\mathbf{e}}e^{-\lambda S_{\mathbf{e}}}\right]}{1-\mathbf{E}_0\left[e^{-\lambda S_{\mathbf{e}}}\right]-\lambda\mathbf{E}_0\left[S_{\mathbf{e}}e^{-\lambda S_{\mathbf{e}}}\right]}\mathcal{L}[u](\lambda) \\
&= 1 + \frac{\alpha}{c_+\Gamma(2-\alpha)} \limsup_{\lambda\downarrow 0} \lambda^{2-\alpha}\mathbf{E}_0\left[S_{\mathbf{e}}e^{-\lambda S_{\mathbf{e}}}\right]\mathcal{L}[u](\lambda).
\end{aligned} \tag{31}$$

When $\alpha \in (\gamma, 2)$, combining Lemma 5 and the fact that $\mathbf{E}_0(S_{\mathbf{e}}) < \infty$, we see that

$$\lambda^{2-\alpha}\mathbf{E}_0\left[S_{\mathbf{e}}e^{-\lambda S_{\mathbf{e}}}\right]\mathcal{L}[u](\lambda) \leqslant \mathbf{E}_0\left[S_{\mathbf{e}}\right]\lambda^{2-\alpha}\left(1 + A\int_1^\infty x^{-\alpha/\gamma}\mathrm{d}x\right) \xrightarrow{\lambda\downarrow 0} 0. \tag{32}$$

When $\alpha \in (1,\gamma]$, using $\alpha \in (1,2)$ and $\gamma \in (1,2]$, we have $1 + \alpha\gamma^{-1} = \alpha + \gamma^{-1}(1-(\alpha-1)(\gamma-1)) > \alpha$. Thus there exists $\delta \in (0,1)$ such that $1 + \alpha\delta/\gamma > \alpha$. Therefore, combining Lemma 5 and $\mathbf{E}_0(S_{\mathbf{e}}) < \infty$,

$$\begin{aligned}
\lambda^{2-\alpha}\mathbf{E}_0\left[S_{\mathbf{e}}e^{-\lambda S_{\mathbf{e}}}\right]\mathcal{L}[u](\lambda) &\leqslant A\mathbf{E}_0\left[S_{\mathbf{e}}\right]\lambda^{2-\alpha}\int_0^\infty e^{-\lambda x}x^{-\alpha\delta/\gamma}\mathrm{d}x \\
&= A\mathbf{E}_0\left[S_{\mathbf{e}}\right]\lambda^{1+\alpha\delta/\gamma-\alpha}\int_0^\infty e^{-x}x^{-\alpha\delta/\gamma}\mathrm{d}x \xrightarrow{\lambda\downarrow 0} 0.
\end{aligned} \tag{33}$$

When $\alpha = 1$, fix a constant $\delta \in (1-1/\gamma, 1)$. Combining [12, (2.1)] and Markov's inequality, we have

$$y^\delta\mathbf{P}_0(S_{\mathbf{e}} \geqslant y) \leqslant \frac{y^\delta}{(1-e^{-1})}\mathbf{E}_0(1-e^{-y^{-1}S_{\mathbf{e}}}) = \frac{y^{\delta-1}\ln y}{(1-e^{-1})}\frac{\mathbf{E}_0(1-e^{-y^{-1}S_{\mathbf{e}}})}{y^{-1}\ln y} \xrightarrow{y\to\infty} 0.$$

Thus there exists $c_\delta > 0$ such that $\mathbf{P}_0(S_{\mathbf{e}} \geqslant y) \leqslant c_\delta y^{-\delta}$ for any $y > 0$, which implies that

$$\begin{aligned}
\lambda\mathbf{E}_0\left[S_{\mathbf{e}}e^{-\lambda S_{\mathbf{e}}}\right]\mathcal{L}[u](\lambda) &= \lambda\int_0^\infty (1-\lambda y)e^{-\lambda y}\mathbf{P}_0(S_{\mathbf{e}} > y)\mathrm{d}y\int_0^\infty e^{-\lambda x}u(x)\mathrm{d}x \\
&\leqslant A\lambda\int_0^\infty e^{-\lambda y}\mathbf{P}_0(S_{\mathbf{e}} \geqslant y)\mathrm{d}y\int_0^\infty e^{-\lambda x}x^{-1/\gamma}\mathrm{d}x \\
&\leqslant Ac_\delta\lambda\int_0^\infty e^{-\lambda y}y^{-\delta}\mathrm{d}y\int_0^\infty e^{-\lambda x}x^{-1/\gamma}\mathrm{d}x \\
&= Ac_\delta\lambda^{\delta+\gamma^{-1}-1}\int_0^\infty e^{-y}y^{-\delta}\mathrm{d}y\int_0^\infty e^{-x}x^{-1/\gamma}\mathrm{d}x \xrightarrow{\lambda\downarrow 0} 0.
\end{aligned} \tag{34}$$

Combining (31), (32), (33) and (34), we get

$$\limsup_{\lambda\downarrow 0} \frac{\mathcal{L}\left[x(\Phi_0(x)+\Phi_R(x))\right](\lambda)}{\lambda^{\alpha-2}} \leqslant \frac{c_+\Gamma(2-\alpha)}{\alpha}. \tag{35}$$

Now we prove the lower bound. Similarly, combining (24) and Lemma 4(ii), we see that for any $\varepsilon > 0$,

$$\begin{aligned}
&\frac{\alpha}{c_+\Gamma(2-\alpha)} \liminf_{\lambda\downarrow 0} \frac{\mathcal{L}\left[x(\Phi_0(x)+\Phi_R(x))\right](\lambda)}{\lambda^{\alpha-2}} \\
&\geqslant 1 - \frac{\alpha}{c_+\Gamma(2-\alpha)} \limsup_{\lambda\downarrow 0} \lambda^{2-\alpha}\mathbf{E}_0\left[1-e^{-\lambda\xi_{\mathbf{e}}^+}\right]\mathcal{L}[xu(x)](\lambda) - \frac{\alpha}{c_+\Gamma(2-\alpha)} \\
&\quad\times \limsup_{\lambda\downarrow 0} \lambda^{2-\alpha}\left(\mathbf{P}_0\left(\xi_{\mathbf{e}} \leqslant 0\right)\mathcal{L}(xu(x))(\lambda) - \int_0^\infty e^{-\lambda x}x\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}<0\}}u(x-\xi_{\mathbf{e}})\right]\mathrm{d}x\right).
\end{aligned} \tag{36}$$

From Lemma 5, it holds that

$$\mathcal{L}[xu(x)](\lambda) \leqslant A\int_0^\infty e^{-\lambda x}x^{1-\gamma^{-1}\alpha}\mathrm{d}x = A\lambda^{\alpha/\gamma-2}\int_0^\infty e^{-x}x^{1-\gamma^{-1}\alpha}\mathrm{d}x.$$

Since $1-\gamma^{-1}\alpha > 1-2 = -1$, we conclude from the above inequality that for any $\alpha\in[1,2)$, there exists A' such that

$$\lambda^{2-\alpha}\mathbf{E}_0\left[1-e^{-\lambda\xi_{\mathbf{e}}^+}\right]\mathcal{L}[xu(x)](\lambda) \leqslant A'\lambda^{\alpha/\gamma-\alpha}\mathbf{E}_0\left[1-e^{-\lambda\xi_{\mathbf{e}}^+}\right].$$

Note that when $\alpha=1$, by [12, (2.1)], $\lim_{\lambda\downarrow 0}\frac{1}{\lambda\ln(\lambda^{-1})}\mathbf{E}_0\left[1-e^{-\lambda\xi_{\mathbf{e}}^+}\right] = \frac{c_+}{\alpha}$, and when $\alpha\in(1,2)$, $\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\mathbf{E}_0\left[1-e^{-\lambda\xi_{\mathbf{e}}^+}\right] = \mathbf{E}_0(\xi_{\mathbf{e}}^+)$. Using the fact that $\alpha/\gamma-\alpha+1 = \gamma^{-1}(1-(\alpha-1)(\gamma-1)) > 0$, we have

$$\limsup_{\lambda\downarrow 0}\lambda^{2-\alpha}\mathbf{E}_0\left[1-e^{-\lambda\xi_{\mathbf{e}}^+}\right]\mathcal{L}[xu(x)](\lambda) = 0. \tag{37}$$

Plugging (37) into (36) yields that

$$\begin{aligned}&\frac{\alpha}{c_+\Gamma(2-\alpha)}\liminf_{\lambda\downarrow 0}\frac{\mathcal{L}\left[x(\Phi_0(x)+\Phi_R(x))\right](\lambda)}{\lambda^{\alpha-2}} \geqslant 1-\frac{\alpha}{c_+\Gamma(2-\alpha)}\\ &\times\limsup_{\lambda\downarrow 0}\lambda^{2-\alpha}\left(\mathbf{P}_0\left(\xi_{\mathbf{e}}\leqslant 0\right)\mathcal{L}(xu(x))(\lambda) - \int_0^\infty e^{-\lambda x}x\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}<0\}}u(x-\xi_{\mathbf{e}})\right]\mathrm{d}x\right).\end{aligned} \tag{38}$$

When $\alpha\in(1,2)$, using the fact that $\mathbf{E}_0(|\xi_{\mathbf{e}}|)<\infty$, we see that

$$\begin{aligned}&\int_0^\infty e^{-\lambda x}x\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}\leqslant 0\}}u(x-\xi_{\mathbf{e}})\right]\mathrm{d}x = \mathbf{E}_0\left(1_{\{\xi_{\mathbf{e}}\leqslant 0\}}\int_{-\xi_{\mathbf{e}}}^\infty e^{-\lambda(x+\xi_{\mathbf{e}})}(x+\xi_{\mathbf{e}})u(x)\mathrm{d}x\right)\\ &\geqslant \mathbf{E}_0\left(1_{\{\xi_{\mathbf{e}}\leqslant 0\}}\int_{-\xi_{\mathbf{e}}}^\infty e^{-\lambda x}(x+\xi_{\mathbf{e}})u(x)\mathrm{d}x\right) \geqslant \mathbf{P}_0\left(\xi_{\mathbf{e}}\leqslant 0\right)\mathcal{L}(xu(x))(\lambda)\\ &\quad - \mathbf{E}_0\left(1_{\{\xi_{\mathbf{e}}\leqslant 0\}}\int_0^{-\xi_{\mathbf{e}}} e^{-\lambda x}xu(x)\mathrm{d}x\right) - \mathbf{E}_0(|\xi_{\mathbf{e}}|)\mathcal{L}(u)(\lambda)\\ &\geqslant \mathbf{P}_0\left(\xi_{\mathbf{e}}\leqslant 0\right)\mathcal{L}(xu(x))(\lambda) - 2\mathbf{E}_0(|\xi_{\mathbf{e}}|)\mathcal{L}(u)(\lambda).\end{aligned} \tag{39}$$

Therefore, combining (32) and (33), we conclude from (39) that

$$\begin{aligned}&\limsup_{\lambda\downarrow 0}\lambda^{2-\alpha}\left(\mathbf{P}_0\left(\xi_{\mathbf{e}}\leqslant 0\right)\mathcal{L}(xu(x))(\lambda) - \int_0^\infty e^{-\lambda x}x\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}<0\}}u(x-\xi_{\mathbf{e}})\right]\mathrm{d}x\right)\\ &\leqslant 2\mathbf{E}_0(|\xi_{\mathbf{e}}|)\limsup_{\lambda\downarrow 0}\lambda^{2-\alpha}\mathcal{L}(u)(\lambda) = 0.\end{aligned}$$

Combining the above inequality with (35) and (38), we get (30) holds for $\alpha\in(1,2)$.

Now we consider the case $\alpha=1$. For any fixed $\delta\in(0,1)$, noticing that $\mathbf{E}_0(|\xi_{\mathbf{e}}|^{1-\delta})<\infty$, we get that

$$\begin{aligned}&\int_0^\infty e^{-\lambda x}x\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}\leqslant 0\}}u(x-\xi_{\mathbf{e}})\right]\mathrm{d}x = \mathbf{E}_0\left(1_{\{\xi_{\mathbf{e}}\leqslant 0\}}\int_{-\xi_{\mathbf{e}}}^\infty e^{-\lambda(x+\xi_{\mathbf{e}})}(x+\xi_{\mathbf{e}})u(x)\mathrm{d}x\right)\\ &\geqslant \mathbf{E}_0\left(1_{\{\xi_{\mathbf{e}}\leqslant 0\}}\int_{-\xi_{\mathbf{e}}}^\infty e^{-\lambda x}(x+\xi_{\mathbf{e}})u(x)\mathrm{d}x\right) = \mathbf{P}_0\left(\xi_{\mathbf{e}}\leqslant 0\right)\mathcal{L}(xu(x))(\lambda)\\ &\quad - \mathbf{E}_0\left(1_{\{\xi_{\mathbf{e}}<0\}}\int_0^{-\xi_{\mathbf{e}}} e^{-\lambda x}xu(x)\mathrm{d}x\right) - \mathbf{E}_0\left(1_{\{\xi_{\mathbf{e}}<0\}}\int_{-\xi_{\mathbf{e}}}^\infty e^{-\lambda x}|\xi_{\mathbf{e}}|u(x)\mathrm{d}x\right)\\ &\geqslant \mathbf{P}_0\left(\xi_{\mathbf{e}}\leqslant 0\right)\mathcal{L}(xu(x))(\lambda) - 2\mathbf{E}_0(|\xi_{\mathbf{e}}|^{1-\delta})\mathcal{L}(x^\delta u(x))(\lambda).\end{aligned} \tag{40}$$

By taking $\delta<\gamma^{-1}$, we get from Lemma 5 and (40) that

$$\begin{aligned}&\limsup_{\lambda\downarrow 0}\lambda\left(\mathbf{P}_0\left(\xi_{\mathbf{e}}\leqslant 0\right)\mathcal{L}(xu(x))(\lambda) - \int_0^\infty e^{-\lambda x}x\mathbf{E}_0\left[1_{\{\xi_{\mathbf{e}}<0\}}u(x-\xi_{\mathbf{e}})\right]\mathrm{d}x\right)\\ &\leqslant 2\mathbf{E}_0(|\xi_{\mathbf{e}}|^{1-\delta})\limsup_{\lambda\downarrow 0}\lambda\mathcal{L}(x^\delta u(x))(\lambda) \leqslant 2A\mathbf{E}_0(|\xi_{\mathbf{e}}|^{1-\delta})\limsup_{\lambda\downarrow 0}\lambda\int_0^\infty e^{-\lambda y}y^{\delta-1/\gamma}\mathrm{d}y\\ &= 2A\mathbf{E}_0(|\xi_{\mathbf{e}}|^{1-\delta})\limsup_{\lambda\downarrow 0}\lambda^{1/\gamma-\delta}\int_0^\infty e^{-y}y^{\delta-1/\gamma}\mathrm{d}y = 0.\end{aligned} \tag{41}$$

Combining (35), (38) and (41), we get that (30) holds when $\alpha = 1$.

The rest of the proof is now similar to the case $\alpha \in (0,1)$. We first consider the case $m = 1$. In this case, $\Phi_0(x) = 0$. Combining (9) and Lemma 3 (i) with $f = u^\gamma$ and $f = u^{\gamma+1}$, we get that

$$\begin{aligned}\mathcal{L}\left[x\Phi_R\right](\lambda) \leqslant (1+\varepsilon)C_2(\gamma)\left(\mathcal{L}\left[xu^\gamma(x)\right](\lambda) + \mathbf{E}_0\left[S_\mathbf{e}\right]\mathcal{L}\left[u^\gamma\right](\lambda)\right) \\ + \frac{1}{\delta^{\gamma+1}}\left(\mathcal{L}\left[xu^{\gamma+1}(x)\right](\lambda) + \mathbf{E}_0\left[S_\mathbf{e}\right]\mathcal{L}\left[u^{\gamma+1}\right](\lambda)\right).\end{aligned} \tag{42}$$

For any $\varepsilon_1 > 0$, since $\lim_{x\to+\infty} u(x) = 0$, there exists $A_2 > 0$ such that $u(x) \leqslant \varepsilon_1$ for $x \geqslant A_2$. Similar to (20), we have

$$\mathcal{L}\left[xu^{\gamma+1}(x)\right](\lambda) \leqslant A_2^2 + \varepsilon_1\mathcal{L}\left[xu^\gamma(x)\right](\lambda). \tag{43}$$

Combining (30), (42) and (43), we get

$$\frac{c_+\Gamma(2-\alpha)}{\alpha} \leqslant \left((1+\varepsilon)C_2(\gamma) + \varepsilon_1/\delta^{\gamma+1}\right)\liminf_{\lambda\downarrow 0}\lambda^{2-\alpha}\mathcal{L}\left[xu^\gamma(x)\right](\lambda).$$

Letting $\varepsilon_1 \to 0$ first and then $\varepsilon \to 0$, we get that

$$\frac{c_+\Gamma(2-\alpha)}{\alpha} \leqslant C_2(\gamma)\liminf_{\lambda\downarrow 0}\lambda^{2-\alpha}\mathcal{L}\left[xu^\gamma(x)\right](\lambda). \tag{44}$$

Next, combining (43), Lemma 3 (ii) with $f = u^\gamma$ and Lemma 3 (i) with $f = u^{\gamma+1}$, it holds that

$$\begin{aligned}&\mathcal{L}\left[x\Phi_R(x)\right](\lambda) \geqslant C_2(\gamma)(1-\varepsilon)\int_0^{+\infty} e^{-\lambda x}x\left(\mathbf{P}_0\left(\xi_\mathbf{e} \geqslant x\right) - \mathbf{P}_0\left(S_\mathbf{e} \geqslant x\right)\right)\mathrm{d}x \\ &+ C_2(\gamma)(1-\varepsilon)\left\{\mathbf{E}_0\left[e^{-\lambda\xi_\mathbf{e}^+}\right]\mathcal{L}\left[xu^\gamma(x)\right](\lambda) - \mathbf{E}_0\left[1_{\{\xi_\mathbf{e}<0\}}\int_0^{-\xi_\mathbf{e}} e^{-\lambda z}zu^\gamma(z)\mathrm{d}z\right] + \mathbf{E}_0\left[\xi_\mathbf{e}e^{-\lambda\xi_\mathbf{e}^+}\right]\mathcal{L}\left[u^\gamma\right](\lambda)\right\} \\ &- \frac{1}{\delta^{\gamma+1}}\left\{A_1^2 + \varepsilon_1\mathcal{L}\left[xu^\gamma(x)\right](\lambda) + \mathbf{E}_0\left[S_\mathbf{e}\right]\mathcal{L}\left[u^{\gamma+1}\right](\lambda)\right\}.\end{aligned}$$

Applying Lemma 3 (iii) with $f = u^\gamma$, we obtain

$$\frac{c_+\Gamma(2-\alpha)}{\alpha} \geqslant \left(C_2(\gamma)(1-\varepsilon) - \varepsilon_1/\delta^{\gamma+1}\right)\limsup_{\lambda\downarrow 0}\lambda^{2-\alpha}\mathcal{L}\left[xu^\gamma(x)\right](\lambda).$$

Letting $\varepsilon_1 \to 0$ first and then $\varepsilon \to 0$, we get that

$$\frac{c_+\Gamma(2-\alpha)}{\alpha} \geqslant C_2(\gamma)\limsup_{\lambda\downarrow 0}\lambda^{2-\alpha}\mathcal{L}\left[xu^\gamma(x)\right](\lambda). \tag{45}$$

Combining (44) and (45), we conclude that

$$\lim_{\lambda\downarrow 0}\frac{\mathcal{L}\left[xu^\gamma(x)\right](\lambda)}{\lambda^{\alpha-2}} = \frac{c_+\Gamma(2-\alpha)}{C_2(\gamma)\alpha}.$$

Finally, by the Tauberian theorem, we get

$$\lim_{x\to+\infty}\frac{1}{x^{2-\alpha}}\int_0^x zu^\gamma(z)\mathrm{d}z = \frac{c_+}{\alpha(2-\alpha)C_2(\gamma)}.$$

The desired result now follows from Karamata's monotone density theorem [2, Theorem 1.7.2].

Next we consider the case $m < 1$. Combining $\Phi_R \geqslant 0$, (23) and Lemma 3 (i) with $f = u$, we get that for any $\varepsilon' > 0$, there exists a constant $A_3 = A_3(\varepsilon')$, such that for all small λ,

$$\mathcal{L}\left[x\Phi_0(x)\right](\lambda) \leqslant \mathcal{L}\left[x(\Phi_0 + \Phi_R)\right](\lambda) \leqslant (1+\varepsilon')(1-m)\left\{\mathcal{L}[xu(x)](\lambda) + \mathbf{E}_0\left[S_\mathbf{e}\right]\mathcal{L}[u](\lambda)\right\} + A_3,$$

which, by (30), implies that

$$(1-m)\limsup_{\lambda\downarrow 0}\lambda^{2-\alpha}\mathcal{L}[xu(x)](\lambda) \leqslant \frac{c_+\Gamma(2-\alpha)}{\alpha} \leqslant (1-m)(1+\varepsilon')\liminf_{\lambda\downarrow 0}\lambda^{2-\alpha}\mathcal{L}[xu(x)](\lambda).$$

Letting $\varepsilon' \downarrow 0$, we get that

$$\lim_{\lambda\downarrow 0}\frac{1}{\lambda^{\alpha-2}}\mathcal{L}\left[xu(x)\right](\lambda) = \frac{c_+\Gamma(2-\alpha)}{\alpha(1-m)}.$$

Finally, by the Tauberian theorem,

$$\lim_{x\to+\infty}\frac{1}{x^{2-\alpha}}\int_0^x zu(z)\mathrm{d}z = \frac{c_+\Gamma(2-\alpha)}{\alpha(1-m)}.$$

The desired result now follows from Karamata's monotone density theorem [2, Theorem 1.7.2]. $\square$

4 Proof of Theorem 2

Define $\widetilde{\xi} = -\xi$, $\widetilde{\tau}_y := \inf\left\{t : \widetilde{\xi}_t \leqslant y\right\}$ and

$$f(u) := \frac{G(u)}{u} = \frac{\sum_{k=0}^{\infty} p_k(1-u)^k - (1-u)}{u}, \quad u \in [0,1].$$

According to [4, Lemma 2.3], u has the following representation:

$$u(x) = \mathbf{E}_x\left(\exp\left\{-\int_0^{\widetilde{\tau}_y} f\left(u\left(\widetilde{\xi}_s\right)\right)\mathrm{d}s\right\}\right)u(y), \quad 0 \leqslant y < x. \tag{46}$$

Similar to [4], we consider the function

$$[0,\infty) \ni x \mapsto \frac{u\left(x + yu(x)^{-\frac{\gamma-1}{\alpha}}\right)}{u(x)},$$

which is bounded between 0 and 1. Therefore, using a diagonalization argument, we can find a subsequence $\{x_k \in [0,\infty)\}$ with $\lim_{k\to\infty} x_k = +\infty$ such that for all $y \geqslant 0, y \in \mathbb{Q}$, the following limits exist:

$$\phi(y) := \lim_{k\to\infty} \frac{u\left(x_k + yu(x_k)^{-\frac{\gamma-1}{\alpha}}\right)}{u(x_k)}. \tag{47}$$

Since $u(x)$ is decreasing, we see that $\phi(0) = 1$ and $\phi(y) \in [0,1]$ for any $y \in \mathbb{Q}\cap[0,\infty)$. Moreover, ϕ is decreasing in $\mathbb{Q}\cap[0,\infty)$. Therefore, for any $y \geqslant 0$, we can define

$$\phi(y) := \sup_{z\in\mathbb{Q}, z\geqslant y} \phi(z) = \lim_{z\in\mathbb{Q}, z\downarrow y} \phi(y).$$

Lemma 6. *The limit (47) holds for all $y \geqslant 0$. Also, it holds that for any $K > 0$,*

$$\lim_{k\to\infty} \sup_{y\in[0,K]} \left|\frac{u\left(x_k + yu(x_k)^{-\frac{\gamma-1}{\alpha}}\right)}{\phi(y)u(x_k)} - 1\right| = 0. \tag{48}$$

Moreover, ϕ satisfies the equation

$$\phi(y) = \mathbf{E}_y\left(\exp\left\{-C_2(\gamma)\int_0^{\widetilde{\tau}_0}\left(\phi(\widetilde{\xi}_s)\right)^{\gamma-1}\mathrm{d}s\right\}\right), \quad y \geqslant 0, \tag{49}$$

where $C_2(\gamma)$ is defined in (11).

Proof. Fix two arbitrary non-negative rational numbers $y_1 < y_2$ and set $z_i(k) = y_i u(x_k)^{-\frac{\gamma-1}{\alpha}}$, $i = 1, 2$. Combining the definition of ϕ and (46), we get that

$$\begin{aligned}
\phi(y_1) \geqslant \phi(y_2) &= \lim_{k\to\infty} \frac{u(x_k + z_2(k))}{u(x_k)} \\
&= \lim_{k\to\infty} \mathbf{E}_{x_k+z_2(k)}\left(\exp\left\{-\int_0^{\widetilde{\tau}_{x_k+z_1(k)}} f\left(u\left(\widetilde{\xi}_s\right)\right)\mathrm{d}s\right\}\right)\frac{u(x_k + z_1(k))}{u(x_k)} \\
&= \phi(y_1)\lim_{k\to\infty} \mathbf{E}_{z_2(k)}\left(\exp\left\{-\int_0^{\widetilde{\tau}_{z_1(k)}} f\left(u\left(x_k + \widetilde{\xi}_s\right)\right)\mathrm{d}s\right\}\right).
\end{aligned} \tag{50}$$

Combining the scaling property of $\widetilde{\xi}$, (11) and (50), we get that there exists $c > 0$ such that

$$\begin{aligned}
\phi(y_1) \geqslant \phi(y_2) &\geqslant \phi(y_1)\lim_{k\to\infty} \mathbf{E}_{z_2(k)}\left(\exp\left\{-c\int_0^{\widetilde{\tau}_{z_1(k)}}\left(u\left(x_k + \widetilde{\xi}_s\right)\right)^{\gamma-1}\mathrm{d}s\right\}\right) \\
&= \phi(y_1)\lim_{k\to\infty} \mathbf{E}_{y_2}\left(\exp\left\{-c\int_0^{u(x_k)^{-(\gamma-1)}\widetilde{\tau}_{y_1}}\left(u\left(x_k + \widetilde{\xi}_{su(x_k)^{\gamma-1}}u(x_k)^{-\frac{\gamma-1}{\alpha}}\right)\right)^{\gamma-1}\mathrm{d}s\right\}\right) \\
&= \phi(y_1)\lim_{k\to\infty} \mathbf{E}_{y_2}\left(\exp\left\{-c\int_0^{\widetilde{\tau}_{y_1}}\left(\frac{u\left(x_k + \widetilde{\xi}_s u(x_k)^{-\frac{\gamma-1}{\alpha}}\right)}{u(x_k)}\right)^{\gamma-1}\mathrm{d}s\right\}\right).
\end{aligned}$$

Since $u\left(x_k+\widetilde{\xi}_s u(x_k)^{-\frac{\gamma-1}{\alpha}}\right)\leqslant u(x_k)$ for all $s\leqslant \widetilde{\tau}_{y_1}$, we get from the above inequality that

$$\phi(y_1)\geqslant \phi(y_2)\geqslant \phi(y_1)\mathbf{E}_{y_2}\left(\exp\left\{-c\widetilde{\tau}_{y_1}\right\}\right)=\phi(y_1)\mathbf{E}_1\left(\exp\left\{-c(y_2-y_1)^{\alpha}\widetilde{\tau}_0\right\}\right), \tag{51}$$

where in the last equality we also used the scaling property of $\widetilde{\xi}$. Therefore, for any $y>0$ and any positive rational number $y_1\leqslant y<y_2$, we have

$$\begin{aligned}\phi(y_2)&=\lim_{k\to\infty}\frac{u\left(x_k+y_2u(x_k)^{-\frac{\gamma-1}{\alpha}}\right)}{u(x_k)}\leqslant \liminf_{k\to\infty}\frac{u\left(x_k+yu(x_k)^{-\frac{\gamma-1}{\alpha}}\right)}{u(x_k)}\\&\leqslant \limsup_{k\to\infty}\frac{u\left(x_k+yu(x_k)^{-\frac{\gamma-1}{\alpha}}\right)}{u(x_k)}\leqslant \lim_{k\to\infty}\frac{u\left(x_k+y_1u(x_k)^{-\frac{\gamma-1}{\alpha}}\right)}{u(x_k)}=\phi(y_1).\end{aligned} \tag{52}$$

Combining (51) and (52), we see that (47) holds for all $y\geqslant 0$.

Taking $y_1=0$ in (51), we see that $\inf_{y\in[0,K]}\phi(y)>0$. Therefore, using an argument similar to that leading to [4, (3.9)], we can get (48).

Now we prove (49). Since $u(x_k+\widetilde{\xi}_s)\leqslant u(x_k)$ for any $s\leqslant\widetilde{\tau}_0$, combining (11) and (46), we see that for any $\varepsilon>0$,

$$\begin{aligned}\phi(y)&=\lim_{k\to\infty}\frac{u\left(x_k+yu(x_k)^{-\frac{\gamma-1}{\alpha}}\right)}{u(x_k)}=\lim_{k\to\infty}\mathbf{E}_{yu(x_k)^{-\frac{\gamma-1}{\alpha}}}\left(\exp\left\{-\int_0^{\widetilde{\tau}_0}f\left(u\left(x_k+\widetilde{\xi}_s\right)\right)\mathrm{d}s\right\}\right)\\&\geqslant \lim_{k\to\infty}\mathbf{E}_{yu(x_k)^{-\frac{\gamma-1}{\alpha}}}\left(\exp\left\{-C_2(\gamma)(1+\varepsilon)\int_0^{\widetilde{\tau}_0}\left(u\left(x_k+\widetilde{\xi}_s\right)\right)^{\gamma-1}\mathrm{d}s\right\}\right).\end{aligned}$$

Applying the scaling property again, we get from the above inequality that

$$\begin{aligned}\phi(y)&\geqslant \lim_{k\to\infty}\mathbf{E}_y\left(\exp\left\{-C_2(\gamma)(1+\varepsilon)\int_0^{\widetilde{\tau}_0}\left(\frac{u\left(x_k+\widetilde{\xi}_su(x_k)^{-\frac{\gamma-1}{\alpha}}\right)}{u(x_k)}\right)^{\gamma-1}\mathrm{d}s\right\}\right)\\&=\mathbf{E}_y\left(\exp\left\{-C_2(\gamma)(1+\varepsilon)\int_0^{\widetilde{\tau}_0}\left(\phi(\widetilde{\xi}_s)\right)^{\gamma-1}\mathrm{d}s\right\}\right),\end{aligned}$$

where in the last equality we used the dominated convergence theorem. Letting $\varepsilon\downarrow 0$, we get the lower bound. The proof of the upper bound is similar and we omit the details. $\square$

As a consequence of Lemma 6, we see that $\phi(y)\in(0,1)$ for all $y>0$.

Proposition 2. *(i) ϕ is the unique solution of of (49).*

(ii) The equation

$$U(z)=\mathbf{E}_z\left(\exp\left\{-C_2(\gamma)\int_0^{\widetilde{\tau}_y}\left(U(\widetilde{\xi}_s)\right)^{\gamma-1}\mathrm{d}s\right\}\right)U(y),\quad z>y\geqslant 0 \tag{53}$$

has a unique solution satisfying the boundary conditions $\lim_{y\downarrow 0}U(y)=+\infty$ *and* $\lim_{y\to\infty}U(y)=0$.

The proof of Proposition 2 is postponed to Section 5.

Lemma 7. *It holds that*

$$\limsup_{x\to\infty}x^{\frac{\alpha}{\gamma-1}}u(x)=\limsup_{x\to\infty}x^{\frac{\alpha}{\gamma-1}}\mathbb{P}\left(M\geqslant x\right)<\infty.$$

and that

$$\liminf_{x\to\infty}x^{\frac{\alpha}{\gamma-1}}u(x)=\liminf_{x\to\infty}x^{\frac{\alpha}{\gamma-1}}\mathbb{P}\left(M\geqslant x\right)>0.$$

Proof. Define $w(x)=x^{\frac{\alpha}{\gamma-1}}u(x)$ and set $A=\liminf_{x\to+\infty}w(x)$ and $B=\limsup_{x\to\infty}w(x)$. It suffices to show that $0<A\leqslant B<\infty$.

Combining (48) and Proposition 2, we get that, for any $K>0$,

$$
\begin{aligned}
0 &= \lim_{x\to+\infty}\sup_{y\in[0,K]}\left|\frac{u\left(x+yu(x)^{-\frac{\gamma-1}{\alpha}}\right)}{\phi(y)u(x)}-1\right| \\
&= \lim_{x\to\infty}\sup_{y\in[0,K]}\left|\frac{w\left(x\left(1+yw(x)^{-\frac{\gamma-1}{\alpha}}\right)\right)}{\phi(y)w(x)\left(1+yw(x)^{-\frac{\gamma-1}{\alpha}}\right)^{\frac{\alpha}{\gamma-1}}}-1\right|. \qquad (54)
\end{aligned}
$$

First we show that $A<\infty$. If $A=\infty$, we define $b_k:=\sup\{x : w(x)<k\}$. Then $b_k\to\infty$ as $k\to\infty$. Using the the definition of b_k and the left-continuity of w, we get that $w(b_k)\leqslant k\leqslant \inf_{z>b_k}w(z)$. Therefore, taking $x=b_k$ in (54), we get that for any $y>0$,

$$
\begin{aligned}
1 &= \lim_{k\to\infty}\frac{w\left(b_k\left(1+yw(x)^{-\frac{\gamma-1}{\alpha}}\right)\right)}{\phi(y)w(b_k)\left(1+yw(b_k)^{-\frac{\gamma-1}{\alpha}}\right)^{\frac{\alpha}{\gamma-1}}} \geqslant \lim_{k\to\infty}\frac{k}{\phi(y)\left(w(b_k)^{\frac{\gamma-1}{\alpha}}+y\right)^{\frac{\alpha}{\gamma-1}}} \\
&\geqslant \lim_{k\to\infty}\frac{k}{\phi(y)\left(k^{\frac{\gamma-1}{\alpha}}+y\right)^{\frac{\alpha}{\gamma-1}}}=\frac{1}{\phi(y)},
\end{aligned}
$$

which is a contraction to Lemma 6. Hence $A<\infty$. Similarly we can show that $B>0$.

Now we show that $B<\infty$. Assume that $A<B=\infty$. Note that for any $K>0$,

$$
\lim_{A_1\to\infty}\phi(K)\left(1+KA_1^{-\frac{\gamma-1}{\alpha}}\right)^{\frac{\alpha}{\gamma-1}}=\phi(K)<1.
$$

Therefore, we may fix an $A_1>A$ and an $\varepsilon>0$ such that

$$
(1+\varepsilon)\phi(K)\left(1+KA_1^{-\frac{\gamma-1}{\alpha}}\right)^{\frac{\alpha}{\gamma-1}}<1. \qquad (55)
$$

Fix another $B_1>A_1$. Define

$$
\begin{aligned}
&a_1:=\inf\{x>0: w(x)<A_1\}, \quad d_1:=\inf\{x>a_1: w(x)>B_1+1\}, \\
&a_k:=\inf\{x>d_{k-1}: w(x)<A_1\}, \quad d_k:=\inf\{x>a_k: w(x)>B_1+k\}, \\
&a_k^*:=\sup\{x\in[a_k,d_k]: w(x)<A_1\}.
\end{aligned}
$$

By (54), for any $\varepsilon,K>0$ satisfying (55), there exists $N>0$ such that when $x>N$.

$$
\sup_{y\in[0,K]}\left|\frac{w\left(x\left(1+yw(x)^{-\frac{\gamma-1}{\alpha}}\right)\right)}{\phi(y)w(x)\left(1+yw(x)^{-\frac{\gamma-1}{\alpha}}\right)^{\frac{\alpha}{\gamma-1}}}-1\right|<\varepsilon. \qquad (56)
$$

By the left continuity of w, taking $x=a_k^*$ in (56), we see that when k is large enough, for all $y\in[0,K]$,

$$
\begin{aligned}
w\left(a_k^*\left(1+yw(a_k^*)^{-\frac{\gamma-1}{\alpha}}\right)\right) &\leqslant (1+\varepsilon)\phi(y)w(a_k^*)\left(1+yw(a_k^*)^{-\frac{\gamma-1}{\alpha}}\right)^{\frac{\alpha}{\gamma-1}} \\
&= (1+\varepsilon)\phi(y)\left(w(a_k^*)^{\frac{\gamma-1}{\alpha}}+y\right)^{\frac{\alpha}{\gamma-1}} \leqslant (1+\varepsilon)\phi(y)\left(A_1^{\frac{\gamma-1}{\alpha}}+K\right)^{\frac{\alpha}{\gamma-1}} \\
&< B_1+k. \qquad (57)
\end{aligned}
$$

Therefore, we see that when k is large enough,

$$
\left\{a_k^*\left(1+yw(a_k^*)^{-\frac{\gamma-1}{\alpha}}\right): y\in[0,K]\right\}\subset[a_k^*,d_k] \quad\Longrightarrow\quad w\left(a_k^*\left(1+Kw(a_k^*)^{-\frac{\gamma-1}{\alpha}}\right)\right)\geqslant A_1. \qquad (58)
$$

However, combining (55) and (57), we have

$$
w\left(a_k^*\left(1+Kw(a_k^*)^{-\frac{\gamma-1}{\alpha}}\right)\right)\leqslant(1+\varepsilon)\phi(K)\left(A_1^{\frac{\gamma-1}{\alpha}}+K\right)^{\frac{\alpha}{\gamma-1}}<A_1,
$$

which contradicts (58). Therefore, $B < \infty$.

Now we prove $A > 0$. If $A = 0 < B$, then combining (51) and (4), we see that for any $y > 0$,

$$\phi(y) \geqslant \mathbf{E}_0\left(\exp\left\{-cy^{\alpha}\widetilde{\tau}_{-1}\right\}\right) = e^{-c_1 y}$$

for some constant $c_1 > 0$, where c is the constant in (51). Note that for any $B_2 > 0$,

$$\begin{aligned}\phi(y)^{\frac{\gamma-1}{\alpha}}\left(1 + yB_2^{-\frac{\gamma-1}{\alpha}}\right) &\geqslant e^{-c_1\frac{\gamma-1}{\alpha}y}\left(1 + yB_2^{-\frac{\gamma-1}{\alpha}}\right) \geqslant \left(1 - c_1\frac{\gamma-1}{\alpha}y\right)\left(1 + yB_2^{-\frac{\gamma-1}{\alpha}}\right) \\ &= 1 + \left(B_2^{-\frac{\gamma-1}{\alpha}} - c_1\frac{\gamma-1}{\alpha}\right)y - c_1\frac{\gamma-1}{\alpha}B_2^{-\frac{\gamma-1}{\alpha}}y^2.\end{aligned}$$

Thus we can take B_2 and K sufficiently small so that for all $y \in (0, K]$,

$$\phi(y)^{\frac{\gamma-1}{\alpha}}\left(1 + yB_2^{-\frac{\gamma-1}{\alpha}}\right) > 1.$$

Let B_2 and K be chosen as above and fix $y \in (0, K]$. Then there exists $\varepsilon > 0$ such that

$$(1-\varepsilon)\phi(y)\left(1 + yB_2^{-\frac{\gamma-1}{\alpha}}\right)^{\frac{\alpha}{\gamma-1}} > 1. \tag{59}$$

Take N large enough so that $N^{-1} < B_2$ and define

$$\begin{aligned}&h_1 := \inf\left\{x > 0 : w(x) > B_2\right\}, \quad j_1 := \inf\left\{x > h_1 : w(x) < \frac{1}{N+1}\right\}, \\ &h_k := \inf\left\{x > j_{k-1} : w(x) > B_2\right\}, \quad j_k := \inf\left\{x > h_k : w(x) < \frac{1}{N+k}\right\}, \\ &h_k^* := \sup\left\{x \in [h_k, j_k] : w(x) > B_2\right\}.\end{aligned}$$

Combining (56) and the left continuity of w, we see that $w(h_k^*) \geqslant B_2$ and that, when k is large enough, for any $y \in [0, K]$,

$$\begin{aligned}w\left(h_k^*\left(1 + yw(h_k^*)^{-\frac{\gamma-1}{\alpha}}\right)\right) &\geqslant (1-\varepsilon)\phi(y)w(h_k^*)\left(1 + yw(h_k^*)^{-\frac{\gamma-1}{\alpha}}\right)^{\frac{\alpha}{\gamma-1}} \\ &= (1-\varepsilon)\phi(y)\left(w(h_k^*)^{\frac{\gamma-1}{\alpha}} + y\right)^{\frac{\alpha}{\gamma-1}} \geqslant (1-\varepsilon)\phi(K)B_2 > \frac{1}{k+N},\end{aligned} \tag{60}$$

which implies that

$$\left\{h_k^*\left(1 + yw(h_k^*)^{-\frac{\gamma-1}{\alpha}}\right) : y \in [0, K]\right\} \subset [h_k^*, j_k] \quad \Longrightarrow \quad w\left(h_k^*\left(1 + Kw(h_k^*)^{-\frac{\gamma-1}{\alpha}}\right)\right) \leqslant B_2. \tag{61}$$

However, combining (59) and (60), we get

$$w\left(h_k^*\left(1 + Kw(h_k^*)^{-\frac{\gamma-1}{\alpha}}\right)\right) \geqslant (1-\varepsilon)\phi(K)B_2\left(1 + KB_2^{-\frac{\gamma-1}{\alpha}}\right)^{\frac{\alpha}{\gamma-1}} > B_2,$$

which contradicts (61). Therefore, $A > 0$ and the proof is copmplete. □

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Define $U^{(x)}(y) := x^{\frac{\alpha}{\gamma-1}}u(xy)$, then it follows from Lemma 7 that for some constant γ_1, γ_2 and A, it holds that

$$\frac{\gamma_1}{y^{\frac{\alpha}{\gamma-1}}} \leqslant U^{(x)}(y) \leqslant \frac{\gamma_2}{y^{\frac{\alpha}{\gamma-1}}}, \quad xy \geqslant A. \tag{62}$$

It follows from (11) that for any $u_0 \in (0,1)$, $f(u)/u^{\gamma-1} \leqslant c$ for all $u \in [0, u_0]$ for some positive constant c. Fixing a $y_0 > 0$. Then by (46), when $x > A/y_0$, we see that for any $z > y > y_0$, under $\mathbf{P}_{xz}$, we have

$u(\widetilde{\xi}_s) \leqslant \frac{\gamma_2}{\widetilde{\xi}_s^{\frac{\alpha}{\gamma-1}}} \leqslant \frac{\gamma_2}{(xy_0)^{\frac{\alpha}{\gamma-1}}}$ on the set $\{s < \widetilde{\tau}_{xy}\}$. Therefore, by (4), for any $y_0 < y < z$,

$$\begin{aligned}
U^{(x)}(z) &= \mathbf{E}_{xz}\left(\exp\left\{-\int_0^{\widetilde{\tau}_{xy}} f\left(u\left(\widetilde{\xi}_s\right)\right)\mathrm{d}s\right\}\right)U^{(x)}(y)\\
&\geqslant \mathbf{E}_{xz}\left(\exp\left\{-c\int_0^{\widetilde{\tau}_{xy}}\left(u\left(\widetilde{\xi}_s\right)\right)^{\gamma-1}\mathrm{d}s\right\}\right)U^{(x)}(y)\\
&\geqslant \mathbf{E}_{xz}\left(\exp\left\{-\frac{c\gamma_2^{\gamma-1}}{(xy_0)^{\alpha}}\widetilde{\tau}_{xy}\right\}\right)U^{(x)}(y) = \mathbf{E}_1\left(\exp\left\{-\frac{c\gamma_2^{\gamma-1}}{(y_0)^{\alpha}}(z-y)^{\alpha}\widetilde{\tau}_0\right\}\right)U^{(x)}(y)\\
&= e^{-c_1(z-y)}U^{(x)}(y),
\end{aligned} \tag{63}$$

for some positive constant c_1. Therefore, combining (62) and (63), we see that when $x > A/y_0$, for any $y_0 < y < z$,

$$\left|U^{(x)}(z) - U^{(x)}(y)\right| = U^{(x)}(y) - U^{(x)}(z) \leqslant U^{(x)}(y)\left(1 - e^{-c_1(z-y)}\right) \leqslant \frac{\gamma_2}{y_0^{\frac{\alpha}{\gamma-1}}}c_1|z-y|. \tag{64}$$

Therefore, combining (62) and a diagonalization argument, we can find sequence from $\{t_k : k \geqslant 1\} \subset (0,\infty)$ with $t_k \to \infty$ such that

$$U(y) = \lim_{k\to\infty} U^{(t_k)}(y), \quad \text{for all} \quad y \in \mathbb{Q} \cap (0,\infty). \tag{65}$$

Moreover, using a standard argument (for example, see [5, Lemma 3.1]) and with the help of (64), one can show that (65) holds for all $y > 0$. taking $x = t_k$ in (62) and letting $k \to \infty$, we see that $U(y)$ is comparable to $y^{-\frac{\alpha}{\gamma-1}}$, which implies that $\lim_{y\downarrow 0} U(y) = \infty$ and that $\lim_{y\to\infty} U(y) = 0$.

Let $z > y \geqslant y_0$. Since $u(\widetilde{\xi}) \leqslant u(xy)$ for all $s \leqslant \widetilde{\tau}_{xy}$ under $\mathbf{P}_{xz}$, we get from (11) that for any $\varepsilon, y_0 > 0$, when x is large enough, we have,

$$\begin{aligned}
U^{(x)}(z) &\geqslant \mathbf{E}_{xz}\left(\exp\left\{-C_2(\gamma)(1+\varepsilon)\int_0^{\widetilde{\tau}_{xy}}\left(u\left(\widetilde{\xi}_s\right)\right)^{\gamma-1}\mathrm{d}s\right\}\right)U^{(x)}(y)\\
&= \mathbf{E}_{z}\left(\exp\left\{-C_2(\gamma)(1+\varepsilon)\int_0^{x^{\alpha}\widetilde{\tau}_{y}}\left(u\left(x\widetilde{\xi}_{s/x^{\alpha}}\right)\right)^{\gamma-1}\mathrm{d}s\right\}\right)U^{(x)}(y)\\
&= \mathbf{E}_{z}\left(\exp\left\{-C_2(\gamma)(1+\varepsilon)\int_0^{\widetilde{\tau}_{y}}x^{\alpha}\left(u\left(x\widetilde{\xi}_s\right)\right)^{\gamma-1}\mathrm{d}s\right\}\right)U^{(x)}(y)\\
&= \mathbf{E}_{z}\left(\exp\left\{-C_2(\gamma)(1+\varepsilon)\int_0^{\widetilde{\tau}_{y}}\left(U^{(x)}\left(\widetilde{\xi}_s\right)\right)^{\gamma-1}\mathrm{d}s\right\}\right)U^{(x)}(y).
\end{aligned}$$

Taking $x = t_k$ in the above inequality and then letting $k \to \infty$, we get that

$$\begin{aligned}
U(z) &\geqslant \mathbf{E}_{z}\left(\exp\left\{-C_2(\gamma)(1+\varepsilon)\int_0^{\widetilde{\tau}_{y}}\left(U\left(\widetilde{\xi}_s\right)\right)^{\gamma-1}\mathrm{d}s\right\}\right)U(y)\\
&\xrightarrow{\varepsilon\to 0} \mathbf{E}_{z}\left(\exp\left\{-C_2(\gamma)\int_0^{\widetilde{\tau}_{y}}\left(U\left(\widetilde{\xi}_s\right)\right)^{\gamma-1}\mathrm{d}s\right\}\right)U(y).
\end{aligned}$$

Similarly, we also have for any $y_0 \leqslant y < z$,

$$U(z) \leqslant \mathbf{E}_{z}\left(\exp\left\{-C_2(\gamma)\int_0^{\widetilde{\tau}_{y}}\left(U\left(\widetilde{\xi}_s\right)\right)^{\gamma-1}\mathrm{d}s\right\}\right)U(y).$$

Therefore, U is the solution of the equation in Proposition 2(ii). Now the constant $C_3(\alpha,\beta,\gamma) = U(1)$ follows immediately, we are done. □

5 Proof of Theorem 3

Repeating the proof of [4, Lemma 2.3], we can get that, in the case $m < 1$, u has the following representation:

$$u(x) = \mathbf{E}_x \left(\exp\left\{ -(1-m)\widetilde{\tau}_y - \int_0^{\widetilde{\tau}_y} f_{sub}\left(u\left(\widetilde{\xi}_s\right)\right) \mathrm{d}s \right\} \right) u(y), \quad x > y \geqslant 0 \tag{66}$$

where

$$f_{sub}(u) = \frac{\sum_{k=0}^{\infty} p_k (1-u)^k - (1-m)(1-u)}{u}$$

is a continuous function of $u \in (0,1]$ with $\lim_{u \to 0+} f_{sub}(u) = 0$. Moreover, by [6, Lemma 2.7], f_{sub} is increasing in u, and that if $\sum_{k=0}^{\infty} k(\log k) p_k < \infty$, then for any $c > 0$, $\int_0^\infty f_{sub}\left(e^{-ct}\right) \mathrm{d}t < \infty$, which is equivalent to

$$\sum_{n=1}^{\infty} f_{sub}\left(e^{-cn}\right) < \infty. \tag{67}$$

Proof of Theorem 3. For simplicity, define $a_0 := \left((1-m)/C_1(\alpha)\right)^{1/\alpha}$. According to (4), we have for any $x > y \geqslant 0$,

$$u(x) \leqslant \mathbf{E}_x \left(\exp\left\{ -(1-m)\widetilde{\tau}_y \right\} \right) u(y) = e^{-a_0(x-y)} u(y).$$

Therefore, we see that $e^{a_0 x} u(x)$ is decreasing in x and that $\lim_{x\to\infty} e^{a_0 x} u(x) \in [0,1]$. Thus, it remains to show that the limit is positive.

Taking $x = n+1, y = n$ in (66), we see that

$$u(n+1) = \mathbf{E}_{n+1} \left(\exp\left\{ -(1-m)\widetilde{\tau}_n - \int_0^{\widetilde{\tau}_n} f_{sub}\left(u\left(\widetilde{\xi}_s\right)\right) \mathrm{d}s \right\} \right) u(n).$$

Since under $\mathbf{P}_{n+1}$, on the set $s < \widetilde{\tau}_n$, we have $\widetilde{\xi}_s \geqslant n$. Therefore, by the monotonicities of f_{sub} and u, we conclude from the above identity that

$$\begin{aligned} u(n+1) &\geqslant \mathbf{E}_{n+1} \left(\exp\left\{ -(1-m)\widetilde{\tau}_n - \int_0^{\widetilde{\tau}_n} f_{sub}\left(u\left(n\right)\right) \mathrm{d}s \right\} \right) u(n) \\ &\geqslant \mathbf{E}_{n+1} \left(\exp\left\{ -\left(1-m+f_{sub}\left(e^{-a_0 n}\right)\right) \widetilde{\tau}_n \right\} \right) u(n) \\ &= \exp\left\{ -H\left(1-m+f_{sub}\left(e^{-a_0 n}\right)\right) \right\} u(n), \end{aligned} \tag{68}$$

where $H(a) := (a/C_1(\alpha))^{1/\alpha}$. Noticing that for $\alpha \in (1,2)$, by Taylor's expansion, we have

$$H(1-m+v) = a_0 + H'(1-m)v + O(v^2), \quad v \to 0.$$

Therefore, there exists $C > 0$ such that for all $v \in (0,1)$, $H(1-m+v) \leqslant a_0 + Cv$. Combining this inequality with (67) and (68), we conclude that

$$\begin{aligned} e^{a_0(n+1)} u(n+1) &\geqslant e^{a_0 n} u(n) \exp\left\{ -C f_{sub}\left(e^{-a_0 n}\right) \right\} \\ &\geqslant \cdots \geqslant u(0) \exp\left\{ -C \sum_{k=0}^{n} f_{sub}\left(e^{-a_0 k}\right) \right\} \\ &\geqslant \exp\left\{ -C \sum_{k=0}^{\infty} f_{sub}\left(e^{-a_0 n}\right) \right\} > 0, \end{aligned}$$

which implies the desired result. $\square$

6 Proof of Proposition 2

The proof of Proposition 2 relies heavily on another important Markov process: super α-stable process. We will briefly introduce this process and some known results.

Let $\mathcal{M}_F(\mathbb{R})$ be the families of finite Borel measures on $\mathbb{R}$. We will use $\mathbf{0}$ to denote the null measure on $\mathbb{R}$. Let $B_b(\mathbb{R})$ and $B_b^+(\mathbb{R})$ be the spaces of bounded Borel functions and non-negative bounded Borel functions on $\mathbb{R}$ respectively. For any $f \in B_b(\mathbb{R})$ and $\mu \in \mathcal{M}_F(\mathbb{R})$, we use $\langle f, \mu \rangle$ to denote the integral of f with respect to μ. For any $\alpha \in (1, 2]$, the function

$$\varphi(\lambda) := C_2(\gamma)\lambda^\gamma$$

is a branching mechanism.

For any $\mu \in \mathcal{M}_F(\mathbb{R})$, we use $X = \{(X_t)_{t\geqslant 0}; \mathbb{P}_\mu\}$ to denote a super α-stable process with spatial motion $\widetilde{\xi}$ and branching mechanism φ, that is, an $\mathcal{M}_F(\mathbb{R})$-valued Markov process such that for any $f \in B_b^+(\mathbb{R})$,

$$-\log \mathbb{E}_\mu \left(\exp\{-\langle f, X_t\rangle\}\right) = \langle v_f(t, \cdot), \mu\rangle,$$

where $(t, x) \mapsto v_f(t, x)$ is the unique locally bounded non-negative solution to

$$v_f(t,x) = \mathbf{E}_x\left(f(\widetilde{\xi}_t)\right) - \mathbf{E}_y\Big(\int_0^t \varphi\left(v_f(t-s, \widetilde{\xi}_s)\right) \mathrm{d}s\Big).$$

According to Dynkin [3], for any open set Q of $\mathbb{R}$, there corresponds a random measure X_Q such that, $\mu \in \mathcal{M}_F(\mathbb{R})$ with supp $\mu \subset Q$, and any $f \in B_b^+(\mathbb{R})$,

$$\mathbb{E}_\mu \left(\exp\{-\langle f, X_Q\rangle\}\right) = \exp\big\{-\langle v_f^Q, \mu\rangle\big\},$$

where $v_f^Q(x)$ is the unique positive solution of the equation

$$v_f^Q(x) = \mathbf{E}_x\left(f(\widetilde{\xi}_\tau)\right) - \mathbf{E}_x \int_0^{\tau_Q} \varphi\big(v_f^Q(\widetilde{\xi}_r)\big)\mathrm{d}r, \tag{69}$$

with $\tau_Q := \inf\Big\{r : \widetilde{\xi}_r \notin Q\Big\}$.

Proof of Proposition 2. Taking $Q = (0, \infty)$ and $f = 1$ in (69), we see that $\phi(x)$ is the unique bounded solution of the following equation:

$$\phi(x) = 1 - \mathbf{E}_x \int_0^{\widetilde{\tau}_0} \varphi\big(\phi(\widetilde{\xi}_r)\big)\mathrm{d}r.$$

Since the above equation is equivalent to (49), we complete the proof of (i).

Now we turn to the proof of (ii). Similarly, let U be an solution to (53) with boundary condition $U(0+) = \infty$ and $U(\infty) = 0$. Noticing that for each fixed y, (53) is equivalent to

$$U(z) = U(y) - \mathbf{E}_z \int_0^{\widetilde{\tau}_y} \varphi\big(U(\widetilde{\xi}_r)\big)\mathrm{d}r, \quad z > y > 0.$$

Therefore, since $\widetilde{\xi}$ is spectrally positive, we see that for $Q = (y, \infty)$, is supported on $\{y\}$. Therefore, from (69), we conclude that

$$U(z) = -\log \mathbb{E}_{\delta_z}\left(\exp\left\{-U(y)X_{(y,\infty)}(\{y\})\right\}\right) = -\log \mathbb{E}_{\delta_{z-y}}\left(\exp\left\{-U(y)X_{(0,\infty)}(\{0\})\right\}\right),$$

where in the last equality we used the spatial homogeneous property of super α-stable process. Therefore, replacing z by $z + y$ first and then letting $y \to 0+$, we conclude that

$$U(z) = \lim_{y\to 0} U(z+y) = -\lim_{y\to 0} \log \mathbb{E}_{\delta_z}\left(\exp\left\{-U(y)X_{(0,\infty)}(\{0\})\right\}\right) = -\log \mathbb{P}_{\delta_z}\left(X_{(0,\infty)}(\{0\}) = 0\right),$$

which implies the desired result. ∎

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Haojie Hou: School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China. Email: houhaojie@bit.edu.cn

Yiyang Jiang: School of Mathematical Sciences, Peking University, Beijing, 100871, P.R. China. Email: jyy.0916@stu.pku.edu.cn

Yan-Xia Ren: LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing, 100871, P.R. China. Email: yxren@math.pku.edu.cn

Renming Song: Department of Mathematics, University of Illinois Urbana-Champaign, Urbana, IL 61801, U.S.A. Email: rsong@illinois.edu