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Convergence rate for a class of supercritical superprocesses[☆]

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Abstract

Suppose $X = \{X_t, t \geq 0\}$ is a supercritical superprocess. Let ϕ be the non-negative eigenfunction of the mean semigroup of X corresponding to the principal eigenvalue $\lambda > 0$. Then $M_t(\phi) = e^{-\lambda t}\langle \phi, X_t\rangle, t \geq 0$, is a non-negative martingale with almost sure limit $M_\infty(\phi)$. In this paper we study the rate at which $M_t(\phi) - M_\infty(\phi)$ converges to 0 as $t \to \infty$ when the process may not have finite variance. Under some conditions on the mean semigroup, we provide sufficient and necessary conditions for the rate in the almost sure sense. Some results on the convergence rate in L^p with $p \in (1, 2)$ are also obtained.

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1. Introduction and main results

Let $\{Z_n, n \geq 0\}$ be a Galton–Watson process with $Z_0 = 1$ and offspring mean $m := EZ_1 > 1$, and let $W_n := m^{-n} Z_n$. Then $\{W_n, n \geq 0\}$ is a non-negative martingale with almost sure limit W_∞. It is well-known that W_n converges to W_∞ in L^1 if and only if $E(Z_1 \log^+ Z_1) < \infty$. In the case $E(Z_1 \log^+ Z_1) < \infty$, it is natural to consider the rate at which $W_\infty - W_n$ converges to 0. In this paper we are mainly concerned with the convergence rate in the almost sure sense, when the process may not have finite variance. This type of results first appeared in Asmussen [2], and then in the book of Asmussen and Hering [3]. The following result is from [3, Theorem II.4.1, p. 36]:

Theorem A. (i) *Let* $p \in (1, 2)$ *and* $1/p + 1/q = 1$. *Then*

$$W_\infty - W_n = o(m^{-n/q}) \quad \textit{a.s. as } n \to \infty \tag{1.1}$$

if and only if $E(Z_1^p) < \infty$.

(ii) *Let* $\alpha > 0$. *Then*

$$\sum_{n=1}^{\infty} n^{\alpha-1}(W_\infty - W_n) \quad \textit{converges a.s.}$$

if and only if $E(Z_1(\log^+ Z_1)^{1+\alpha}) < \infty$.

(iii) *Let* $\alpha > 0$. *Then* $W_\infty - W_n = o(n^{-\alpha})$ *a.s. as* $n \to \infty$ *if and only if*

$$E\Big[Z_1\Big(\log Z_1 - \log n\Big)1_{\{Z_1 > n\}}\Big] = o\left((\log n)^{-\alpha}\right), \quad \textit{as } n \to \infty.$$

Asmussen [2] also discussed corresponding results for finite type Galton–Watson processes, and continuous time Galton–Watson processes. For multigroup branching diffusions on bounded domains, convergence rate corresponding to Theorem A(i) is considered in [3, VIII.13]. A sufficient condition, corresponding to $E(Z_1^p) < \infty$, is given for (1.1) to hold, see [3, Theorem VIII.13.2, p.343]. The goal of this paper is to prove the counterparts of the results in Theorem A for a class of superprocesses.

Before we give our model and results, we first review some related work in the literature. For any $p > 1$, the L^p convergence rate of $W_n - W_\infty$ to 0 is obtained in Liu [32, Proposition 1.3]. Huang and Liu [17] obtained L^p convergence rates for similar martingales in quenched and annealed senses for branching processes in random environment. In [1,18,19], a class of non-negative intrinsic martingales W_n for supercritical branching random walks were investigated. Let W_∞ be the almost sure limit of W_n as $n \to \infty$. Necessary and sufficient conditions for the L^p-convergence, $p > 1$, of the series

$$\sum_{n=1}^{\infty} e^{an}(W_\infty - W_n), \quad a > 0, \tag{1.2}$$

were obtained in [1], which may be viewed as the exponential rate of convergence of $E|W_\infty - W_n|^p$ to 0 as $n \to \infty$. In [18], sufficient conditions for the almost sure convergence of the series

$$\sum_{n=0}^{\infty} f(n)(W_\infty - W_n) \tag{1.3}$$

were obtained, where f is a function regularly varying at ∞ with index larger than -1. [19] investigated sufficient conditions for (1.2) to converge in the almost sure sense. For general supercritical indecomposable multi-type branching processes, sufficient conditions for polynomial rate of convergence in the sense of convergence in probability were given in [20].

We now introduce the setup of this paper. We always assume that E is a locally compact separable metric space. We will use $E_\partial := E \cup \{\partial\}$ to denote the one-point compactification of E. We will use $\mathcal{B}(E)$ and $\mathcal{B}(E_\partial)$ to denote the Borel σ-fields on E and E_∂ respectively. $\mathcal{B}_b(E)$ (respectively $\mathcal{B}^+(E)$, respectively $\mathcal{B}_b^+(E)$) will denote the set of all bounded (respectively non-negative, respectively bounded and non-negative) real-valued Borel functions on E. All functions f on E will be automatically extended to E_∂ by setting $f(\partial) = 0$.

We will always assume that $\xi = \{(\xi_t)_{t\geq 0}; \Pi_x, x \in E\}$ is a Hunt process on E and $\zeta = \inf\{t > 0 : \xi_t = \partial\}$ is the lifetime of ξ. We use $(P_t)_{t\geq 0}$ to denote the semigroup of ξ acting on functions defined on E and $(\overline{P}_t)_{t\geq 0}$ to denote the semigroup of ξ acting on functions defined on E_∂. We mention in passing that it is important that we take E_∂ to be the one-point compactification of E. For example, if E is a bounded smooth domain of $\mathbb{R}^d$, ξ is the killed Brownian motion in E and ∂ was added as an isolated point, then ξ will not be a Hunt process. Let *the branching mechanism* ψ be a function on $E \times \mathbb{R}_+$ given by

$$\psi(x, z) = -\beta(x)z + \frac{1}{2}\alpha(x)z^2 + \int_{(0,\infty)} (e^{-zr} - 1 + zr)\pi(x, dr), \quad x \in E, z \geq 0, \tag{1.4}$$

where $\alpha \geq 0$ and β are both in $\mathcal{B}_b(E)$, and π is a kernel from $(E, \mathcal{B}(E))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ satisfying

$$\int_0^\infty (r \wedge r^2)\pi(\cdot, dr) \in \mathcal{B}_b^+(E). \tag{1.5}$$

Note that this assumption implies that, for any fixed $z > 0$, $\psi(\cdot, z)$ is bounded on E. We extend ψ to a branching mechanism $\overline{\psi}$ on E_∂ by defining $\overline{\psi}(\partial, z) = 0$ for all $z \geq 0$.

Let $\mathcal{M}(E)$ (resp. $\mathcal{M}(E_\partial)$) denote the space of all finite Borel measures on E (resp. E_∂) equipped with the topology of weak convergence. Any $\mu \in \mathcal{M}(E)$ will be identified with its zero extension in $\mu \in \mathcal{M}(E_\partial)$. Denote by $\mathbf{0}$ the null measure on E and E_∂. Write $\mathcal{M}^0(E) = \mathcal{M}(E) \setminus \{\mathbf{0}\}$ and $\mathcal{M}^0(E_\partial) = \mathcal{M}(E_\partial) \setminus \{\mathbf{0}\}$. For any $\mu \in \mathcal{M}(E_\partial)$ and $f \in \mathcal{B}(E_\partial)$, we use $\langle f, \mu\rangle$ or $\mu(f)$ to denote the integral of f with respect to μ whenever the integral is well-defined. For $f \in \mathcal{B}_b^+(E_\partial)$, there is a unique locally bounded non-negative map $(t, x) \mapsto \overline{V}_t f(x)$ on $\mathbb{R}_+ \times E_\partial$ such that

$$\overline{V}_t f(x) + \Pi_x\Big[\int_0^t \overline{\psi}\big(\xi_s, \overline{V}_{t-s} f(\xi_s)\big)ds\Big] = \Pi_x[f(\xi_t)], \quad t \geq 0, x \in E_\partial. \tag{1.6}$$

Here, local boundedness of the map $(t, x) \mapsto \overline{V}_t f(x)$ means that for any $T > 0$,

$$\sup_{0\leq t\leq T, x\in E} \overline{V}_t f(x) < \infty.$$

Similarly, for $f \in \mathcal{B}_b^+(E)$, there is a unique locally bounded non-negative map $(t, x) \mapsto V_t f(x)$ on $\mathbb{R}_+ \times E$ such that

$$V_t f(x) + \Pi_x\Big[\int_0^{t\wedge\zeta} \psi\big(\xi_s, V_{t-s} f(\xi_s)\big)ds\Big] = \Pi_x[f(\xi_t)1_{\{t<\zeta\}}], \quad t \geq 0, x \in E. \tag{1.7}$$

Then we can define a Markov transition semigroup $(Q_t)_{t\geq 0}$ on $\mathcal{M}(E_\partial)$ by

$$\int_{\mathcal{M}(E_\partial)} e^{-\nu(f)} Q_t(\mu, d\nu) = e^{-\mu(\overline{V}_t f)}, \quad \mu \in \mathcal{M}(E_\partial), f \in \mathcal{B}_b^+(E_\partial).$$

If $\overline{X} = \{(\overline{X}_t)_{t\geq 0}; \mathbb{P}_\mu, \mu \in \mathcal{M}(E_\partial)\}$ is a Markov process in $\mathcal{M}(E_\partial)$ with transition semigroup $(\underline{Q}_t)_{t\geq 0}$, we call it a Dawson–Watanabe superprocess with parameters $(\xi, \overline{\psi})$. Since ξ is Hunt, $\overline{X}$ has a Hunt realization. We will always assume that $\overline{X}$ is a Hunt process. See [31, Section 2.3 and Theorem 5.11] for more details. For other sources on the constructions of superprocesses and more basic facts on superprocesses we refer the reader to [8,10,14,15,31,36]. Let $\iota(\mu)$ be the restriction of a measure $\mu \in \mathcal{M}(E_\partial)$ to E and $X_t = \iota(\overline{X}_t)$. It follows from the proof of [31, Theorem 5.12] that $X = \{(X_t)_{t\geq 0}; \mathbb{P}_\mu, \mu \in \mathcal{M}(E)\}$ is an $\mathcal{M}(E)$-valued Markov process such that

$$\mathbb{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)}, \quad t \geq 0, f \in \mathcal{B}_b^+(E).$$

However, since we have taken E_∂ to be the one-point-compactification of E, X is in general not a Hunt process and does not have good regularity properties. Since $X_t(f) = \overline{X}_t(f)$ for any function f on E and we are only interested in quantities of the form $X_t(f)$, we can work with the Hunt process $\overline{X}$ when necessary.

The process X may be loosely described as a scaling limit of branching particle systems as follows. Let $\mu \in \mathcal{M}^0(E)$. Suppose that, at time zero, a random number of particles are set in E, according to a Poisson random measure with intensity $N\mu$. The particles move independently according to the law of ξ in E. A given particle lives an exponential amount of time with mean lifetime b_N, and upon its death gives birth to a random number of offspring. The offspring wander and propagate in the same fashion. Offspring are born at the death site of their parent, and the distribution $(p_k^N(x); k \geq 0)$ of the number of offspring is allowed to depend on the death site x, and on the parameter N. The mass distribution of particles alive at time t may be viewed as a random measure $X_t^{(N)}$ (each particle being given weight $1/N$). Under suitable hypotheses this sequence of measure-valued process converges in distribution, as $N \to \infty$, to a limit measure-valued Markov process X with $X_0 = \mu$. The typical conditions are $b_N \to 0$ and

$$\lim_{N\to\infty} \left[\sum_{k=0}^{\infty} p_k^{(N)}(x)(1 - z/N)^k - (1 - z/N)\right](N/b_N) = \psi(x, z), \quad z \geq 0.$$

For details on superprocesses as limits of branching particle systems we refer to Dynkin [10] and Li [31].

When the initial value is $\delta_x, x \in E$, we write $\mathbb{P}_x$ for $\mathbb{P}_{\delta_x}$. We use $(P_t^\beta)_{t\geq 0}$ to denote the following Feynman–Kac semigroup

$$P_t^\beta f(x) = \Pi_x\left(\exp\left(\int_0^t \beta(\xi_s)ds\right) f(\xi_t)1_{\{t<\zeta\}}\right), \quad x \in E, f \in \mathcal{B}_b^+(E).$$

Then it is known (see [31, Proposition 2.27]) that for any $\mu \in \mathcal{M}(E)$,

$$\mathbb{P}_\mu[X_t(f)] = \mu(P_t^\beta f), \quad t \geq 0, f \in \mathcal{B}_b^+(E). \tag{1.8}$$

$(P_t^\beta)_{t\geq 0}$ is called the mean semigroup of X. For this mean semigroup, we will always assume that

Assumption 1. There exist a constant $\lambda > 0$, a positive function $\phi \in \mathcal{B}_b(E)$ and a probability measure ν with full support on E such that for any $t \geq 0$, $P_t^\beta \phi = e^{\lambda t}\phi$, $\nu P_t^\beta = e^{\lambda t}\nu$ and $\nu(\phi) = 1$.

Denote by $L_1^+(\nu)$ the collection of non-negative Borel functions on E which are integrable with respect to the measure ν. We further assume that the following assumption holds:

Assumption 2. For all $t > 0$, $x \in E$, and $f \in L_1^+(\nu)$, it holds that $P_t^\beta f(x) = e^{\lambda t}\phi(x)\nu(f)(1 + C_{t,x,f})$ for some $C_{t,x,f} \in \mathbb{R}$, and that $\lim_{t\to\infty} c_t = 0$, where $c_t := \sup_{x\in E, f\in L_1^+(\nu)} |C_{t,x,f}|$.

Note that $\lim_{t\to\infty} c_t = 0$ implies that there exists $t_0 > 0$ such that

$$\sup_{t>t_0} \sup_{x\in E, f\in L_1^+(\nu)} |C_{t,x,f}| < \infty.$$

Without loss of generality, throughout this paper we will assume $t_0 = 1$.

Here are two classes of examples satisfying Assumptions 1 and 2. For more examples, see [35, Section 1.3] and [37, Section 1.4].

Example 1.1. Suppose that E is the closure of a bounded connected C^2 open set in $\mathbb{R}^d$ and that m denotes the Lebesgue measure on E. Let ξ be the reflecting Brownian motion in E. Then ξ has a transition density $p(t, x, y)$, with respect to the Lebesgue measure, which is a strictly positive, continuous and symmetric function of (x, y) for any $t > 0$ and that there exists $c > 0$ such that

$$p(t, x, y) \le c\, t^{-d/2}, \qquad (t, x, y) \in (0, \infty) \times E \times E.$$

The largest eigenvalue of the generator of the semigroup $\{P_t : t \ge 0\}$ of ξ is $\widetilde{\lambda}_0 = 0$ and the corresponding eigenfunction $\widetilde{\phi}_0$ is a positive constant. Using this and the argument on [37, pp. 241–243], one can easily see that Assumptions 1 and 2 are satisfied.

Example 1.2. Suppose that E is an open subset of $\mathbb{R}^d$ with finite Lebesgue measure and that m denotes the Lebesgue measure on E. Let ξ be the subprocesses in E of any of the subordinate Brownian motions studied in [23,24]. Brownian motion and isotropic α-stable processes, $\alpha \in (0, 2)$, are special cases of subordinate Brownian motions. Then it is known (see [4,5]) that ξ has a transition density $p(t, x, y)$, with respect to the Lebesgue measure, which is a strictly positive, continuous, bounded, symmetric function of (x, y) for any $t > 0$. It follows from [22] that the semigroup $\{P_t : t \ge 0\}$ of ξ is intrinsic ultracontractive and that the eigenfunction $\widetilde{\phi}_0$ corresponding to the largest eigenvalue of the generator of $\{P_t : t \ge 0\}$ is bounded. Using this and the argument on [37, pp. 241–243], one can easily see that Assumptions 1 and 2 are satisfied.

Define

$$M_t(\phi) := e^{-\lambda t}\langle \phi, X_t\rangle, \quad t \ge 0. \tag{1.9}$$

It follows from (1.8) and Assumption 1 that $\{M_t(\phi), t \ge 0\}$ is a non-negative càdlàg martingale, see (2.11). By the martingale convergence theorem, $M_t(\phi)$ has an almost sure limit as $t \to \infty$. We denote this limit as $M_\infty(\phi)$. In this paper, we study the rate at which $M_t(\phi)$ converges to $M_\infty(\phi)$ as $t \to \infty$.

To state our results we need to introduce some notation. Define a new kernel $\pi^\phi(x, dr)$ from $(E, \mathcal{B}(E))$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that for any non-negative Borel function f on $\mathbb{R}_+$,

$$\int_0^\infty f(r)\pi^\phi(x, dr) = \int_0^\infty f(r\phi(x))\pi(x, dr), \quad x \in E. \tag{1.10}$$

By (1.5) and the boundedness of ϕ, π^ϕ satisfies

$$\int_0^\infty (r \wedge r^2)\pi^\phi(x, dr) = \phi(x)\Big[\int_0^{1/\phi(x)} r^2\phi(x)\pi(x, dr) + \int_{1/\phi(x)}^\infty r\pi(x, dr)\Big]$$

$$\leq \phi(x)\left(\|\phi\int_0^1 r^2\pi(\cdot,dr)\|_\infty + 2\|\int_{1\wedge 1/\|\phi\|_\infty}^\infty r\pi(x,dr)\|_\infty\right).$$

Let $C := \|\phi\int_0^1 r^2\pi(\cdot,dr)\|_\infty + 2\|\int_{1\wedge 1/\|\phi\|_\infty}^\infty r\pi(x,dr)\|_\infty$. Then

$$\int_0^\infty (r\wedge r^2)\pi^\phi(x,dr) \leq C\phi(x). \tag{1.11}$$

In [33], we studied the relationship between $M_\infty(\phi)$ being a non-degenerate random variable and the following function l:

$$l(y) := \int_1^\infty r\ln r\pi^\phi(y,dr), \quad y\in E, \tag{1.12}$$

and established an $L\log L$ criterion (see Proposition 1.3) for a class of superdiffusions with $\alpha = 0$.

Proposition 1.3 (*[33, Theorem 1.1]*). *Suppose that Assumptions 1–2 hold and $\mu\in\mathcal{M}^0(E)$. Then $\mathbb{P}_\mu(M_\infty(\phi)) = \langle\phi,\mu\rangle$ if and only if the following $L\log L$ condition holds:*

$$\int_E l(y)\nu(dy) < \infty. \tag{1.13}$$

Moreover, if (1.13) holds, then for any $\mu\in\mathcal{M}^0(E)$,

$$\{M_\infty(\phi) > 0\} = \{X_t > 0,\ \forall t > 0\}, \quad \mathbb{P}_\mu\text{-a.s.}$$

Otherwise, $M_\infty(\phi) = 0$, $\mathbb{P}_\mu$-a.s. for any $\mu\in\mathcal{M}^0(E)$.

If $\alpha\neq 0$, from the stochastic integral representation (2.11) of the martingale $M_t(\phi)$, one can see that the continuous part (the part corresponding to the term $\frac{1}{2}\alpha(x)z^2$ of the branching mechanism ψ) is a martingale bounded in $L^2(\mathbb{P}_\mu)$, so it converges almost surely under our setting. And thus the criterion (1.13) still holds for the superprocesses with general branching mechanism ψ.

The non-degeneracy of the limit $M_\infty(\phi)$ is closely related to the convergence of $e^{-\lambda t}X_t$ as a process in $\mathcal{M}(E)$, which can be regarded as a law of large numbers. The earliest work along this line dates back to 1970's when Watanabe [42,43] studied the asymptotic properties of branching symmetric diffusions. Since then, many people have worked on this topic, see, for instance, Chen and Shiozawa [7] for branching Hunt processes, Engländer et al. [12] for branching diffusions, Engländer and Kyprianou [13] for branching diffusions and superdiffusions, and Eckhoff, Kyprianou and Winkel [11] for superprocesses. In the paper mentioned above, the branching Markov processes or superprocesses, under some conditions, increase exponentially. Recently, some papers, for example, Kouritzin and Ren [27] and Wang [41], investigated the strong law of large numbers of super-stable processes and super-Brownian motion with branching mechanism $\psi(x,z) = -\beta z + \frac{1}{2}\alpha z^2$ with $\beta,\alpha > 0$ being constants. It turns out that in this case $X_t(f)$ grows with exponential rate multiplied by a polynomial of time t, where f is some test function. Kouritzin and Lê [25] studied the long-time behavior of α-stable ($\alpha\in(0,2]$) Fleming–Viot process, and got asymptotic expansions for $t^{\frac{d}{\alpha}}X_t(f)$ with f being some test function. Similar expansion for super-Brownian motion was given in Lê [29]. The law of large numbers for branching Gaussian processes, including fractional Brownian motion and fractional Ornstein–Uhlenbeck processes with large Hurst parameters, was obtained in Kouritzin et al. [26].

In the Appendix, we will state an almost sure convergence of the superprocesses which is used in our proof. We will give a proof since the result is not available in the literature.

Throughout this paper, we assume that (1.13) holds. Thus $M_t(\phi)$ converges to $M_\infty(\phi)$ $\mathbb{P}_\mu$-almost surely and in $L^1(\mathbb{P}_\mu)$ for any $\mu \in \mathcal{M}^0(E)$.

Since $M_s(\phi)$ is right continuous, for any $a^* > 0$, we can define

$$A_t(a^*) = \int_0^t e^{\frac{\lambda s}{a^*}} \big(M_\infty(\phi) - M_s(\phi)\big) ds, \qquad t \in [0, \infty). \tag{1.14}$$

Note that

$$A_t(a^*) - A_1(a^*) = \int_1^t e^{\frac{\lambda s}{a^*}} \big(M_\infty(\phi) - M_s(\phi)\big) ds, \quad t \geq 1.$$

The convergence of $A_t(a^*) - A_1(a^*)$ as $t \to \infty$ is related to the rate at which $M_\infty(\phi) - M_t(\phi)$ converges to 0 as $t \to \infty$. Our first result is the following criterion for the L^p convergence rate of $M_\infty(\phi) - M_t(\phi)$ to 0 as $t \to \infty$. We use the usual notation $\|\cdot\|_p$ to denote the L^p norm with $p \geq 1$.

Theorem 1.4. *Assume that Assumptions 1–2 and (1.13) hold. Let $1 < a < p \leq 2$ and $\frac{1}{a} + \frac{1}{a^*} = 1$.*

(1) *If*

$$\int_E \nu(dy) \int_1^\infty r^p \pi^\phi(y, dr) < \infty, \tag{1.15}$$

then for any $\mu \in \mathcal{M}^0(E)$, $(A_t(a^*) - A_1(a^*))$ converges in $L^p(\mathbb{P}_\mu)$ and $\mathbb{P}_\mu$-almost surely as $t \to \infty$.

(2) *If for some $\mu \in \mathcal{M}^0(E)$, $(A_t(a^*) - A_1(a^*))$ converges in $L^p(\mathbb{P}_\mu)$ as $t \to \infty$, then it must converge $\mathbb{P}_\mu$-almost surely and (1.15) holds.*

(3) *If (1.15) holds, then $\|M_\infty(\phi) - M_t(\phi)\|_p = o(e^{-\frac{\lambda t}{a^*}})$ as $t \to \infty$.*

(4) *If $\|M_\infty(\phi) - M_t(\phi)\|_p = o(1)$ as $t \to \infty$, then (1.15) holds.*

For a Galton–Watson process Z, it is proved in [32, Proposition 1.3] that if $E(Z_1^p) < \infty$ for some $p > 1$, then there exists some $c > 0$ such that

$$\|W_n - W_\infty\|_p \leq \begin{cases} cm^{-\frac{1}{q}n}, & \text{if } p \in (1, 2], \\ cm^{-\frac{1}{2}n}, & \text{if } p > 2, \end{cases}$$

where $1/q + 1/p = 1$. The above result implies that $\|W_n - W_\infty\|_p = o(m^{-\frac{n}{a^*}})$ for $1 < a < p \leq 2$, which corresponds to our Theorem 1.4(3), and $\|W_n - W_\infty\|_p = o(\rho^{-n})$ for any $\rho < m^{1/2}$.

If $\int_1^\infty r^2 \pi(x, dr)$ is bounded (which implies that (1.15) holds for $p = 2$), by the central limit theorem (see [38, Theorem 1.4]), $e^{-\lambda t/2}(M_t(\phi) - M_\infty(\phi))$ converges to $Z\sqrt{M_\infty(\phi)}$ with Z being a normal random variable with mean zero and independent of $M_\infty(\phi)$. In [38], the mean semigroup P_t^β is assumed to be symmetric with respect to some measure m, and the assumptions on $(P_t^\beta)_{t\geq 0}$ are slightly different, but the central limit theorem also holds in the nonsymmetric case, see [39] for the corresponding results for branching Markov processes. A question related to Theorem 1.4 is whether the results still holds for $a = p < 2$. The following theorem gives necessary and sufficient conditions for the almost sure convergence for the case of $a = p < 2$.

Theorem 1.5. *Suppose that Assumptions 1–2 and (1.13) hold. Let $1 < p < 2$, $1/p + 1/q = 1$.*

(1) *If* (1.15) *holds, then for any* $\mu \in \mathcal{M}^0(E)$, *as* $t \to \infty$, $A_t(q)$ *converges* $\mathbb{P}_\mu$*-a.s. and*

$$M_t(\phi) - M_\infty(\phi) = o(e^{-\frac{\lambda t}{q}}), \quad \mathbb{P}_\mu\text{-a.s.}$$

(2) *Suppose there exist* $B > 0$ *and* $T_0 > 0$ *such that*

$$\sup_{x\in E} \frac{1}{\phi(x)} \int_t^\infty \pi^\phi(x, dr) \le B \int_E \nu(dy) \int_t^\infty \pi^\phi(y, dr), \qquad t > T_0. \tag{1.16}$$

If

$$\int_E \nu(dy) \int_1^\infty r^p \pi^\phi(y, dr) = \infty, \tag{1.17}$$

then for any $\mu \in \mathcal{M}^0(E)$, $M_t(\phi) - M_\infty(\phi) = o(e^{-\frac{\lambda t}{q}})$ $\mathbb{P}_\mu$*-a.s. does not hold as* $t \to \infty$.

Theorem 1.6. *Assume that Assumptions 1–2 and* (1.13) *hold.*

(1) *For any* $\gamma > 0$,

$$\int_E \nu(dx) \int_1^\infty r(\ln r)^{\gamma+1} \pi^\phi(x, dr) < \infty \tag{1.18}$$

implies that, for any $\mu \in \mathcal{M}^0(E)$,

$$\int_0^t s^{\gamma-1}(M_\infty(\phi) - M_s(\phi))ds$$

converges $\mathbb{P}_\mu$*-almost surely as* $t \to \infty$, *and*

$$M_\infty(\phi) - M_t(\phi) = o(t^{-\gamma}), \qquad \mathbb{P}_\mu\text{-a.s.}$$

If (1.18) *holds with* $\gamma \ge 1$, *then* $\int_0^\infty (M_\infty(\phi) - M_t(\phi))dt$ *exists* $\mathbb{P}_\mu$*-almost surely for any* $\mu \in \mathcal{M}^0(E)$.

(2) *Suppose that there exist* $b > 0, T_1 > 0$ *and a Borel set* $F \subset E$ *with* $\nu(F) > 0$ *such that*

$$\inf_{x\in F} \frac{1}{\phi(x)} \int_t^\infty r\pi^\phi(x, dr) \ge b \int_E \nu(dx) \int_t^\infty r\pi^\phi(x, dr), \qquad t > T_1. \tag{1.19}$$

If there is $\gamma \in (0, \infty)$ *such that*

$$\int_E \nu(dx) \int_1^\infty r(\ln r)^{\gamma+1} \pi^\phi(x, dr) = \infty, \tag{1.20}$$

then for any $\mu \in \mathcal{M}^0(E)$, $\int_0^t s^{\gamma-1}(M_\infty(\phi) - M_s(\phi))ds$ *does not converge* $\mathbb{P}_\mu$*-a.s. as* $t \to \infty$. *If, as* $t \to \infty$,

$$\int_E \nu(dx) \int_t^\infty r(\ln r - \ln t)\pi^\phi(x, dr) = o\left((\ln t)^{-\gamma}\right) \tag{1.21}$$

does not hold, then $M_\infty(\phi) - M_t(\phi) = o(t^{-\gamma})$, $\mathbb{P}_\mu$*-a.s. does not hold as well.*

It was noted in [3, Theorem II.4.1, p.36] that (1.18) implies (1.21), and (1.21) implies that

$$\int_E \nu(dx) \int_1^\infty r(\ln r)^{\gamma+1-\varepsilon} \pi^\phi(x, dr) < \infty \quad \text{for all } 0 < \varepsilon \le \gamma.$$

This says that (1.18) is slightly stronger than (1.21).

We make a few remarks about (1.16) and (1.19). Note that by definition,

$$\int_t^\infty \pi^\phi(x,dr) = \int_{t/\phi(x)}^\infty \pi(x,dr) = \pi\left(x,(t/\phi(x),\infty)\right).$$

If $\pi(x,dr)=\gamma(x)r^{-1-\alpha}dr$ with $\alpha\in(1,2)$ and γ a bounded non-negative Borel function, then

$$\int_t^\infty \pi^\phi(x,dr) = \frac{1}{\alpha}\gamma(x)t^{-\alpha}\phi(x)^\alpha.$$

Hence

$$\frac{1}{\phi(x)}\int_t^\infty \pi^\phi(x,dr) = \frac{1}{\alpha}\gamma(x)t^{-\alpha}\phi(x)^{\alpha-1}$$

and

$$\int_E \nu(dx)\int_t^\infty \pi^\phi(x,dr) = \frac{t^{-\alpha}}{\alpha}\int_E \nu(dx)\gamma(x)\phi(x)^\alpha.$$

Since γ and ϕ are bounded, (1.16) is satisfied. Similarly, by definition,

$$\int_t^\infty r\pi^\phi(x,dr) = \phi(x)\int_{t/\phi(x)}^\infty r\pi(x,dr).$$

Hence, we similarly have

$$\frac{1}{\phi(x)}\int_t^\infty r\pi^\phi(x,dr) = \frac{1}{\alpha-1}\gamma(x)\phi(x)^{\alpha-1}t^{1-\alpha}$$

and

$$\int_E \nu(dx)\int_t^\infty r\pi^\phi(x,dr) = \frac{t^{1-\alpha}}{\alpha-1}\int_E \gamma(x)\phi(x)^\alpha\nu(dx).$$

Thus (1.19) is satisfied if $\nu(\{x:\gamma(x)>0\})>0$. It is easy to generalize the remarks above on (1.16) and (1.19) to the case when $\pi(x,dr)=\gamma(x)r^{-1-\alpha}s(r)dr$ with $\alpha\in(1,2)$, γ a bounded non-negative Borel function, and s a local bounded non-negative Borel function $(0,\infty)$ which is slowly varying at ∞.

The remainder of this article is organized as follows. In Section 2.1, we present a stochastic integral representation of superprocesses which will be used in later sections. In Section 2.2, we introduce the spine decomposition of superprocesses which is used in the proof of Lemma 3.5. The main results are proved in Section 3. Lemma 3.5 plays a key role in the proof of Theorem 1.5.

In this paper, we use the convention that an expression of the type $a\lesssim b$ means that there exists a positive constant N which is independent of a and b such that $a\le Nb$. Moreover, if $a\lesssim b$ and $b\lesssim a$, we shall write $a\asymp b$.

2. Superprocesses

2.1. Stochastic integral representation of superprocesses

Without loss of generality, we assume that our process $\overline{X}$ is the coordinate process on

$$\mathbb{D} := \{w=(w_t)_{t\ge 0} : w \text{ is an } \mathcal{M}(E_\partial)\text{-valued càdlàg function on } [0,\infty).\}.$$

We assume that $(\mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0})$ is the natural filtration on $\mathbb{D}$, completed as usual with the $\mathcal{F}_\infty$-measurable and $\mathbb{P}_\mu$-negligible sets for every $\mu \in \mathcal{M}(E_\partial)$. Let $\mathbb{W}_0^+$ be the family of $\mathcal{M}(E_\partial)$-valued càdlàg functions on $(0, \infty)$ with $\mathbf{0}$ as a trap and with $\lim_{t\downarrow 0} w_t = \mathbf{0}$. $\mathbb{W}_0^+$ can be regarded as a subset of $\mathbb{D}$.

Throughout this paper assume that $\mathbb{P}_x(X_t(1) = 0) > 0$ for any $x \in E$ and $t > 0$, which implies that there exists a unique family of σ-finite measures $\{\mathbb{N}_x; x \in E\}$ on $\mathbb{W}_0^+$ such that for any $\mu \in \mathcal{M}(E)$, if $\mathcal{N}(dw)$ is a Poisson random measure on $\mathbb{W}_0^+$ with intensity measure

$$\mathbb{N}_\mu(dw) := \int_E \mathbb{N}_x(dw)\mu(dx),$$

then the process defined by

$$\widetilde{X}_0 = \mu, \quad \widetilde{X}_t = \int_{\mathbb{W}_0^+} w_t \mathcal{N}(dw), \quad t > 0$$

is a realization of the superprocess $\overline{X} = \{(\overline{X}_t)_{t\geq 0}; \mathbb{P}_\mu, \mu \in \mathcal{M}(E)\}$. Furthermore, $\mathbb{N}_x(\langle f, w_t\rangle) = \mathbb{P}_x\langle f, X_t\rangle$ for any $f \in \mathcal{B}^+(E)$ (see [31, Theorem 8.22] and [40, Section 2.2]). $\{\mathbb{N}_x; x \in E\}$ can be regarded as measures on $\mathbb{D}$ carried by $\mathbb{W}_0^+$.

Let us recall the stochastic integral representation of superprocesses, for more details see [16] or [31, Chapter 7]. Let $(\mathbf{A}, \mathfrak{D}(\mathbf{A}))$ be the weak infinitesimal generator of ξ as defined in [15, Section 4]. For any $f \in \mathfrak{D}(\mathbf{A})$,

$$\frac{P_t f(x) - f(x)}{t} \to \mathbf{A} f(x), \quad \text{bounded and pointwisely as } t \to 0.$$

We will use the standard notation $\triangle \overline{X}_s = \overline{X}_s - \overline{X}_{s-}$ for the jump of $\overline{X}$ at time s. Let $C_0^2(\mathbb{R})$ denote the set of all twice continuously differentiable functions on $\mathbb{R}$ vanishing at infinity. It is known (cf. [15, Theorem 1.5]) that the superprocess $\overline{X}$ is a solution to the following martingale problem: for any $\varphi \in \mathfrak{D}(\mathbf{A})$ and $h \in C_0^2(\mathbb{R})$,

$$\begin{aligned} &h(\langle\varphi, \overline{X}_t\rangle) - h(\langle\varphi, \mu\rangle) - \int_0^t h'(\langle\varphi, \overline{X}_s\rangle)\langle(\mathbf{A}+\beta)\varphi, \overline{X}_s\rangle ds \\ &- \frac{1}{2}\int_0^t h''(\langle\varphi, \overline{X}_s\rangle)\langle\alpha\varphi^2, \overline{X}_s\rangle ds \\ &- \int_0^t \int_E \int_{(0,\infty)} \big(h(\langle\varphi, \overline{X}_s\rangle + r\varphi(x)) - h(\langle\varphi, \overline{X}_s\rangle) \\ &- h'(\langle\varphi, \overline{X}_s\rangle) r\varphi(x) \big)\, \pi(x, dr)\overline{X}_{s-}(dx)ds \end{aligned} \tag{2.1}$$

is a $\mathbb{P}_\mu$-martingale for any $\mu \in \mathcal{M}^0(E_\partial)$.

By [16, Proposition 2.1] (also see [31, Theorem 7.13]), for any $\varphi \in \mathfrak{D}(\mathbf{A})$ and $\mu \in \mathcal{M}^0(E_\partial)$,

$$\langle\varphi, \overline{X}_t\rangle = \langle\varphi, \mu\rangle + S_t^J(\varphi) + S_t^C(\varphi) + \int_0^t \langle(\mathbf{A}+\beta)\varphi, \overline{X}_s\rangle ds, \tag{2.2}$$

where $S_t^C(\varphi)$ is a continuous $\mathbb{P}_\mu$-local martingale and $S_t^J(\varphi)$ is a $\mathbb{P}_\mu$-pure jump martingale. The quadratic variation process of the continuous local martingale $S_t^C(\varphi)$ is given by

$$\langle S^C(\varphi)\rangle_t = \int_0^t \langle\alpha\varphi^2, \overline{X}_s\rangle ds. \tag{2.3}$$

Next, we characterize the pure jump martingale $(S_t^J(\varphi), t \geq 0)$. Let J denote the set of all jump times of $\overline{X}$ and δ denote the Dirac measure. From the last part of (2.1), we see that the only possible jumps of $\overline{X}$ are point measures of the form $r\delta_x$ with $r > 0$ and $x \in E_\partial$, see [30, Section 2.3]. Thus the predictable compensator of the random measure (for the definition of the predictable compensator of a random measure, see, for instance [8, p. 107])

$$N := \sum_{s\in J} \delta_{(s,\triangle \overline{X}_s)}$$

is a random measure $\widehat{N}$ on $\mathbb{R}_+ \times \mathcal{M}(E_\partial)$ such that for any nonnegative predictable function F on $\mathbb{R}_+ \times \Omega \times \mathcal{M}(E_\partial)$,

$$\int_0^\infty \int_{\mathcal{M}(E_\partial)} F(s,\omega,\upsilon)\widehat{N}(ds,d\upsilon) = \int_0^\infty ds \int_E \overline{X}_{s-}(dx) \int_0^\infty F(s,\omega,r\delta_x)\pi(x,dr), \tag{2.4}$$

where $\pi(x, dr)$ is the kernel of the branching mechanism ψ. Therefore we have

$$\mathbb{P}_\mu\left[\sum_{s\in J} F(s,\omega,\triangle \overline{X}_s)\right] = \mathbb{P}_\mu \int_0^\infty ds \int_E \overline{X}_{s-}(dx) \int_0^\infty F(s,\omega,r\delta_x)\pi(x,dr). \tag{2.5}$$

See [8, p.111].

Let F be a Borel function on $\mathbb{R}_+ \times \mathcal{M}(E_\partial)$ satisfying

$$\mathbb{P}_\mu\left[\left(\sum_{s\in[0,t],s\in J} F(s,\Delta\overline{X}_s)^2\right)^{1/2}\right] < \infty, \qquad \text{for all } \mu \in \mathcal{M}(E_\partial).$$

Then the stochastic integral of F with respect to the compensated random measure $N - \widehat{N}$

$$\int_0^t \int_{\mathcal{M}(E_\partial)} F(s,\upsilon)(N-\widehat{N})(ds,d\upsilon)$$

can be defined (cf. [30] and the references therein) as the unique purely discontinuous martingale (vanishing at time 0) whose jumps are indistinguishable from $1_J(s)F(s,\Delta\overline{X}_s)$.

Suppose that g is a Borel function on $\mathbb{R}_+ \times E$. Define

$$F_g(s,\upsilon) := \int_E g(s,x)\upsilon(dx), \qquad \upsilon \in \mathcal{M}(E_\partial), \tag{2.6}$$

whenever the integral above makes sense. When g is a bounded Borel function on $\mathbb{R}_+ \times E$, for any $\mu \in \mathcal{M}(E)$,

$$\begin{aligned}
&\mathbb{P}_\mu\left[\left(\textstyle\sum_{s\in[0,t],s\in J} F_g(s,\Delta\overline{X}_s)^2\right)^{1/2}\right] = \mathbb{P}_\mu\left[\left(\textstyle\sum_{s\in[0,t],s\in J}\left(\int_E g(s,x)(\Delta\overline{X}_s)(dx)\right)^2\right)^{1/2}\right]\\
\le\ & \|g\|_\infty \mathbb{P}_\mu\left[\left(\sum_{s\in[0,t],s\in J}\langle 1,\Delta\overline{X}_s\rangle^2 1_{\{\langle 1,\Delta\overline{X}_s\rangle\le 1\}} + \sum_{s\in[0,t],s\in J}\langle 1,\Delta\overline{X}_s\rangle^2 1_{\{\langle 1,\Delta\overline{X}_s\rangle> 1\}}\right)^{1/2}\right]\\
\le\ & \|g\|_\infty \mathbb{P}_\mu\left[\left(\sum_{s\in[0,t],s\in J}\langle 1,\Delta\overline{X}_s\rangle^2 1_{\{\langle 1,\Delta\overline{X}_s\rangle\le 1\}}\right)^{1/2}\right]\\
&+\|g\|_\infty \mathbb{P}_\mu\left[\left(\sum_{s\in[0,t],s\in J}\langle 1,\Delta\overline{X}_s\rangle^2 1_{\{\langle 1,\Delta\overline{X}_s\rangle> 1\}}\right)^{1/2}\right].
\end{aligned}$$

Using the first two displays on [30, p.203], we get

$$\mathbb{P}_\mu\left[\left(\sum_{s\in[0,t],s\in J} F_g(s,\Delta\overline{X}_s)^2\right)^{1/2}\right]<\infty. \tag{2.7}$$

Therefore, if g is bounded on $\mathbb{R}_+\times E$, then the integral $\int_0^t\int_{\mathcal{M}(E_\partial)}F_g(s,\upsilon)(N-\widehat{N})(ds,d\upsilon)$ is well defined and is a martingale. Define the martingale measure $S^J(ds,dx)$ by

$$\int_0^t\int_E g(s,x)S^J(ds,dx):=\int_0^t\int_{\mathcal{M}(E_\partial)}F_g(s,\upsilon)(N-\widehat{N})(ds,d\upsilon). \tag{2.8}$$

Thus the pure jump martingale $S_t^J(\varphi)$ in (2.2) can be written as

$$S_t^J(\varphi)=\int_0^t\int_E\varphi(x)S^J(ds,dx).$$

A martingale measure $S^C(ds,dx)$ can be defined (see [16] or [36] for the precise definition) so that the continuous martingale in (2.2) can be expressed as

$$S_t^C(\varphi)=\int_0^t\int_E\varphi(x)S^C(ds,dx).$$

Summing up these two martingale measures, we get a martingale measure

$$M(ds,dx)=S^J(ds,dx)+S^C(ds,dx). \tag{2.9}$$

Using this, [16, Proposition 2.14] and applying a limit argument, one can show that for any bounded Borel function g on E,

$$\langle g,X_t\rangle=\langle P_t^\beta g,\mu\rangle+\int_0^t\int_E P_{t-s}^\beta g(x)S^J(ds,dx)+\int_0^t\int_E P_{t-s}^\beta g(x)S^C(ds,dx). \tag{2.10}$$

In particular, taking $g=\phi$ in (2.10), where ϕ is the positive eigenfunction of P_t^β given in Assumption 1, we get the expression for the martingale $(M_t(\phi))_{t\geq 0}$:

$$M_t(\phi)=\langle\phi,\mu\rangle+\int_0^t e^{-\lambda s}\int_E\phi(x)S^J(ds,dx)+\int_0^t e^{-\lambda s}\int_E\phi(x)S^C(ds,dx). \tag{2.11}$$

Therefore the limit $M_\infty(\phi)$ of $M_t(\phi)$ can be written as

$$M_\infty(\phi)=\langle\phi,\mu\rangle+\int_0^\infty e^{-\lambda s}\int_E\phi(x)S^J(ds,dx)+\int_0^\infty e^{-\lambda s}\int_E\phi(x)S^C(ds,dx). \tag{2.12}$$

For the jump part above, we always handle the 'small jumps' and the 'large jumps' separately. Now we give the precise definitions of 'small jumps' and 'large jumps'. Given $\rho\in(0,\infty]$, a jump at time s is called 'small' if $0<\Delta\overline{X}_s(\phi)<e^{\frac{\lambda}{\rho}s}$, and 'large' if $\triangle\overline{X}_s(\phi)\geq e^{\frac{\lambda}{\rho}s}$, where $\triangle\overline{X}_s(\phi)=r\phi(x)$ when $\triangle\overline{X}_s=r\delta_x$ with $r>0$ and $x\in E$. Define

$$N^{(1,\rho)}:=\sum_{0<\triangle\overline{X}_s(\phi)<e^{\frac{\lambda}{\rho}s}}\delta_{(s,\triangle\overline{X}_s)},\qquad N^{(2,\rho)}:=\sum_{\triangle\overline{X}_s(\phi)\geq e^{\frac{\lambda}{\rho}s}}\delta_{(s,\triangle\overline{X}_s)},$$

and denote the compensators of $N^{(1,\rho)}$ and $N^{(2,\rho)}$ by $\widehat{N}^{(1,\rho)}$ and $\widehat{N}^{(1,\rho)}$ respectively. Then for any non-negative Borel function F on $\mathbb{R}_+\times\mathcal{M}(E_\partial)$,

$$\int_0^\infty\int_{\mathcal{M}(E_\partial)}F(s,\upsilon)\widehat{N}^{(1,\rho)}(ds,d\upsilon)=\int_0^\infty ds\int_E\overline{X}_{s-}(dx)\int_0^{e^{\frac{\lambda}{\rho}s}}F(s,r\phi(x)^{-1}\delta_x)\pi^\phi(x,dr)$$

and

$$\int_0^\infty \int_{\mathcal{M}(E_\partial)} F(s,\upsilon)\widehat{N}^{(2,\rho)}(ds,d\upsilon) = \int_0^\infty ds \int_E \overline{X}_{s-}(dx) \int_{e^{\frac{\lambda}{\rho}s}}^\infty F(s, r\phi(x)^{-1}\delta_x)\pi^\phi(x,dr),$$

where π^ϕ was defined in (1.10). Let $J^{(1,\rho)}$ denote the set of jump times of $N^{(1,\rho)}$, and let $J^{(2,\rho)}$ denote the set of jump times of $N^{(2,\rho)}$. Then

$$\int_0^\infty \int_{\mathcal{M}(E_\partial)} F(s,\upsilon)N^{(1,\rho)}(ds,d\upsilon) = \sum_{s\in J^{(1,\rho)}} F(s,\triangle\overline{X}_s),$$

$$\int_0^\infty \int_{\mathcal{M}(E_\partial)} F(s,\upsilon)N^{(2,\rho)}(ds,d\upsilon) = \sum_{s\in J^{(2,\rho)}} F(s,\triangle\overline{X}_s).$$

Similar to the way we constructed $S^J(ds,dx)$ from $N(ds,d\upsilon)$, we can construct two martingale measures $S^{(1,\rho)}(ds,dx)$ and $S^{(2,\rho)}(ds,dx)$ respectively from $N^{(1,\rho)}(ds,d\upsilon)$ and $N^{(2,\rho)}(ds,d\upsilon)$. Then for any bounded Borel function g on $\mathbb{R}_+\times E$, we can obtain the following martingales, for $t>0$,

$$S_t^{(1,\rho)}(g) = \int_0^t \int_E g(s,x)S^{(1,\rho)}(ds,dx) = \int_0^t \int_{\mathcal{M}(E_\partial)} F_g(s,\upsilon)(N^{(1,\rho)}-\widehat{N}^{(1,\rho)})(ds,d\upsilon) \tag{2.13}$$

and

$$S_t^{(2,\rho)}(g) = \int_0^t \int_E g(s,x)S^{(2,\rho)}(ds,dx) = \int_0^t \int_{\mathcal{M}(E_\partial)} F_g(s,\upsilon)(N^{(2,\rho)}-\widehat{N}^{(2,\rho)})(ds,d\upsilon), \tag{2.14}$$

where $F_g(s,\upsilon) = \int_E g(s,x)\upsilon(dx)$.

2.2. Spine decomposition of superprocesses

Recall that $\{(\xi_t)_{t\geq 0}; \Pi_x, x\in E\}$ is the spatial motion. Let $(\mathcal{F}_t^\xi)_{t\geq 0}$ be the natural filtration of $(\xi_t)_{t\geq 0}$. For each $x\in E$, let $\widetilde{\Pi}_x$ be the probability measure defined by

$$\frac{d\widetilde{\Pi}_x|_{\mathcal{F}_t^\xi}}{d\Pi_x|_{\mathcal{F}_t^\xi}} = \frac{e^{\int_0^t \beta(\xi_s)ds}\phi(\xi_t)1_{\{t<\zeta\}}}{e^{\lambda t}\phi(x)}, \quad t\geq 0. \tag{2.15}$$

It can be verified (see [21] for example) that the process $\{(\xi_t)_{t\geq 0}; \widetilde{\Pi}_x, x\in E\}$ is a time homogeneous Markov process. For any $\mu\in\mathcal{M}(E)$, define $(\phi\mu)(dx) := \phi(x)\mu(dx)$. For any $\mu\in\mathcal{M}^0(E)$, we define

$$\Pi_\mu(\cdot) := \mu(E)^{-1}\int_E \Pi_x(\cdot)\mu(dx) \qquad \text{and} \qquad \widetilde{\Pi}_\mu(\cdot) := \mu(E)^{-1}\int_E \widetilde{\Pi}_x(\cdot)\mu(dx).$$

For any $\mu\in\mathcal{M}^0(E)$, we define the probability measure $\widetilde{\mathbb{P}}_\mu$ by

$$\frac{d\widetilde{\mathbb{P}}_\mu|_{\mathcal{F}_t}}{d\mathbb{P}_\mu|_{\mathcal{F}_t}} = \frac{M_t(\phi)}{\mu(\phi)}, \quad t\geq 0. \tag{2.16}$$

To prove Theorem 1.5, we will use the spine decomposition of $(X_t;\widetilde{\mathbb{P}}_\mu)$, which says that $(X_t;\widetilde{\mathbb{P}}_\mu)$ has the same distribution as the sum of three different kinds of immigration along a spine process. This decomposition has been used for studying limit properties of superprocesses, see for example Engländer and Kyprianou [13], Liu et al. [33] in the case of $\alpha = 0$, Kyprianou et al. [28] and Chen et al. [6] to mention some. Now we state the spine decomposition. On some probability space with probability measure $\mathbb{Q}_\mu$, we have the following processes:

(i) $\{(\xi_t)_{t\geq 0}; \mathbb{Q}_\mu\}$ is a Markov process, called the spine process, with

$$\{(\xi_t)_{t\geq 0}; \mathbb{Q}_\mu\} \overset{d}{=} \{(\xi_t)_{t\geq 0}; \widetilde{\Pi}_{\phi\mu}\};$$

(ii) Conditioned on $(\xi_t)_{t\geq 0}$, *the continuum immigration* $\{(X^{C,\sigma})_{\sigma\in\mathcal{D}^C}; \mathbb{Q}_\mu(\cdot|(\xi_t)_{t\geq 0})\}$ is a $\mathbb{D}$-valued point process such that

$$\mathrm{n}(ds, dw) := \sum_{\sigma\in\mathcal{D}^C} \delta_{(\sigma, X^{C,\sigma})}(ds, dw) \tag{2.17}$$

is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{D}$ with intensity

$$\mathbf{n}(ds, dw) := \alpha(\xi_s)ds \cdot \mathbb{N}_{\xi_s}(dw),$$

where $\mathcal{D}^C$ stands for the jumping time set of the point process $(X^{C,\sigma})$. Given ξ, $\{X^{C,\sigma} : \sigma \in \mathcal{D}^C\}$ are independent.

(iii) Conditioned on $(\xi_t)_{t\geq 0}$, *the discrete immigration* $\{(X^{J,\sigma})_{\sigma\in\mathcal{D}^J}; \mathbb{Q}_\mu(\cdot|(\xi_t)_{t\geq 0})\}$ is a $\mathbb{D}$-valued point process such that $\mathrm{m}(ds, dw) := \sum_{\sigma\in\mathcal{D}^J} \delta_{(\sigma, X^{J,\sigma})}(ds, dw)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{D}$ with intensity

$$\mathbf{m}(ds, dw) := ds \cdot \int_{(0,\infty)} y\mathbb{P}_{y\delta_{\xi_s}}(X \in dw)\pi(\xi_s, dy); \tag{2.18}$$

and $\mathcal{D}^J$ stands for the jumping time set of the point process $(X^{J,\sigma})$. Given ξ, $\{X^{J,\sigma}, \sigma \in \mathcal{D}^J\}$ are independent, and independent of $\{X^{C,\sigma} : \sigma \in \mathcal{D}^C\}$.

(iv) $\{(X_t)_{t\geq 0}; \mathbb{Q}_\mu\}$, known as *the immigration at time* $t = 0$, is a copy of the superprocess $\{(X_t)_{t\geq 0}; \mathbb{P}_\mu\}$, and is independent of $(\xi_t)_{t\geq 0}$, $(X^{C,\sigma})_{\sigma\in\mathcal{D}^C}$ and $(X^{J,\sigma})_{\sigma\in\mathcal{D}^J}$.

$\{(\xi_t)_{t\geq 0}, (X^{C,\sigma})_{\sigma\in\mathcal{D}^C}, (X^{J,\sigma})_{\sigma\in\mathcal{D}^J}, (X_t)_{t\geq 0}; \mathbb{Q}_\mu\}$ is called a spine decomposition of $\{(X_t)_{t\geq 0}; \widetilde{\mathbb{P}}_\mu\}$. The spine decomposition theorem says that

$$\left(X_t; \widetilde{\mathbb{P}}_\mu\right) \overset{d}{=} \left(X_t + \sum_{\sigma\in\mathcal{D}^C\cap[0,t]} X^{C,\sigma}_{t-\sigma} + \sum_{\sigma\in\mathcal{D}^J\cap[0,t]} X^{J,\sigma}_{t-\sigma}; \mathbb{Q}_\mu\right), \tag{2.19}$$

see, for instance, [33] for the case $\alpha = 0$, [11, Proposition 2.17] and [6, Proposition 3.1] for general branching mechanism ψ. Put

$$Z^C_t := \sum_{\sigma\in\mathcal{D}^C\cap[0,t]} X^{C,\sigma}_{t-\sigma} \quad \text{and} \quad Z^J_t := \sum_{\sigma\in\mathcal{D}^J\cap[0,t]} X^{J,\sigma}_{t-\sigma}, \qquad t > 0.$$

Then the spine representation (2.19) of X can be simplified as for any $t \geq 0$,

$$\left(X_t; \widetilde{\mathbb{P}}_\mu\right) \overset{d}{=} \left(X_t + Z^C_t + Z^J_t; \mathbb{Q}_\mu\right). \tag{2.20}$$

3. Proofs of main results

3.1. Some lemmas

Recall the definition (1.14), i.e.,

$$A_t(q) = \int_0^t e^{\frac{\lambda s}{q}}(M_\infty(\phi) - M_s(\phi))ds, \quad t \in [0, \infty).$$

Let $A_\infty(q)$ denote the almost sure limit of $A_t(q)$ as $t \to \infty$ whenever it exists. Recall the definition (2.9) of the martingale measure

$$M(ds, dx) = S^J(ds, dx) + S^C(ds, dx).$$

For any $p > 0$, $g(s, x) := e^{\frac{-\lambda s}{p}} \phi(x)$ is bounded on $\mathbb{R}_+ \times E$, and thus we can define a martingale $(\widetilde{A}_t(p))_{t\geq 0}$ by

$$\widetilde{A}_t(p) = \int_0^t e^{\frac{-\lambda s}{p}} \int_E \phi(x) M(ds, dx), \qquad t \geq 0. \tag{3.1}$$

When the almost sure limit of this martingale exists as $t \to \infty$, we denote the limit by $\widetilde{A}_\infty(p)$ and write it in the integral form $\widetilde{A}_\infty(p) := \int_0^\infty e^{\frac{-\lambda s}{p}} \int_E \phi(x) M(ds, dx)$. It follows from the representation (2.11) of $M_t(\phi)$ that $M_t(\phi) - \langle \phi, \mu \rangle = \widetilde{A}_t(1)$. When the moment condition (1.13) holds, we have shown that the limit $M_\infty(\phi)$ exists. Thus we have the following expression of $M_\infty(\phi) - M_t(\phi)$ in terms of the martingale measure $M(ds, dx)$:

$$M_\infty(\phi) - M_t(\phi) = \widetilde{A}_\infty(1) - \widetilde{A}_t(1) = \int_t^\infty e^{-\lambda s} \int_E \phi(x) M(ds, dx).$$

Lemma 3.1. *Assume that (1.13) holds. Suppose $p \in (1, 2]$, $1/p + 1/q = 1$, $r > 1$, and $\mu \in \mathcal{M}^0(E)$.*

(1) $\widetilde{A}_t(p)$ converges $\mathbb{P}_\mu$-almost surely as $t \to \infty$ if and only if $A_t(q)$ converges $\mathbb{P}_\mu$-almost surely and $M_\infty(\phi) - M_t(\phi) = o\left(e^{-\frac{\lambda t}{q}}\right)$, $\mathbb{P}_\mu$-almost surely as $t \to \infty$. In this case, we have

$$A_\infty(q) = \frac{q \widetilde{A}_\infty(p)}{\lambda} - \frac{q}{\lambda}(M_\infty(\phi) - M_0(\phi)), \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.2}$$

(2) $A_t(q) - A_1(q)$ is in $L^r(\mathbb{P}_\mu)$ and converges in $L^r(\mathbb{P}_\mu)$ as $t \to \infty$ if and only if $\widetilde{A}_t(p) - \widetilde{A}_1(p)$ is in $L^r(\mathbb{P}_\mu)$ and converges in $L^r(\mathbb{P}_\mu)$ as $t \to \infty$. In this case, we have

$$A_\infty(q) - A_1(q) = \frac{q}{\lambda}(\widetilde{A}_\infty(p) - \widetilde{A}_1(p)) - \frac{q}{\lambda} e^{\frac{\lambda}{q}} (M_\infty(\phi) - M_1(\phi)) \quad \text{in } L^r(\mathbb{P}_\mu), \tag{3.3}$$

where $A_\infty(q) - A_1(q)$ (resp. $\widetilde{A}_\infty(p) - \widetilde{A}_1(p)$) is the $L^r(\mathbb{P}_\mu)$-limit of $A_t(q) - A_1(q)$ (resp. $\widetilde{A}_t(p) - \widetilde{A}_1(p)$) as $t \to \infty$.

Proof. The assumption (1.13) implies the uniform integrability of $M_t(\phi)$, $t \in [0, \infty]$. Consequently, $M_t(\phi)$ is bounded on $[0, \infty]$ $\mathbb{P}_\mu$-almost surely. By the bounded convergence theorem and the stochastic Fubini theorem for martingale measures (c.f. [31, Theorem 7.24]), for any $T > 0$,

$$\begin{aligned}
A_T(q) &= \lim_{l \to \infty} \int_0^T e^{\frac{\lambda t}{q}} dt \int_t^l e^{-\lambda s} \int_E \phi(x) M(ds, dx) \\
&= \lim_{l \to \infty} \int_0^l e^{-\lambda s} \int_E \phi(x) M(ds, dx) \int_0^{s \wedge T} e^{\frac{\lambda t}{q}} dt \\
&= \frac{q}{\lambda} \lim_{l \to \infty} \int_0^l e^{-\lambda s} (e^{\frac{\lambda (s \wedge T)}{q}} - 1) \int_E \phi(x) M(ds, dx) \\
&= \frac{q}{\lambda} \int_0^T e^{\frac{-\lambda s}{p}} \int_E \phi(x) M(ds, dx) + \frac{q}{\lambda} e^{\frac{\lambda T}{q}} \big(M_\infty(\phi) - M_T(\phi)\big) - \frac{q}{\lambda}(M_\infty(\phi) - M_0(\phi)) \\
&= \frac{q}{\lambda} \widetilde{A}_T(p) + \frac{q}{\lambda} e^{\frac{\lambda T}{q}} \big(M_\infty(\phi) - M_T(\phi)\big) - \frac{q}{\lambda}(M_\infty(\phi) - M_0(\phi)).
\end{aligned} \tag{3.4}$$

Note that $A'_t(q) := \dfrac{dA_t(q)}{dt} = e^{\frac{\lambda t}{q}}\big(M_\infty(\phi) - M_t(\phi)\big)$ for almost every $t \in (0, \infty)$. Therefore (3.4) can be rewritten as

$$-\frac{\lambda}{q}A_t(q) + A'_t(q) = (M_\infty(\phi) - M_0(\phi)) - \widetilde{A}_t(p), \quad \text{a.e. } t > 0. \tag{3.5}$$

From this we get that

$$e^{-\frac{\lambda}{q}t}A_t(q) = \frac{q}{\lambda}(M_\infty(\phi) - M_0(\phi))(1 - e^{-\frac{\lambda}{q}t}) - \int_0^t e^{-\frac{\lambda}{q}s}\widetilde{A}_s(p)ds, \quad \text{a.e. } t > 0.$$

Combining this with (3.5), we get that for almost every $t \in (0, \infty)$,

$$\begin{aligned} e^{-\frac{\lambda}{q}t}A'_t(q) &= (M_\infty(\phi) - M_0(\phi))(1 - e^{-\frac{\lambda}{q}t}) - \frac{\lambda}{q}\int_0^t e^{-\frac{\lambda}{q}s}\widetilde{A}_s(p)ds \\ &\quad + e^{-\frac{\lambda}{q}t}(M_\infty(\phi) - M_0(\phi)) - e^{-\frac{\lambda}{q}t}\widetilde{A}_t(p) \\ &= (M_\infty(\phi) - M_0(\phi)) - e^{-\frac{\lambda}{q}t}\widetilde{A}_t(p) - \frac{\lambda}{q}\int_0^t e^{-\frac{\lambda}{q}s}\widetilde{A}_s(p)ds. \end{aligned} \tag{3.6}$$

Since for a.e. $t > 0$, $e^{-\frac{\lambda}{q}t}A'_t(q) = M_\infty(\phi) - M_t(\phi)$, we have for almost all $T, t > 0$,

$$e^{-\frac{\lambda}{q}t}A'_t(q) - e^{-\frac{\lambda}{q}(T+t)}A'_{T+t}(q) = M_{T+t}(\phi) - M_t(\phi).$$

Using (3.6), we get that for almost all $T, t > 0$,

$$\begin{aligned} &e^{\frac{\lambda}{q}T}\,(M_{T+t}(\phi) - M_T(\phi)) \\ &= e^{-\frac{\lambda}{q}t}\widetilde{A}_{T+t}(p) - \widetilde{A}_T(p) + \frac{\lambda}{q}\int_0^t e^{-\frac{\lambda}{q}s}\widetilde{A}_{T+s}ds \\ &= (e^{-\frac{\lambda}{q}t} - 1)\widetilde{A}_{T+t}(p) + (\widetilde{A}_{T+t}(p) - \widetilde{A}_T(p)) + \frac{\lambda}{q}\int_0^t e^{-\frac{\lambda}{q}s}\widetilde{A}_{T+s}(p)ds. \end{aligned} \tag{3.7}$$

Since $(1 - e^{-\frac{\lambda}{q}t})\widetilde{A}_{T+t} = \dfrac{\lambda}{q}\int_0^t e^{-\frac{\lambda}{q}s}\widetilde{A}_{T+t}ds$, (3.7) can be written as: for almost all $T, t > 0$,

$$e^{\frac{\lambda}{q}T}\,(M_{T+t}(\phi) - M_T(\phi)) = \widetilde{A}_{T+t}(p) - \widetilde{A}_T(p) - \frac{\lambda}{q}\int_0^t e^{-\frac{\lambda}{q}s}\left(\widetilde{A}_{T+t}(p) - \widetilde{A}_{T+s}(p)\right)ds. \tag{3.8}$$

(1) If $A_T(q)$ converges $\mathbb{P}_\mu$-almost surely as $T \to \infty$ and $\lim_{T\to\infty} e^{\frac{\lambda T}{q}}\big(M_\infty(\phi) - M_T(\phi)\big) = 0$ $\mathbb{P}_\mu$-a.s. then by (3.4), $\widetilde{A}_T(p)$ converges $\mathbb{P}_\mu$-almost surely as $T \to \infty$, and (3.2) follows. Conversely, if $\widetilde{A}_T(p)$ converges $\mathbb{P}_\mu$-a.s. as $T \to \infty$, then for any $\varepsilon > 0$, there is $\widetilde{T}(\omega) > 0$ such that for $T > \widetilde{T}(\omega)$ and $t, s \geq 0$, $\big|\widetilde{A}_{T+t}(p) - \widetilde{A}_{T+s}(p)\big| < \varepsilon$. Using (3.8) and the right continuity of $M_t(\phi)$, we have for any $t, T > 0$,

$$\begin{aligned} \left|e^{\frac{\lambda}{q}T}\,(M_{T+t}(\phi) - M_T(\phi))\right| &\leq |\widetilde{A}_{T+t}(p) - \widetilde{A}_T(p)| + \frac{\lambda}{q}\int_0^t e^{-\frac{\lambda}{q}s}\left|\widetilde{A}_{T+t}(p) - \widetilde{A}_{T+s}(p)\right|ds \\ &\leq \varepsilon + \varepsilon\frac{\lambda}{q}\int_0^t e^{-\frac{\lambda}{q}s}ds \leq 2\varepsilon. \end{aligned}$$

Letting $t \to \infty$, we get that for $T > \widetilde{T}$, $\left|e^{\frac{\lambda}{q}T}\left(M_\infty(\phi) - M_T(\phi)\right)\right| \le 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have

$$\mathbb{P}_\mu\left(\lim_{T\to\infty} e^{\frac{\lambda}{q}T}\left(M_\infty(\phi) - M_T(\phi)\right) = 0\right) = 1.$$

Thus by (3.4), $A_T(q)$ converges $\mathbb{P}_\mu$-almost surely as $T \to \infty$.

(2) Now we consider the $L^r(\mathbb{P}_\mu)$ convergence. For any $T \ge 1$, note that

$$\begin{aligned}&\int_T^{T+1} e^{\frac{\lambda}{q}t}(M_\infty(\phi) - M_t(\phi))dt\\ &= \int_T^{T+1} e^{\frac{\lambda}{q}t}(M_\infty(\phi) - M_{T+1}(\phi))dt + \int_T^{T+1} e^{\frac{\lambda}{q}t}(M_{T+1}(\phi) - M_t(\phi))dt\\ &= \frac{q}{\lambda}(M_\infty(\phi) - M_{T+1}(\phi))e^{\frac{\lambda}{q}(T+1)}\left(1 - e^{-\lambda/q}\right)\\ &\quad + \mathbb{P}_\mu\left(\int_T^{T+1} e^{\frac{\lambda}{q}t}(M_\infty(\phi) - M_t(\phi))dt\Big|\mathcal{F}_{T+1}\right),\end{aligned}$$

which can be written as

$$\begin{aligned}&A_{T+1}(q) - A_T(q)\\ &= \frac{q}{\lambda}(M_\infty(\phi) - M_{T+1}(\phi))e^{\frac{\lambda}{q}(T+1)}\left(1 - e^{-\lambda/q}\right) + \mathbb{P}_\mu\left(A_{T+1}(q) - A_T(q)\big|\mathcal{F}_{T+1}\right).\end{aligned} \tag{3.9}$$

By Jensen's inequality,

$$\mathbb{P}_\mu\left|\mathbb{P}_\mu\left(A_{T+1}(q) - A_T(q)\big|\mathcal{F}_{T+1}\right)\right|^r \le \mathbb{P}_\mu\left|A_{T+1}(q) - A_T(q)\right|^r.$$

If $A_t(q) - A_1(q)$ is in $L^r(\mathbb{P}_\mu)$ and has an $L^r(\mathbb{P}_\mu)$ limit as $t \to \infty$, then by (3.9), $\{(M_\infty(\phi) - M_T(\phi))e^{\frac{\lambda}{q}T}, T \ge 1\}$ is bounded in $L^r(\mathbb{P}_\mu)$. We obtain from (3.4) that the martingale $\{(\widetilde{A}_t(p) - \widetilde{A}_1(p)); t \ge 1\}$ is $L^r(\mathbb{P}_\mu)$ bounded as well. Thus the martingale $\widetilde{A}_t(p) - \widetilde{A}_1(p)$ has an $L^r(\mathbb{P}_\mu)$ limit as $t \to \infty$.

Conversely, if $\widetilde{A}_t(p) - \widetilde{A}_1(p)$ is in $L^r(\mathbb{P}_\mu)$ and has an $L^r(\mathbb{P}_\mu)$ limit $\widetilde{A}_\infty(p) - \widetilde{A}_1(p)$ as $t \to \infty$, then thanks to (3.8) and Jensen's inequality, for $t \ge 1$,

$$\begin{aligned}&\mathbb{P}_\mu\left[\left|e^{\frac{\lambda t}{q}}\left(M_\infty(\phi) - M_t(\phi)\right)\right|^r\right]\\ &\lesssim \mathbb{P}_\mu|\widetilde{A}_\infty(p) - \widetilde{A}_t(p)|^r + \frac{\lambda}{q}\int_0^\infty e^{-\frac{\lambda}{q}s}\mathbb{P}_\mu\left|\widetilde{A}_\infty(p) - \widetilde{A}_{t+s}(p)\right|^r ds.\end{aligned} \tag{3.10}$$

Applying the dominated convergence theorem to the second term of the right-hand above, we get

$$\lim_{t\to\infty} \mathbb{P}_\mu\left|e^{\frac{\lambda t}{q}}\left(M_\infty(\phi) - M_t(\phi)\right)\right|^r = 0. \tag{3.11}$$

By Minkowski's inequality, for any $t_1, t_2 \ge 1$, in (3.4), we deduce that

$$\begin{aligned}&\|A_{t_1}(q) - A_{t_2}(q)\|_r\\ &\le \frac{q}{\lambda}\left[\|\widetilde{A}_{t_1}(p) - \widetilde{A}_{t_2}(p)\|_r + \|e^{\frac{\lambda t_1}{q}}(M_\infty(\phi) - M_{t_1}(\phi))\|_r + \|e^{\frac{\lambda t_2}{q}}(M_\infty(\phi) - M_{t_2}(\phi))\|_r\right].\end{aligned}$$

Therefore, $A_t(q) - A_1(q)$ is in $L^r(\mathbb{P}_\mu)$ and has an $L^r(\mathbb{P}_\mu)$ limit as $t \to \infty$. $\square$

Lemma 3.2. *Suppose $\mu \in \mathcal{M}^0(E)$ and $\gamma > 0$ is a constant. Define for $t \geq 0$,*

$$\widetilde{C}_t(\gamma) = \int_0^t e^{-\lambda s} s^\gamma \int_E \phi(x) M(ds, dx), \quad C_t(\gamma) = \int_0^t s^{\gamma-1}(M_\infty(\phi) - M_s(\phi)) ds.$$

Then $\widetilde{C}_t(\gamma)$ converges $\mathbb{P}_\mu$-almost surely as $t \to \infty$ if and only if $C_t(\gamma)$ converges and $t^{\gamma-1}(M_\infty(\phi) - M_t(\phi)) = o(t^{-1})$, $\mathbb{P}_\mu$-almost surely as $t \to \infty$. When $\widetilde{C}_t(\gamma)$ (resp. $C_t(\gamma)$) converges as $t \to \infty$, we denote its limit by $\widetilde{C}_\infty(\gamma)$ (resp. $C_\infty(\gamma)$). Then we have

$$\gamma \widetilde{C}_\infty(\gamma) = C_\infty(\gamma), \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.12}$$

Proof. The proof is similar to that of Lemma 3.1. Similar to (3.4), we have for any $T > 0$,

$$\begin{aligned} C_T(\gamma) &= \frac{1}{\gamma} \lim_{l \to \infty} \int_0^l e^{-\lambda s} (s \wedge T)^\gamma \int_E \phi(x) M(ds, dx) \\ &= \frac{1}{\gamma} \int_0^T e^{-\lambda s} s^\gamma \int_E \phi(x) M(ds, dx) + \frac{1}{\gamma} T^\gamma \big(M_\infty(\phi) - M_T(\phi)\big) \\ &= \frac{1}{\gamma} \widetilde{C}_T(\gamma) + \frac{1}{\gamma} T^\gamma \big[M_\infty(\phi) - M_T(\phi)\big]. \end{aligned} \tag{3.13}$$

Suppose $C_t(\gamma)$ converges and $t^{\gamma-1}(M_\infty(\phi) - M_t(\phi)) = o(t^{-1})$ as $t \to \infty$ $\mathbb{P}_\mu$-almost surely, then using (3.13), we get $\widetilde{C}_T(\gamma)$ converges $\mathbb{P}_\mu$-almost surely and (3.12) holds. We now deduce the almost sure convergence of $C_t(\gamma)$ and $t^\gamma\big[M_\infty(\phi) - M_t(\phi)\big]$ from the a.s. convergence of $\widetilde{C}_t(\gamma)$. From (3.13), we get

$$T^\gamma \big[M_\infty(\phi) - M_T(\phi)\big] = \gamma C_T(\gamma) - \widetilde{C}_T(\gamma)$$

and

$$T^\gamma \big[M_\infty(\phi) - M_{T+t}(\phi)\big] = \frac{T^\gamma}{(T+t)^\gamma} \big[\gamma C_{T+t}(\gamma) - \widetilde{C}_{T+t}(\gamma)\big].$$

It follows from the two displays above that

$$\begin{aligned} T^\gamma \big[M_{T+t}(\phi) - M_T(\phi)\big] = \gamma T^\gamma &\left[\frac{C_T(\gamma)}{T^\gamma} - \frac{C_{T+t}(\gamma)}{(T+t)^\gamma}\right] \\ &+ \left[\frac{T^\gamma}{(T+t)^\gamma} \widetilde{C}_{T+t}(\gamma) - \widetilde{C}_T(\gamma)\right]. \end{aligned} \tag{3.14}$$

Noticing that $t^\gamma(M_\infty(\phi) - M_t(\phi)) = t \dfrac{dC_t(\gamma)}{dt} = t C_t'(\gamma)$ for a.e. $t > 0$, one has that (3.13) can be written as

$$\gamma C_t(\gamma) - t C_t'(\gamma) = \widetilde{C}_t(\gamma), \qquad \text{a.e. } t > 0.$$

From this we get that for any $t, T > 0$,

$$\frac{C_T(\gamma)}{T^\gamma} - \frac{C_{T+t}(\gamma)}{(T+t)^\gamma} = \int_T^{T+t} \frac{\widetilde{C}_s(\gamma)}{s^{\gamma+1}} ds. \tag{3.15}$$

Simple calculations yield

$$\frac{T^\gamma}{(T+t)^\gamma} \widetilde{C}_{T+t}(\gamma) - \widetilde{C}_T(\gamma) = \widetilde{C}_{T+t}(\gamma) - \widetilde{C}_T(\gamma) - \gamma T^\gamma \int_T^{T+t} \frac{\widetilde{C}_{T+t}(\gamma)}{s^{\gamma+1}} ds. \tag{3.16}$$

Combining (3.14)–(3.16), we get

$$T^{\gamma}\left[M_{T+t}(\phi)-M_{T}(\phi)\right]=\widetilde{C}_{T+t}(\gamma)-\widetilde{C}_{T}(\gamma)-\gamma T^{\gamma}\int_{T}^{T+t}\frac{\widetilde{C}_{T+t}(\gamma)-\widetilde{C}_{s}(\gamma)}{s^{\gamma+1}}ds. \tag{3.17}$$

If $\widetilde{C}_t(\gamma)$ converges $\mathbb{P}_\mu$-almost surely as $t\to\infty$, by using the argument in the proof of Lemma 3.1(1) after (3.8), we get

$$\lim_{t\to\infty} t^{\gamma}\left[M_{\infty}(\phi)-M_{t}(\phi)\right]=0,\quad \mathbb{P}_{\mu}\text{-a.s.} \tag{3.18}$$

Therefore, it follows from (3.13) that $C_t(\gamma)$ converges $\mathbb{P}_\mu$-almost surely and (3.12) holds as well. □

Remark 3.3. Suppose that $L(ds,dx)$ is a random measure on $[0,\infty)\times E$ such that, as $t\to\infty$,

$$L_t:=\int_0^t e^{-\lambda s}\int_E \phi(x)L(ds,dx)\to L_\infty,\quad \mathbb{P}_\mu\text{-a.s.}$$

where L_∞ is a finite random variable. Using arguments similar to those in the proof of Lemma 3.2, we can show that for any $\gamma>0$, $\int_0^T e^{-\lambda s}s^{\gamma}\int_E\phi(x)L(ds,dx)$ converges $\mathbb{P}_\mu$-almost surely as $T\to\infty$ if and only if

$$\int_0^T t^{\gamma-1}dt\int_t^\infty e^{-\lambda s}\int_E\phi(x)L(ds,dx)=\int_0^T t^{\gamma-1}(L_\infty-L_t)dt$$

converges and $L_\infty-L_T=o(T^{-\gamma})$, $\mathbb{P}_\mu$-almost surely as $T\to\infty$.

Lemma 3.4. *Assume that Assumptions 1–2 and (1.13) hold. Let $1\le a<p\le 2$.*

(1) *If (1.15) holds, then for any $\mu\in\mathcal{M}^0(E)$, $(\widetilde{A}_t(a)-\widetilde{A}_1(a))$ is in $L^p(\mathbb{P}_\mu)$ and converges in $L^p(\mathbb{P}_\mu)$ and therefore $\mathbb{P}_\mu$-almost surely as $t\to\infty$.*
(2) *Suppose that for some $\mu\in\mathcal{M}^0(E)$, $(\widetilde{A}_t(a)-\widetilde{A}_1(a))$ is in $L^p(\mathbb{P}_\mu)$ and converges in $L^p(\mathbb{P}_\mu)$ as $t\to\infty$, then it must converge $\mathbb{P}_\mu$-almost surely as $t\to\infty$ and (1.15) holds.*

Proof. (1) Suppose condition (1.15) holds. From the definition (3.1) of $\widetilde{A}_t(a)$ and (2.9), we only need to consider the convergences of

$$\int_1^t e^{-\frac{\lambda}{a}s}\int_E\phi(x)S^J(ds,dx)\qquad\text{and}\qquad\int_1^t e^{-\frac{\lambda}{a}s}\int_E\phi(x)S^C(ds,dx)$$

as $t\to\infty$. Recall that definitions of $S_t^{(1,\infty)}$ and $S_t^{(2,\infty)}$ given in (2.13) and (2.14) with $\rho=\infty$. For the "small jump" part, by the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned}\mathbb{P}_\mu\left[(\sup_{t\ge0}S_t^{(1,\infty)}(e^{-\frac{\lambda}{a}\cdot}\phi))^2\right]&\lesssim\mathbb{P}_\mu\left(\int_0^\infty\int_{\mathcal{M}(E_\partial)}F^2_{e^{-\frac{\lambda}{a}\cdot}\phi}(s,\upsilon)\widehat{N}^{(1,\infty)}(ds,d\upsilon)\right)\\&=\mathbb{P}_\mu\int_0^\infty ds\int_E X_s(dx)\int_0^1F^2_{e^{-\frac{\lambda}{a}\cdot}\phi}(s,r\phi(x)^{-1}\delta_x)\pi^\phi(x,dr)\\&=\int_0^\infty e^{-\frac{2\lambda}{a}s}\langle P_s^\beta\left(\int_0^1r^2\pi^\phi(\cdot,dr)\right),\mu\rangle ds.\end{aligned}$$

Thanks to (1.11), we have

$$\mathbb{P}_\mu\left[\left(\sup_{t\geq 0} S_t^{(1,\infty)}(e^{-\frac{\lambda}{a}\cdot}\phi)\right)^2\right] \lesssim \int_0^\infty e^{-\frac{2\lambda}{a}s}\langle P_s^\beta\phi, \mu\rangle ds$$
$$= \frac{a}{(2-a)\lambda}\int_E \phi(y)\mu(dy) < \infty, \tag{3.19}$$

where in the last equality, we used the fact that $e^{-\lambda s}P_s^\beta\phi = \phi$ for $s \geq 0$. Applying the Burkholder–Davis–Gundy inequality to $S_t^{(2,\infty)}(e^{-\frac{\lambda}{a}\cdot}\phi)$, we obtain that for $1 < p \leq 2$,

$$\mathbb{P}_\mu\left[\left|\sup_{t\geq 1}\left(S_t^{(2,\infty)}(e^{-\frac{\lambda}{a}\cdot}\phi) - S_1^{(2,\infty)}(e^{-\frac{\lambda}{a}\cdot}\phi)\right)\right|^p\right] \lesssim \mathbb{P}_\mu\Bigg[\sum_{s\in[1,\infty)\cap J^{(2,\infty)}} F_{e^{-\frac{\lambda}{a}\cdot}\phi}(s, \Delta\overline{X}_s)^2\Bigg]^{\frac{p}{2}}$$
$$\leq \mathbb{P}_\mu\Bigg[\sum_{s\in[1,\infty)\cap J^{(2,\infty)}} F_{e^{-\frac{\lambda}{a}\cdot}\phi}(s, \Delta\overline{X}_s)^p\Bigg]$$
$$= \mathbb{P}_\mu\int_1^\infty ds\int_E X_s(dx)\int_1^\infty F^p_{e^{-\frac{\lambda}{a}\cdot}\phi}(s, r\phi(x)^{-1}\delta_x)\pi^\phi(x,dr)$$
$$= \int_1^\infty e^{-\frac{p\lambda}{a}s}\langle P_s^\beta\left(\int_1^\infty r^p\pi^\phi(\cdot,dr)\right), \mu\rangle ds.$$

Set $h(x) := \int_1^\infty r^p\pi^\phi(x,dr)$. Condition (1.15) says that $h \in L_1^+(\nu)$. Note that $p > a$. If $\mu \in \mathcal{M}^0(E)$, by Assumption 2,

$$\int_1^\infty e^{-\frac{p\lambda}{a}s}\langle P_s^\beta\left(\int_1^\infty r^p\pi^\phi(\cdot,dr)\right), \mu\rangle ds$$
$$\lesssim \mu(\phi)\nu(h)\Big(1+\sup_{t>1,x\in E}|C_{t,x,h}|\Big)\int_1^\infty e^{-(\frac{p}{a}-1)\lambda s}ds < \infty.$$

Therefore,

$$\mathbb{P}_\mu\left[\left|\sup_{t\geq 1}\left(S_t^{(2,\infty)}(e^{-\frac{\lambda}{a}\cdot}\phi) - S_1^{(2,\infty)}(e^{-\frac{\lambda}{a}\cdot}\phi)\right)\right|^p\right] < \infty. \tag{3.20}$$

Combining (3.19) and (3.20), we get

$$\mathbb{P}_\mu\left[\left|\sup_{t\geq 1}\int_1^t e^{-\frac{\lambda}{a}s}\int_E\phi(x)S^J(ds,dx)\right|^p\right] < \infty. \tag{3.21}$$

We also have

$$\sup_{t\geq 0}\mathbb{P}_\mu\left[\left(\int_0^t e^{-\frac{\lambda}{a}s}\int_E\phi(x)S^C(ds,dx)\right)^2\right] = \mathbb{P}_\mu\int_0^\infty e^{-\frac{2}{a}\lambda s}\langle\alpha\phi^2, X_s\rangle ds$$
$$= \int_0^\infty e^{-\frac{2}{a}\lambda s}ds\int_E P_s^\beta(\alpha\phi^2)(y)\mu(dy) \leq \frac{a\|\alpha\phi\|_\infty}{(2-a)\lambda}\langle\phi,\mu\rangle < \infty. \tag{3.22}$$

Consequently, by (3.21) and (3.22), $\sup_{t\geq 1}\mathbb{P}_\mu\left(|\widetilde{A}_t(a)-\widetilde{A}_1(a)|^p\right) < \infty$. Thus $\widetilde{A}_t(a)-\widetilde{A}_1(a)$ is in $L^p(\mathbb{P}_\mu)$ and converges in $L^p(\mathbb{P}_\mu)$ and $\mathbb{P}_\mu$-almost surely as $t\to\infty$.

(2) Suppose that, for some $\mu \in \mathcal{M}^0(E)$, $\widetilde{A}_t(a)-\widetilde{A}_1(a) \to \widetilde{A}_\infty(a)-\widetilde{A}_1(a)$ in $L^p(\mathbb{P}_\mu)$ as $t\to\infty$. Then $\sup_{t\geq 0}\mathbb{P}_\mu\left(|\widetilde{A}_t(a)-\widetilde{A}_1(a)|^p\right) < \infty$. Since $\widetilde{A}_t(a)$ is a $\mathbb{P}_\mu$-martingale, by the L^p convergence theorem for martingales, it must converge $\mathbb{P}_\mu$-almost surely as $t\to\infty$. By

Jensen's inequality, for any $t > 1$,

$$\mathbb{P}_\mu(|\widetilde{A}_t(a) - \widetilde{A}_1(a)|^p) = \mathbb{P}_\mu\left(\left|\mathbb{P}_\mu\left(\widetilde{A}_\infty(a) - \widetilde{A}_1(a)\middle|\mathcal{F}_t\right)\right|^p\right) \le \mathbb{P}_\mu(|\widetilde{A}_\infty(a) - \widetilde{A}_1(a)|^p) < \infty.$$

We have shown in (3.19) and (3.22) that

$$\mathbb{P}_\mu\left[\left(S_t^{(1,\infty)}(e^{-\frac{\lambda}{a}\cdot}\phi)\right)^2\right] < \infty \qquad \text{and} \qquad \mathbb{P}_\mu\left[\left(\int_0^t e^{-\frac{\lambda}{a}s}\int_E \phi(x)S^C(ds,dx)\right)^2\right] < \infty.$$

Therefore, by the definition of $\widetilde{A}_t(a)$ given in (3.1), we have that for any $t \ge 0$,

$$\mathbb{P}_\mu\left(\left|S_t^{(2,\infty)}(e^{-\frac{\lambda}{a}\cdot}\phi) - S_1^{(2,\infty)}(e^{-\frac{\lambda}{a}\cdot}\phi)\right|^p\right) < \infty. \tag{3.23}$$

Note that it follows from (3.10) that $\mathbb{P}_\mu(|M_\infty(\phi) - M_1(\phi)|^p) < \infty$ when $\mathbb{P}_\mu(|\widetilde{A}_\infty(a) - \widetilde{A}_1(a)|^p) < \infty$. Therefore,

$$\mathbb{P}_\mu\left[M_\infty(\phi)^p\middle|\mathcal{F}_1\right] \lesssim \mathbb{P}_\mu\left[|M_\infty(\phi) - M_1(\phi)|^p\middle|\mathcal{F}_1\right] + M_1(\phi)^p < \infty.$$

Since $\{M_t(\phi); t \ge 1\}$ is a martingale with respect to $(\mathcal{F}_t)_{t\ge 1}$ under $\mathbb{P}_\mu(\cdot|\mathcal{F}_1)$, we have almost surely

$$\mathbb{P}_\mu\left[\sup_{t\ge 1} M_t(\phi)^p\middle|\mathcal{F}_1\right] \lesssim \mathbb{P}_\mu\left[M_\infty(\phi)^p\middle|\mathcal{F}_1\right] < \infty.$$

Thus for the compensator $\widehat{N}^{(2,\infty)}$ of the "big jumps", we have $\mathbb{P}_\mu$-almost surely

$$\begin{aligned}
&\mathbb{P}_\mu\left[\left(\int_1^t \int_{\mathcal{M}(E_\partial)} F_{e^{-\frac{\lambda}{a}\cdot}\phi}(s,\upsilon)\widehat{N}^{(2,\infty)}(ds,d\upsilon)\right)^p\middle|\mathcal{F}_1\right] \\
&= \mathbb{P}_\mu\left[\left(\int_1^t e^{-\frac{\lambda}{a}s}ds\int_E X_s(dx)\int_1^\infty r\pi^\phi(x,dr)\right)^p\middle|\mathcal{F}_1\right] \\
&\lesssim \mathbb{P}_\mu\left[\left(\int_1^t e^{\frac{\lambda}{a^*}s}M_s(\phi)ds\right)^p\middle|\mathcal{F}_1\right] \le f_a^p(t)\mathbb{P}_\mu\left(\sup_{s\le t} M_s(\phi)^p\middle|\mathcal{F}_1\right] \\
&\lesssim f_a^p(t)\mathbb{P}_\mu\left(M_t(\phi)^p\middle|\mathcal{F}_1\right] \le f_a^p(t)\mathbb{P}_\mu\left(M_\infty(\phi)^p\middle|\mathcal{F}_1\right] < \infty, \quad t \ge 1,
\end{aligned} \tag{3.24}$$

where $\dfrac{1}{a^*} + \dfrac{1}{a} = 1$ and $f_a(t) = \begin{cases} \dfrac{a^*}{\lambda}e^{\frac{\lambda}{a^*}t}, & a > 1 \\ t, & a = 1, \end{cases}$ and in the first inequality we used the moment assumption (1.5) and (1.11). It follows from (3.23) and (3.24) that for any $t > 1$,

$$\mathbb{P}_\mu\left[\left(\int_1^t \int_{\mathcal{M}(E_\partial)} F_{e^{-\frac{\lambda}{a}\cdot}\phi}(s,\upsilon)N^{(2,\infty)}(ds,d\upsilon)\right)^p\middle|\mathcal{F}_1\right] < \infty.$$

Since $p > 1$, $\{X_t\}$ is Markov and $F_{e^{-\lambda\cdot}\phi}(s,\upsilon) \ge 0$, for any $t > 0$,

$$\begin{aligned}
\infty &> \mathbb{P}_{X_1}\left[\left(\int_0^t \int_{\mathcal{M}(E_\partial)} F_{e^{-\frac{\lambda}{a}\cdot}\phi}(s+1,\upsilon)N^{(2,\infty)}(ds,d\upsilon)\right)^p\right] \\
&= \mathbb{P}_{X_1}\left[\left(\sum_{s\in(0,t]\bigcap J^{(2,+\infty)}} F_{e^{-\frac{\lambda}{a}\cdot}\phi}(s+1,\Delta\overline{X}_s)\right)^p\right] \\
&\ge \mathbb{P}_{X_1}\left[\sum_{s\in(0,t]\bigcap J^{(2,+\infty)}} \left(F_{e^{-\frac{\lambda}{a}\cdot}\phi}(s+1,\Delta\overline{X}_s)\right)^p\right]
\end{aligned}$$

$$= \mathbb{P}_{X_1}\Big(\int_0^t e^{-\frac{p\lambda}{a}(s+1)}ds \int_E X_s(dx) \int_1^\infty r^p \pi^\phi(x,dr)\Big)$$
$$= \int_0^t e^{-\frac{p\lambda}{a}(s+1)}ds \int_E P_s^\beta\Big(\int_1^\infty r^p \pi^\phi(\cdot,dr)\Big)(y) X_1(dy),$$

which implies that for any $T \geq 1$,

$$\int_T^{1+T} e^{-\frac{p\lambda}{a}s}ds \int_E P_s^\beta h(y) X_1(dy) < \infty,$$

where $h(x) = \int_1^\infty r^p \pi^\phi(x,dr)$ as before. Since ν is a probability measure, $h \wedge L \in L_1^+(\nu)$ for any $L > 0$. Let $c_t = \sup_{x\in E, f\in L_1^+(\nu)} |C_{t,x,f}|$ be as in Assumption 2. Then $\lim_{t\to\infty} c_t = 0$. Choose $T > 0$ such that when $t > T$, $c_t \leq 1/2$. Applying Assumption 2 again, we get that for $t > T$,

$$e^{\lambda t}\phi(y)\nu(h\wedge L) \leq 2P_t^\beta h(y).$$

Integrating both sides of the above inequality with respect to $e^{-\frac{p}{a}\lambda t}dt X_1(dy)$ on $[T, T+1]\times E$, we obtain

$$\frac{\nu(h\wedge L)X_1(\phi)}{(\frac{p}{a}-1)\lambda} e^{-(\frac{p}{a}-1)\lambda T}\left(1 - e^{-(\frac{p}{a}-1)\lambda}\right) \leq \int_T^{T+1} e^{-\frac{p}{a}\lambda t}dt \int_E \nu(h\wedge L) e^{\lambda t}\phi(y) X_1(dy)$$
$$\leq 2\int_T^{1+T} e^{-\frac{p}{a}\lambda t}dt \int_E P_s^\beta h(y) X_1(dy) < \infty.$$

Since the last term in the above inequality does not depend on L, letting $L \to \infty$ we get

$$\nu(h) = \int_E \nu(dx) \int_1^\infty r^p \pi^\phi(x,dr) < \infty.$$

The proof is now complete. □

Lemma 3.5. *Suppose (1.16) holds and T_0 is the constant in (1.16). Then there is a constant $K > 0$ such that, for any $\mu \in \mathcal{M}^0(E)$, and any $t, n, a, b > 0$ satisfying $0 < b < a$ and $e^{bn} > T_0$,*

$$\widetilde{\mathbb{P}}_\mu(\langle\phi, X_t\rangle > e^{an}) \leq 3\langle\phi,\mu\rangle e^{\lambda t - an} + K e^{\lambda t-(a-b)n} + Kt\int_E \nu(dy)\int_{e^{bn}}^\infty r\pi^\phi(y,dr).$$

Proof. It follows from the spine decomposition that

$$\widetilde{\mathbb{P}}_\mu(\langle\phi, X_t\rangle > e^{an}) = \mathbb{Q}_\mu\Big(\langle\phi, X_t\rangle + \langle\phi, Z_t^C\rangle + \langle\phi, Z_t^J\rangle > e^{an}\Big)$$
$$\leq \mathbb{Q}_\mu\left(\langle\phi, X_t\rangle > \frac{1}{3}e^{an}\right) + \mathbb{Q}_\mu\left(\langle\phi, Z_t^C\rangle > \frac{1}{3}e^{an}\right) + \mathbb{Q}_\mu\left(\langle\phi, Z_t^J\rangle > \frac{1}{3}e^{an}\right)$$
$$\leq \frac{3\mathbb{Q}_\mu\langle\phi, X_t\rangle}{e^{an}} + \frac{3\mathbb{Q}_\mu\langle\phi, Z_t^C\rangle}{e^{an}} + \mathbb{Q}_\mu\left(\langle\phi, Z_t^J\rangle > \frac{1}{3}e^{an}\right). \tag{3.25}$$

Noting that $\mathbb{Q}_\mu\langle\phi, X_t\rangle = \mathbb{P}_\mu\langle\phi, X_t\rangle = e^{\lambda t}\langle\phi,\mu\rangle$, we get

$$\frac{3\mathbb{Q}_\mu\langle\phi, X_t\rangle}{e^{an}} \leq 3\langle\phi,\mu\rangle e^{\lambda t - an}.$$

Since $\mathbb{Q}_\mu\langle\phi, Z_t^C\rangle = \widetilde{\Pi}_{\phi\mu}\left[\int_0^t \alpha(\xi_s)\phi(\xi_s)e^{\lambda(t-s)}ds\right] \leq \frac{2\|\alpha\phi\|_\infty e^{\lambda t}}{\lambda}$, it follows that

$$\frac{3\mathbb{Q}_\mu\langle\phi, Z_t^C\rangle}{e^{an}} \leq \frac{6\|\alpha\phi\|_\infty e^{\lambda t}e^{-an}}{\lambda}. \tag{3.26}$$

Recall that the process $(X_t^{J,\sigma})_{t\geq 0}$ for $\sigma \in \mathcal{D}^J$ is the discrete immigration at time σ in the spine decomposition. Let $m_\sigma = X_0^{J,\sigma}(E)$ be the mass immigrated at time σ at the spine position ξ_σ. From the construction of Z_t^J, we can estimate the third term in (3.25) as follows:

$$\begin{aligned}\mathbb{Q}_\mu\left(\langle\phi, Z_t^J\rangle > \frac{1}{3}e^{an}\right) &\leq \mathbb{Q}_\mu\left(\sum_{\substack{\sigma\in\mathcal{D}^J\cap[0,t]\\ m_\sigma\phi(\xi_\sigma)>e^{bn}}}\langle\phi, X_{t-\sigma}^{J,\sigma}\rangle > \frac{1}{6}e^{an}\right)\\ &\quad + \mathbb{Q}_\mu\left(\sum_{\substack{\sigma\in\mathcal{D}^J\cap[0,t]\\ m_\sigma\phi(\xi_\sigma)\leq e^{bn}}}\langle\phi, X_{t-\sigma}^{J,\sigma}\rangle > \frac{1}{6}e^{an}\right).\end{aligned}$$

By the Markov inequality, (1.10) and (2.18),

$$\begin{aligned}&\mathbb{Q}_\mu\left(\sum_{\substack{\sigma\in\mathcal{D}^J\cap[0,t]\\ m_\sigma\phi(\xi_\sigma)\leq e^{bn}}}\langle\phi, X_{t-\sigma}^{J,\sigma}\rangle > \frac{1}{6}e^{an}\right) \leq 6e^{-an}\mathbb{Q}_\mu\left(\sum_{\substack{\sigma\in\mathcal{D}^J\cap[0,t]\\ m_\sigma\phi(\xi_\sigma)\leq e^{bn}}}\langle\phi, X_{t-\sigma}^{J,\sigma}\rangle\right)\\ &= 6e^{-an}\widetilde{\Pi}_{\phi\mu}\left[\int_0^t e^{\lambda(t-s)}\phi^{-1}(\xi_s)ds\int_0^{e^{bn}} r^2\pi^\phi(\xi_s, dr)\right]\\ &= 6e^{-an}\widetilde{\Pi}_{\phi\mu}\int_0^t e^{\lambda(t-s)}\phi^{-1}(\xi_s)ds\left(\int_0^1 r^2\pi^\phi(\xi_s, dr) + \int_1^{e^{bn}} r^2\pi^\phi(\xi_s, dr)\right)\\ &\leq 6e^{-(a-b)n}\widetilde{\Pi}_{\phi\mu}\int_0^t e^{\lambda(t-s)}\phi^{-1}(\xi_s)ds\int_0^\infty (r\wedge r^2)\pi^\phi(\xi_s, dr).\end{aligned} \tag{3.27}$$

Thus by (1.11) and (3.27),

$$\mathbb{Q}_\mu\left(\sum_{\substack{\sigma\in\mathcal{D}^J\cap[0,t]\\ m_\sigma\phi(\xi_\sigma)\leq e^{bn}}}\langle\phi, X_{t-\sigma}^{J,\sigma}\rangle > \frac{1}{6}e^{an}\right) \leq \frac{6C}{\lambda}e^{\lambda t-(a-b)n} := Ae^{\lambda t-(a-b)n}, \tag{3.28}$$

where $A = \frac{6C}{\lambda}$ and C is the constant in (1.11). It is obvious that A is independent of μ, t, a and b. When the event $\left\{\sum_{\substack{\sigma\in\mathcal{D}^J\cap[0,t]\\ m_\sigma\phi(\xi_\sigma)>e^{bn}}}\langle\phi, X_{t-\sigma}^{J,\sigma}\rangle > \frac{1}{6}e^{an}\right\}$ occurs, $\sharp\{\sigma \in \mathcal{D}^J\cap[0, t]; m_\sigma\phi(\xi_\sigma) >$

$e^{bn}\} \geq 1$. Thus

$$\mathbb{Q}_\mu\left(\sum_{\substack{\sigma\in\mathcal{D}^J\cap[0,t]\\ m_\sigma\phi(\xi_\sigma)>e^{bn}}}\langle\phi, X^{J,\sigma}_{t-\sigma}\rangle > \frac{1}{6}e^{an}\right) \leq \mathbb{Q}_\mu\left(\sum_{\substack{\sigma\in\mathcal{D}^J\cap[0,t]\\ m_\sigma\phi(\xi_\sigma)>e^{bn}}} 1 \geq 1\right)$$
$$\leq \mathbb{Q}_\mu\left(\sum_{\substack{\sigma\in\mathcal{D}^J\cap[0,t]\\ m_\sigma\phi(\xi_\sigma)>e^{bn}}} 1\right) = \widetilde{\Pi}_{\phi\mu}\left[\int_0^t ds\int_{e^{bn}}^\infty r\pi^\phi(\xi_s, dr)\right].$$

When $e^{bn} \geq T_0$, by (1.16), we have that $\int_{e^{bn}}^\infty r\pi^\phi(\xi_s, dr) \leq B\phi(\xi_s)\int_E \nu(dy)\int_{e^{bn}}^\infty r\pi^\phi(y, dr)$ for some constant $B > 0$, and thus,

$$\widetilde{\Pi}_{\phi\mu}\left[\int_0^t ds\int_{e^{bn}}^\infty r\pi^\phi(\xi_s, dr)\right] \leq B\widetilde{\Pi}_{\phi\mu}\left[\int_0^t \phi(\xi_s)ds\int_E \nu(dy)\int_{e^{bn}}^\infty r\pi^\phi(y, dr)\right].$$

Therefore,

$$\mathbb{Q}_\mu\left(\sum_{\substack{\sigma\in\mathcal{D}^J\cap[0,t]\\ m_\sigma\phi(\xi_\sigma)>e^{bn}}}\langle\phi, X^{\mathrm{m},\sigma}_{t-\sigma}\rangle > \frac{1}{6}e^{an}\right) \leq Bt\|\phi\|_\infty\int_E \nu(dy)\int_{e^{bn}}^\infty r\pi^\phi(y, dr). \tag{3.29}$$

Put $K = \dfrac{6\|\alpha\phi\|_\infty}{\lambda} + A + B\|\phi\|_\infty$, which is independent of μ, t, a and b. Combining (3.25), (3.26), (3.28) and (3.29), we obtain

$$\widetilde{\mathbb{P}}_\mu(\langle\phi, X_t\rangle > e^{an}) \leq 3\langle\phi,\mu\rangle e^{\lambda t-an} + Ke^{\lambda t-(a-b)n} + Kt\int_E \nu(dy)\int_{e^{bn}}^\infty r\pi^\phi(y, dr). \quad \square$$

3.2. Proofs of main results

In this subsection, we give the proofs of our main results.

Proof of Theorem 1.4. (1) Suppose (1.15) holds. Using Lemma 3.4(1) with $1 < a < p \leq 2$, $(\widetilde{A}_t(a) - \widetilde{A}_1(a))$ converges in $L^p(\mathbb{P}_\mu)$ and $\mathbb{P}_\mu$-almost surely as $t \to \infty$. Then by Lemma 3.1(2), $(A_t(a^*) - A_1(a^*))$ converges in $L^p(\mathbb{P}_\mu)$ as $t \to \infty$.

(2) Suppose that for some $\mu \in \mathcal{M}^0(E)$, $(A_t(a^*) - A_1(a^*))$ converges in $L^p(\mathbb{P}_\mu)$ as $t \to \infty$. By Lemma 3.1(2), $(\widetilde{A}_t(a) - \widetilde{A}_1(a))$ converges in $L^p(\mathbb{P}_\mu)$ as $t \to \infty$. Applying Lemma 3.4(2), we get that it converges $\mathbb{P}_\mu$-almost surely as $t \to \infty$ and (1.15) holds. It now follows from Lemma 3.1(1) that $(A_t(a^*) - A_1(a^*))$ converges $\mathbb{P}_\mu$-almost surely as $t \to \infty$.

(3) According to (1), $(\widetilde{A}_t(a^*) - \widetilde{A}_1(a^*))$ converges in $L^p(\mathbb{P}_\mu)$ as $t \to \infty$. Repeating the argument leading to (3.11), we get

$$\lim_{t\to\infty}\mathbb{P}_\mu\left|e^{\frac{\lambda t}{a^*}}\left(M_\infty(\phi) - M_t(\phi)\right)\right|^p = 0.$$

Thus the assertion of (3) holds.

(4) This is the result of Lemma 3.4(2) with $a = 1$. $\square$

Proof of Theorem 1.5. (1) Suppose (1.15) holds and $\mu \in \mathcal{M}^0(E)$. Since $(\widetilde{A}_t(p))$ is a martingale, it converges $\mathbb{P}_\mu$-almost surely as $t \to \infty$ if it is uniformly integrable. Note that

$$\widetilde{A}_t(p) = \int_0^1 e^{\frac{-\lambda s}{p}} \int_E \phi(x) M(ds, dx) + \int_1^t e^{\frac{-\lambda s}{p}} \int_E \phi(x) M(ds, dx), \qquad t \geq 1.$$

We only need to consider the convergence of

$$\int_1^t e^{\frac{-\lambda s}{p}} \int_E \phi(x) M(ds, dx)$$

as $t \to \infty$.

For the "small jumps" part, we have, for $t > 0$,

$$\begin{aligned}
&\mathbb{P}_\mu \left[\Big(\int_1^t e^{-\frac{\lambda}{p} s} \int_E \phi(x) S^{(1,p)}(ds, dx) \Big)^2 \right] \\
&\lesssim \int_1^t e^{\frac{-2\lambda}{p} s} ds \int_E P_s^\beta \Big(\int_0^{e^{\frac{\lambda}{p} s}} r^2 \pi^\phi(\cdot, dr) \Big)(y) \mu(dy) \\
&\lesssim \mu(\phi) \int_1^\infty e^{(1-\frac{2}{p})\lambda s} ds \int_E \nu(dx) \int_0^{e^{\frac{\lambda}{p} s}} r^2 \pi^\phi(x, dr) \\
&\leq \int_1^\infty e^{\lambda s(1-\frac{2}{p})} ds \int_E \nu(dx) \int_0^1 r^2 \pi^\phi(x, dr) \\
&\quad + \int_1^\infty e^{\lambda s(1-\frac{2}{p})} ds \int_E \nu(dx) \int_1^{e^{\frac{\lambda}{p} s}} r^2 \pi^\phi(x, dr) \\
&= I + II,
\end{aligned}$$

where in the second inequality we used Assumption 2. Since $p < 2$, we have $I < \infty$. When (1.15) holds, by Fubini's theorem, we get

$$\begin{aligned}
II &\leq \int_E \nu(dx) \int_1^\infty r^2 \pi^\phi(x, dr) \int_{\frac{p}{\lambda} \ln r}^\infty e^{\lambda s(1-\frac{2}{p})} ds \\
&\lesssim \int_E \nu(dx) \int_1^\infty r^p \pi^\phi(x, dr) < \infty.
\end{aligned}$$

It follows that

$$\sup_{t>1} \mathbb{P}_\mu \left[\Big(\int_1^t e^{-\frac{\lambda}{p} s} \int_E \phi(x) S^{(1,p)}(ds, dx) \Big)^2 \right] < \infty. \tag{3.30}$$

For the "big jumps" part, by (2.14),

$$\begin{aligned}
&\Big| \int_1^t e^{-\frac{\lambda}{p} s} \int_E \phi(x) S^{(2,p)}(ds, dx) \Big| \\
&\leq 2 \int_1^t ds \int_E X_s(dx) \int_{e^{\lambda s/p}}^t F_{e^{-\frac{\lambda}{p} \cdot} \phi}(s, r\phi(x)^{-1} \delta_x) \pi^\phi(x, dr).
\end{aligned}$$

Then using Assumption 2 and Fubini's theorem again, we get

$$\begin{aligned}
&\mathbb{P}_\mu \sup_{t>1} \Big| \int_1^t e^{-\frac{\lambda}{p} s} \int_E \phi(x) S^{(2,p)}(ds, dx) \Big| \\
&\leq 2 \mathbb{P}_\mu \int_1^\infty ds \int_E X_s(dx) \int_{e^{\lambda s/p}}^\infty F_{e^{-\frac{\lambda}{p} \cdot} \phi}(s, r\phi(x)^{-1} \delta_x) \pi^\phi(x, dr)
\end{aligned}$$

$$\lesssim \int_1^\infty e^{\frac{\lambda}{q}s} ds \int_E \nu(dx) \int_{e^{\frac{\lambda}{p}s}}^\infty r\pi^\phi(x, dr)$$
$$\lesssim \int_E \nu(dx) \int_1^\infty r^p \pi^\phi(x, dr) < \infty. \tag{3.31}$$

For the continuum part, we have the following estimates:

$$\sup_{t>1} \mathbb{P}_\mu \left[\left(\int_1^t e^{-\frac{\lambda}{p}s} \int_E \phi(x) S^C(ds, dx) \right)^2 \right] = \mathbb{P}_\mu \int_1^\infty e^{-\frac{2\lambda}{p}s} ds \int_E \alpha(x)\phi(x)^2 X_s(dx)$$
$$\lesssim \int_1^\infty e^{-\lambda s(2/p-1)} ds \int_E \alpha(x)\phi(x)^2 \nu(dx).$$

Since $p < 2$,

$$\sup_{t>1} \mathbb{P}_\mu \left[\left(\int_1^t e^{-\frac{\lambda}{p}s} \int_E \phi(x) S^C(ds, dx) \right)^2 \right] < \infty. \tag{3.32}$$

Combining (3.30)–(3.32), we obtain that $\int_1^t e^{\frac{-\lambda s}{p}} \int_E \phi(x) M(ds, dx)$ is uniformly integrable. Thus $\widetilde{A}_t(p)$ converges $\mathbb{P}_\mu$-almost surely as $t \to \infty$. By Lemma 3.1, $A_t(q)$ converges $\mathbb{P}_\mu$-almost surely as $t \to \infty$ and

$$M_\infty(\phi) - M_t(\phi) = o\left(e^{-\frac{\lambda}{q}t}\right), \qquad \mathbb{P}_\mu\text{-a.s. as } t \to \infty.$$

(2) Now we suppose

$$\int_E \nu(dy) \int_1^\infty r^p \pi^\phi(y, dr) = \infty \tag{3.33}$$

and $\mu \in \mathcal{M}^0(E)$. By Assumption 2, there is $t_0 > 0$ such that for any $f \in L_1^+(\nu)$ and $t > t_0$,

$$P_t^\beta f(x) \ge \frac{1}{2} e^{\lambda t} \phi(x) \nu(f), \quad x \in E. \tag{3.34}$$

Without loss of generality, we assume $t_0 = 1/2$. Set $\rho_t = e^{\lambda t/q}(M_\infty(\phi) - M_t(\phi))$, $t > 0$. For any $n \in \mathbb{N}$ and $1/2 \le t \le 1$, note that

$$\Delta \rho_{n+t} = -e^{-\lambda(n+t)/p} \Delta \overline{X}_{n+t}(\phi),$$

and thus $\Delta \overline{X}_{n+t}(\phi) > 2e^{\lambda n/p}$ implies that

$$|\rho_{n+t}| > e^{-\lambda/p} \quad \text{or} \quad \left|\rho_{(n+t)-}\right| > e^{-\lambda/p}.$$

Define

$$B_n = \left\{ \sum_{\frac{1}{2} \le t < 1} 1_{\{\Delta \overline{X}_{n+t}(\phi) > 2e^{\lambda n/p}\}} > 0 \right\}.$$

Then we have $\{B_n, \text{i.o.}\}$ implies $\rho_t = o(1)$ does not hold as $t \to \infty$. Therefore, we only need to prove $\mathbb{P}_\mu(B_n, \text{i.o.}) > 0$. If we can prove that

$$\sum_{n=1}^\infty \mathbb{P}_\mu\left(B_n \middle| \mathcal{F}_n\right) = \infty, \qquad \text{a.s. on } \{M_\infty(\phi) > 0\}, \tag{3.35}$$

then by the second conditional Borel–Cantelli lemma (see, [9, Theorem 5.3.2]),

$$\mathbb{P}_\mu(B_n, \text{i.o.}) = \mathbb{P}_\mu\Big(\sum_{n=1}^{\infty} \mathbb{P}_\mu\left(B_n \Big| \mathcal{F}_n\right) = \infty\Big) \geq \mathbb{P}_\mu\Big(M_\infty(\phi) > 0\Big) > 0.$$

Therefore, we only need to prove (3.35).

To prove (3.35), we will estimate the probability $\mathbf{P}(Y > 0)$ for the non-negative random variable $Y := \sum_{n+\frac{1}{2}\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}}$ defined on some probability space with probability $\mathbf{P}$. Our basic idea is to use the inequality $\mathbf{P}(Y > 0) \geq \frac{(\mathbf{P}Y)^2}{\mathbf{P}Y^2}$, which follows trivially from the Cauchy–Schwarz inequality. However, Y may not have second moment. Thus we consider $\sum_{n+1/2\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}\bigcap C_n^A(t)}$ for some appropriate events $C_n^A(t)$. We will prove (3.35) in 4 steps.

Step 1. We first prove that

$$\sum_{n=1}^{\infty} \mathbb{P}_\mu\Big(\sum_{n+\frac{1}{2}\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}} \Big| \mathcal{F}_n\Big) = \infty \quad \text{a.s. on} \quad \{M_\infty(\phi) > 0\}. \tag{3.36}$$

Using (3.34) with $t_0 = 1/2$, we have

$$\begin{aligned} &\mathbb{P}_\mu\Big(\sum_{n+\frac{1}{2}\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}} \Big| \mathcal{F}_n\Big) = \Big\langle \int_{\frac{1}{2}}^{1} ds\, P_s^\beta\Big(\int_{2e^{\lambda n/p}}^{\infty} \pi^\phi(\cdot, dr)\Big), X_n\Big\rangle \\ &\gtrsim \langle \phi, X_n\rangle \int_E \nu(dx) \int_{2e^{\lambda n/p}}^{\infty} \pi^\phi(x, dr). \end{aligned} \tag{3.37}$$

Therefore,

$$\sum_{n=1}^{\infty} \mathbb{P}_\mu\Big(\sum_{n+\frac{1}{2}\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}} \Big| \mathcal{F}_n\Big) \gtrsim \sum_{n=1}^{\infty} \langle \phi, X_n\rangle \int_E \nu(dx) \int_{2e^{\lambda n/p}}^{\infty} \pi^\phi(x, dr). \tag{3.38}$$

Since $\lim_{n\to\infty} e^{-\lambda n}\langle \phi, X_n\rangle = M_\infty(\phi)$ almost surely, on the event $\{M_\infty(\phi) > 0\}$, the convergence of the right-hand side of (3.38) is equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} e^{\lambda n} \int_E \nu(dx) \int_{2e^{\lambda n/p}}^{\infty} \pi^\phi(x, dr).$$

Since $e^{\lambda s}$ is increasing and $\int_{2e^{\lambda s/p}}^{\infty} \pi^\phi(x, dr)$ is decreasing in s, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} e^{\lambda n} \int_E \nu(dx) \int_{2e^{\lambda n/p}}^{\infty} \pi^\phi(x, dr) \\ &\geq \int_0^\infty e^{\lambda s} ds \int_E \nu(dx) \int_{2e^{\lambda/p} e^{\lambda s/p}}^{\infty} \pi^\phi(x, dr) \\ &= \int_E \nu(dx) \int_{2e^{\lambda/p}}^{\infty} \pi^\phi(x, dr) \int_0^{\frac{p}{\lambda}\ln r - \frac{p}{\lambda}\ln(2e^{\lambda/p})} e^{\lambda s} ds \\ &\gtrsim \int_E \nu(dx) \int_{2e^{\lambda/p}}^{\infty} r^p \pi^\phi(x, dr). \end{aligned}$$

Therefore (3.36) follows from (3.33).

Step 2. Suppose $A > 0$ is an arbitrary fixed constant. For any integer $n \geq 1$, define

$$C_n^A(t) = \left\{ \frac{\langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \overline{X}_{t-} \rangle}{e^{\lambda n} \langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \nu \rangle} < L \right\}, \quad n + \frac{1}{2} \leq t < n+1, \tag{3.39}$$

and

$$C_n^{>}(t) = \left\{ \frac{\langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \overline{X}_{t-} \rangle}{e^{\lambda n} \langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \nu \rangle} > L/2 \right\}, \quad \frac{1}{2} \leq t < 1, \tag{3.40}$$

where L is chosen large enough so that for any $\mu \in \mathcal{M}^0(E)$ satisfying $\langle \phi, \mu \rangle < A$,

$$\mathbb{P}_\mu \left((C_n^A(t))^c \right) < \frac{1}{4}, \quad n \geq 1, \; n + \frac{1}{2} \leq t < n, \tag{3.41}$$

and for any $\mu \in \mathcal{M}^0(E)$ satisfying $\langle \phi, \mu \rangle < A e^{\lambda n}$,

$$\mathbb{P}_\mu \left(C_n^{>}(t) \right) < \frac{1}{2}, \quad n \geq 1, \; t \in [\frac{1}{2}, 1]. \tag{3.42}$$

The existence of such an L is guaranteed by

$$\begin{aligned} &\mathbb{P}_\mu \left(\frac{\langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \overline{X}_{t-} \rangle}{e^{\lambda n} \langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \nu \rangle} > L \right) \leq \frac{\mathbb{P}_\mu \left(\langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \overline{X}_{t-} \rangle \right)}{L e^{\lambda n} \langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \nu \rangle} \\ &\leq \frac{(1+c_t) e^{\lambda t} \langle \phi, \mu \rangle}{L e^{\lambda n}} \leq \frac{A e^{\lambda} (1 + c_t)}{L}, \end{aligned}$$

if $n + \frac{1}{2} \leq t < n+1$ and $\langle \phi, \mu \rangle < A$, or if $\frac{1}{2} \leq t < 1$ and $\langle \phi, \mu \rangle < A e^{\lambda n}$. The first inequality above is the Markov inequality, and c_t is the quantity in Assumption 2 which is bounded for $t > 1/2$. Thus L can be chosen large enough to assure both (3.41) and (3.42) hold. In this step, we will prove that there is $N \in \mathbb{N}$ such that when $n > N$, $\mathbb{P}_\mu$-almost surely on $\{M_n(\phi) \leq A\}$,

$$\begin{aligned} &\mathbb{P}_\mu \Big(\sum_{n+1/2 \leq t < n+1} 1_{\{\Delta \overline{X}_t(\phi) > 2e^{\lambda n/p}\} \bigcap C_n^A(t)} \Big| \mathcal{F}_n \Big) \\ &\geq \frac{1}{4} \mathbb{P}_\mu \left(\sum_{n+1/2 \leq t < n+1} 1_{\{\Delta \overline{X}_t(\phi) > 2e^{\lambda n/p}\}} \middle| \mathcal{F}_n \right). \end{aligned} \tag{3.43}$$

We divide X_n into $[e^{\lambda n}]$ disjoint parts each with value $[e^{\lambda n}]^{-1} X_n$. For $i = 1, 2, \ldots, [e^{\lambda n}]$, let $(X_s^{(i)}, 0 \leq s < 1)$ be the superprocess with the ith part as its initial mass. By the branching property of superprocesses, $X_s^{(i)}$, $i = 1, 2, \ldots, [e^{\lambda n}]$, are independent and identically distributed as $\mathbb{P}_{[e^{\lambda n}]^{-1} X_n}$ under $\mathbb{P}_\mu \left(\cdot \middle| \mathcal{F}_n \right) = \mathbb{P}_{X_n}(\cdot)$. Thus for any $i = 1, 2, \ldots, [e^{\lambda n}]$,

$$\begin{aligned} (C_n^A(s))^c &= \left\{ \frac{\langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \overline{X}_{s-} \rangle}{e^{\lambda n} \langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \nu \rangle} > L \right\} \\ &\subset \left\{ \frac{\langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \overline{X}_{s-}^{(i)} \rangle}{e^{\lambda n} \langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \nu \rangle} > L/2 \right\} \bigcup \left\{ \frac{\sum_{j \neq i} \langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \overline{X}_{s-}^{(j)} \rangle}{e^{\lambda n} \langle \int_{2e^{\lambda n/p}} \pi^\phi(\cdot, dr), \nu \rangle} > L/2 \right\} \\ &:= C_n^{(i)}(s) \bigcup C_n^{(\neq i)}(s), \quad n + \frac{1}{2} \leq s < n+1, n \geq 1. \end{aligned}$$

Consider the conditional expectation:

$$
\begin{aligned}
E_n &:= \mathbb{P}_\mu\Big(\sum_{n+\frac{1}{2}\le t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}\setminus C_n^A(t)} \Big| \mathcal{F}_n\Big) \\
&= \mathbb{P}_{X_n}\left(\int_{\frac{1}{2}}^1 1_{(C_n^A(s))^c} ds \int_E \overline{X}_{s-}(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x,dr)\right) \\
&= \sum_{i=1}^{[e^{\lambda n}]} \mathbb{P}_{X_n}\left(\int_{\frac{1}{2}}^1 1_{(C_n^A(s))^c} ds \int_E \overline{X}^{(i)}_{s-}(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x,dr)\right) \\
&\le \sum_{i=1}^{[e^{\lambda n}]} \mathbb{P}_{X_n}\left(\int_{\frac{1}{2}}^1 (1_{C_n^{(i)}(s)} + 1_{C_n^{(\ne i)}(s)}) ds \int_E \overline{X}^{(i)}_{s-}(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x,dr)\right) \\
&\le \sum_{i=1}^{[e^{\lambda n}]} \mathbb{P}_{X_n}\left(\int_{\frac{1}{2}}^1 1_{C_n^{(i)}(s)} ds \int_E \overline{X}^{(i)}_{s-}(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x,dr)\right) \\
&\qquad + \sum_{i=1}^{[e^{\lambda n}]} \mathbb{P}_{X_n}\left(\int_{\frac{1}{2}}^1 1_{C_n^{(\ne i)}(s)} ds \int_E \overline{X}^{(i)}_{s-}(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x,dr)\right) \\
&:= I_n^{(1)} + I_n^{(2)}.
\end{aligned}
$$

Since $X^{(i)}$ and $X^{(\ne i)} = \sum_{j\ne i} X^{(j)}$ are independent,

$$
\begin{aligned}
1_{\{M_n(\phi)\le A\}} I_n^{(2)} &= \sum_{i=1}^{[e^{\lambda n}]} I_{\{M_n(\phi)\le A\}} \mathbb{P}_{X_n}\left(\int_{\frac{1}{2}}^1 1_{C_n^{(\ne i)}(s)} ds \int_E \overline{X}^{(i)}_{s-}(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x,dr)\right) \\
&= \sum_{i=1}^{[e^{\lambda n}]} \int_{\frac{1}{2}}^1 ds 1_{\{M_n(\phi)\le A\}} \mathbb{P}_{X_n}\big(C_n^{(\ne i)}(s)\big) \mathbb{P}_{X_n}\left(\int_E \overline{X}^{(i)}_{s-}(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x,dr)\right) \\
&\le \sum_{i=1}^{[e^{\lambda n}]} \int_{\frac{1}{2}}^1 ds 1_{\{M_n(\phi)\le A\}} \mathbb{P}_{X_n}\big(C_n^{>}(s)\big) \mathbb{P}_{X_n}\left(\int_E \overline{X}^{(i)}_{s-}(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x,dr)\right).
\end{aligned}
$$

On the event $\{M_n(\phi) \le A\}$, we have $\langle \phi, X_n\rangle \le Ae^{\lambda n}$. Therefore, from the setting (3.42) of $C_n^{>}(s)$ for $n \ge 1$ and $1/2 \le s < 1$, it follows that

$$
\mathbb{P}_{X_n}\big(C_n^{>}(s)\big) \le 1/2.
$$

As a consequence,

$$
\begin{aligned}
1_{\{M_n(\phi)\le A\}} I_n^{(2)} &\le 1_{\{M_n(\phi)\le A\}} \frac{1}{2} \mathbb{P}_{X_n}\left(\int_{\frac{1}{2}}^1 ds \int_E \overline{X}_{s-}(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x,dr)\right) \\
&\le 1_{\{M_n(\phi)\le A\}} \frac{1}{2} \mathbb{P}_\mu\left(\sum_{n+\frac{1}{2}\le t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}} \Big| \mathcal{F}_n\right).
\end{aligned}
\tag{3.44}
$$

As for $I_n^{(1)}$, when $2e^{\frac{\lambda}{p}n} > T_0$, by assumption (1.16),

$$
\int_E \overline{X}^{(i)}_{s-}(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x,dr) \le B\langle \phi, \overline{X}^{(i)}_{s-}\rangle \int_E \nu(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x,dr), \quad s \in [\frac{1}{2}, 1).
$$

Therefore,

$$C_n^{(i)}(s) \subset \{\langle \phi, \overline{X}_{s-}^{(i)} \rangle \geq \frac{L}{2B} e^{\lambda n}\}, \quad s \in [\frac{1}{2}, 1)$$

and

$$\begin{aligned} I_n^{(1)} &= \sum_{i=1}^{[e^{\lambda n}]} \mathbb{P}_{X_n} \left(\int_{\frac{1}{2}}^1 1_{C_n^{(i)}(s)} ds \int_E \overline{X}_{s-}^{(i)}(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x, dr) \right) \\ &\leq B e^{\lambda n} \int_E \nu(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x, dr) \mathbb{P}_{[e^{\lambda n}]^{-1} X_n} \int_{\frac{1}{2}}^1 \langle \phi, \overline{X}_{s-} \rangle 1_{\{\langle \phi, \overline{X}_{s-} \rangle \geq \frac{L}{2B} e^{\lambda n}\}} ds \\ &= B e^{\lambda n} \int_E \nu(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x, dr) \int_{\frac{1}{2}}^1 \mathbb{P}_{[e^{\lambda n}]^{-1} X_n} \langle \phi, X_s \rangle 1_{\{\langle \phi, X_s \rangle \geq \frac{L}{2B} e^{\lambda n}\}} ds \\ &= B e^{\lambda n} [e^{\lambda n}]^{-1} X_n(\phi) \int_E \nu(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x, dr) \int_{\frac{1}{2}}^1 e^{\lambda s} ds \widetilde{\mathbb{P}}_{[e^{\lambda n}]^{-1} X_n} \left(\langle \phi, X_s \rangle \geq \frac{L}{2B} e^{\lambda n} \right). \end{aligned}$$

Note that we may choose L large enough that $\frac{L}{2B} \geq 1$. From Lemma 3.5, for any $0 < b < \lambda$, and any $1/2 < s < 1$, on the set $\{M_n(\phi) \leq A\}$, there is a constant $\widetilde{K} > 0$ such that almost surely

$$\begin{aligned} &\widetilde{\mathbb{P}}_{[e^{\lambda n}]^{-1} X_n} \left(\langle \phi, X_s \rangle \geq \frac{L}{2B} e^{\lambda n} \right) \leq \widetilde{\mathbb{P}}_{[e^{\lambda n}]^{-1} X_n} \left(\langle \phi, X_s \rangle \geq e^{\lambda n} \right) \\ &\leq 3 [e^{\lambda n}]^{-1} \langle \phi, X_n \rangle e^{\lambda s - \lambda n} + K e^{\lambda s - (\lambda - b) n} + K s \int_E \nu(dy) \int_{e^{bn}}^\infty r \pi^\phi(y, dr) \\ &\leq \widetilde{K} e^{-(\lambda - b) n} + \widetilde{K} \int_E \nu(dy) \int_{e^{bn}}^\infty r \pi^\phi(y, dr). \end{aligned}$$

Therefore,

$$\begin{aligned} &1_{\{M_n(\phi) \leq A\}} I_n^{(1)} \\ &\lesssim 1_{\{M_n(\phi) \leq A\}} X_n(\phi) \left(\int_E \nu(dx) \int_{2e^{\frac{\lambda}{p}n}}^\infty \pi^\phi(x, dr) \right) \left[e^{-(\lambda - b) n} + \int_E \nu(dy) \int_{e^{bn}}^\infty r \pi^\phi(y, dr) \right] \\ &\lesssim 1_{\{M_n(\phi) \leq A\}} \mathbb{P}_\mu \Big(\sum_{n + \frac{1}{2} \leq t < n+1} 1_{\{\triangle \overline{X}_t(\phi) > 2 e^{\lambda n / p}\}} \Big| \mathcal{F}_n \Big) \left[e^{-(\lambda - b) n} + \int_E \nu(dy) \int_{e^{bn}}^\infty r \pi^\phi(y, dr) \right], \end{aligned}$$

where the last inequality follows from (3.37). Since $\lim_{n \to \infty} e^{-(\lambda - b) n} + \int_E \nu(dy) \int_{e^{bn}}^\infty r \pi^\phi(y, dr) = 0$, we can choose $N > 0$ such that when $n \geq N$, we have $e^{bn} > T_0$ and

$$1_{\{M_n(\phi) \leq A\}} I_n^{(1)} \leq \frac{1}{4} 1_{\{M_n(\phi) \leq A\}} \mathbb{P}_\mu \Big(\sum_{n + \frac{1}{2} \leq t < n+1} 1_{\{\triangle \overline{X}_t(\phi) > 2 e^{\lambda n / p}\}} \Big| \mathcal{F}_n \Big), \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.45}$$

Combining (3.44) and (3.45), we get, when $n > N$, on $\{M_n(\phi) \leq A\}$,

$$E_n \leq I_n^{(1)} + I_n^{(2)} \leq \frac{3}{4} 1_{\{M_n(\phi) \leq A\}} \mathbb{P}_\mu \Big(\sum_{n + \frac{1}{2} \leq t < n+1} 1_{\{\triangle \overline{X}_t(\phi) > 2 e^{\lambda n / p}\}} \Big| \mathcal{F}_n \Big), \quad \mathbb{P}_\mu\text{-a.s.}$$

Therefore, when $n > N$, on $\{M_n(\phi) \le A\}$,

$$\begin{aligned}
&\mathbb{P}_\mu\Big(\sum_{n+1/2\le t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}\bigcap C_n^A(t)}\Big|\mathcal{F}_n\Big)\\
&\ge \mathbb{P}_\mu\Big(\sum_{n+1/2\le t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}}\Big|\mathcal{F}_n\Big) - E_n\\
&\ge \frac{1}{4}\mathbb{P}_\mu\left(\sum_{n+1/2\le t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}}\Big|\mathcal{F}_n\right), \quad \mathbb{P}_\mu\text{-a.s.}
\end{aligned}$$

This proves (3.43).

Step 3. In this step, we prove that, on $\{M_\infty(\phi) > 0, \sup_n M_n(\phi) \le A\}$,

$$\sum_{n=1}^\infty \mathbb{P}_\mu\left(\sum_{n+1/2\le t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\}\bigcap C_n^A(t)} > 0\Big|\mathcal{F}_n\right) = \infty, \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.46}$$

Let N be a number large enough so that (3.43) almost surely holds on $\{M_n(\phi) \le A\}$ for any $n \ge N$. Then on the event $\{M_\infty(\phi) > 0, \sup_n M_n(\phi) \le A\}$,

$$\begin{aligned}
&\sum_{n=N}^m 1_{\{M_n(\phi)\le A\}}\mathbb{P}_\mu\Big(\sum_{n+1/2\le t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}\bigcap C_n^A(t)}\Big|\mathcal{F}_n\Big)\\
&\ge \frac{1}{4}\sum_{n=N}^m 1_{\{M_n(\phi)\le A\}}\mathbb{P}_\mu\left(\sum_{n+1/2\le t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}}\Big|\mathcal{F}_n\right)\\
&= \frac{1}{4}\sum_{n=N}^m \mathbb{P}_\mu\left(\sum_{n+1/2\le t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}}\Big|\mathcal{F}_n\right), \quad \mathbb{P}_\mu\text{-a.s.}
\end{aligned}$$

By (3.36), letting $m \to \infty$ in the display above, we get that, on $\{\sup_{n>1} M_n(\phi) \le A, M_\infty(\phi) > 0\}$,

$$\sum_{n=1}^\infty \mathbb{P}_\mu\Big(\sum_{n+1/2\le t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\lambda n/p}\}\bigcap C_n^A(t)}\Big|\mathcal{F}_n\Big) = \infty, \quad \mathbb{P}_\mu\text{-a.s.}$$

Let $\widetilde{C}_n(t) := \left\{\triangle \overline{X}_t(\phi) > 2e^{\frac{\lambda n}{p}}\right\}\bigcap C_n^A(t)$. Now we consider the second moments:

$$\begin{aligned}
&\mathbb{P}_\mu\left[\Big(\sum_{n+1/2\le t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\}\bigcap C_n^A(t)}\Big)^2\Big|\mathcal{F}_n\right]\\
&= 2\mathbb{P}_\mu\Big[\sum_{n+1/2\le t_1<t_2<n+1} 1_{\widetilde{C}_n(t_1)}1_{\widetilde{C}_n(t_2)}\Big|\mathcal{F}_n\Big] + \mathbb{P}_\mu\Big[\sum_{n+1/2\le t<n+1} 1_{\left\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\right\}\bigcap C_n^A(t)}\Big|\mathcal{F}_n\Big].
\end{aligned}$$

Define $I_n^{(3)} := 2\mathbb{P}_\mu\Big[\sum_{n+1/2\le t_1<t_2<n+1} 1_{\widetilde{C}_n(t_1)}1_{\widetilde{C}_n(t_2)}\Big|\mathcal{F}_n\Big]$, then

$$\begin{aligned}
I_n^{(3)} &= 2\mathbb{P}_\mu\Big[\sum_{n+1/2\le t_1<n+1} 1_{\widetilde{C}_n(t_1)}\mathbb{P}_\mu\left(\sum_{t_1<t_2<n+1} 1_{\widetilde{C}_n(t_2)}\Big|\mathcal{F}_{t_1}\right)\Big|\mathcal{F}_n\Big]\\
&\le 2\mathbb{P}_\mu\Big[\sum_{n+1/2\le t_1<n+1} 1_{\widetilde{C}_n(t_1)}\mathbb{P}_{X_{t_1}}\left(\int_{t_1}^{n+1} 1_{C_n^A(s)}ds\int_E \overline{X}_{s-}(dx)\int_{2e^{\lambda n/p}}^\infty \pi^\phi(x,dr)\right)\Big|\mathcal{F}_n\Big]
\end{aligned}$$

$$\leq 2Le^{\lambda n}\int_E \nu(dx)\int_{2e^{\lambda n/p}}^\infty \pi^\phi(x,dr)\cdot \mathbb{P}_\mu\Big[\sum_{n+1/2\leq t_1<n+1} 1_{\widetilde{C}_n(t_1)}\Big|\mathcal{F}_n\Big]$$
$$\lesssim e^{\lambda n}\langle \phi, X_n\rangle^{-1}\mathbb{P}_\mu^2\Big[\sum_{n+1/2\leq t_1<n+1} 1_{\left\{\triangle \overline{X}_{t_1}(\phi)>2e^{\frac{\lambda n}{p}}\right\}}\Big|\mathcal{F}_n\Big],$$

where the last inequality comes from (3.37) and L is the constant in the definition of $C_n^A(s)$, see (3.39). Consequently

$$\mathbb{P}_\mu\left[\Big(\sum_{n+1/2\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\}\bigcap C_n^A(t)}\Big)^2\Big|\mathcal{F}_n\right]$$
$$\lesssim \frac{1}{M_n(\phi)}\mathbb{P}_\mu^2\Big[\sum_{n+1/2\leq t<n+1} 1_{\left\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\right\}}\Big|\mathcal{F}_n\Big]$$
$$+\ \mathbb{P}_\mu\Big[\sum_{n+1/2\leq t<n+1} 1_{\left\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\right\}\bigcap C_n^A(t)}\Big|\mathcal{F}_n\Big]. \tag{3.47}$$

Now by the Cauchy–Schwarz inequality, when $n>N$, on the event $\{M_\infty(\phi)>0, \sup_n M_n(\phi) \leq A\}$,

$$\mathbb{P}_\mu\left(\sum_{n+1/2\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\}\bigcap C_n^A(t)}>0\Big|\mathcal{F}_n\right)$$
$$\geq \frac{\mathbb{P}_\mu^2\left(\sum_{n+1/2\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\}\bigcap C_n^A(t)}\Big|\mathcal{F}_n\right)}{\mathbb{P}_\mu\left(\left(\sum_{n+1/2\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\}\bigcap C_n^A(t)}\right)^2\Big|\mathcal{F}_n\right)}$$
$$\gtrsim \frac{\frac{1}{16}\mathbb{P}_\mu^2\left(\sum_{n+1/2\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\}}\Big|\mathcal{F}_n\right)}{M_n(\phi)^{-1}\mathbb{P}_\mu^2\Big[\sum_{n+1/2\leq t_1<n+1} 1_{\left\{\triangle \overline{X}_{t_1}(\phi)>2e^{\frac{\lambda n}{p}}\right\}}\Big|\mathcal{F}_n\Big]+\mathbb{P}_\mu\Big[\sum_{n+1/2\leq t_1<n+1} 1_{\left\{\triangle \overline{X}_{t_1}(\phi)>2e^{\frac{\lambda n}{p}}\right\}}\Big|\mathcal{F}_n\Big]}$$
$$\gtrsim \frac{\mathbb{P}_\mu\left(\sum_{n+1/2\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\}}\Big|\mathcal{F}_n\right)}{M_n(\phi)^{-1}\mathbb{P}_\mu\Big[\sum_{n+1/2\leq t_1<n+1} 1_{\left\{\triangle \overline{X}_{t_1}(\phi)>2e^{\frac{\lambda n}{p}}\right\}}\Big|\mathcal{F}_n\Big]+1}$$
$$\gtrsim M_n(\phi)\bigwedge \mathbb{P}_\mu\left(\sum_{n+1/2\leq t<n+1} 1_{\{\triangle \overline{X}_t(\phi)>2e^{\frac{\lambda n}{p}}\}}\Big|\mathcal{F}_n\right),\quad \mathbb{P}_\mu\text{-a.s.}$$

where the second inequality comes from (3.43) and (3.47), and the last inequality comes from the fact that $\frac{x}{y+1}\geq\frac{x}{2(y\wedge 1)}\gtrsim\frac{x}{y}\wedge x$ for any $x,y>0$. Since we are working on $\{M_\infty(\phi)>0\}$ and we have proved (3.36), (3.46) follows from the inequalities above.

Step 4. By the second conditional Borel–Cantelli lemma (see, [9, Theorem 5.3.2]), (3.46) implies that

$$\mathbb{P}_\mu\left(B_n \text{ i.o.}\Big|M_\infty(\phi)>0, \sup_n M_n(\phi)\leq A\right)=1.$$

Note that $\sup_{n\geq 1} M_n(\phi) < \infty$ $\mathbb{P}_\mu$-almost surely. The above equation holds for any constant $A > 0$. Letting $A \to \infty$, we get (3.35). Consequently,

$$\limsup_{t\to\infty} \left| e^{\lambda t/q} \left(M_\infty(\phi) - M_t(\phi)\right)\right| \geq e^{-\lambda/p}$$

with positive probability. The proof is complete. □

For $\gamma > 0$, define

$$f(s) = e^{\lambda s} s^{-\gamma}, s > 0. \tag{3.48}$$

Direct computation shows that

$$f'(s) = f(s)(\lambda - \gamma s^{-1}).$$

Thus when $s > \gamma/\lambda$, $f(s)$ is a strictly increasing function. If g is the inverse function of f on $(\gamma/\lambda, \infty)$, then

$$(g(r))' = \frac{1}{r(\lambda - \gamma g(r)^{-1})}.$$

It is obvious that

$$\lim_{r\to\infty} g(r) = \infty.$$

Therefore, there is a constant $R > \gamma/\lambda$ such that for $r > R$,

$$\frac{1}{\lambda r} \leq (g(r))' \leq \frac{2}{\lambda r}. \tag{3.49}$$

Consequently, when $r \to \infty$,

$$g(r) \asymp \ln r. \tag{3.50}$$

Proof of Theorem 1.6. (1) The main idea is similar to that of the proof of Theorem 1.5. We will use Lemma 3.2 and different truncating functions to analyze the convergency of $\widetilde{C}_t(\gamma)$. First, for the continuous part, by the Burkholder–Davis–Gundy inequality,

$$\begin{aligned}
&\mathbb{P}_\mu\left[\left(\sup_{t>1} \int_1^t e^{-\lambda s} s^\gamma \int_E \phi(x) S^C(ds, dx)\right)^2\right] \\
&\lesssim \sup_{t>1} \mathbb{P}_\mu \int_1^t e^{-2\lambda s} s^{2\gamma} ds \int_E \alpha(x)\phi(x)^2 X_s(dx) \\
&\lesssim \int_1^\infty e^{-\lambda s} s^{2\gamma} ds \int_E \alpha(x)\phi(x)^2 \nu(dx) < \infty.
\end{aligned} \tag{3.51}$$

For the jump part, we still handle the ‘small jumps’ and the ‘large jumps’ separately. Define

$$N^{(1)} := \sum_{0<\triangle \overline{X}_s(\phi)<e^{\lambda s}s^{-\gamma}} \delta_{(s,\triangle \overline{X}_s)} \quad \text{and} \quad N^{(2)} := \sum_{\triangle \overline{X}_s(\phi)\geq e^{\lambda s}s^{-\gamma}} \delta_{(s,\triangle \overline{X}_s)},$$

and denote the compensators of $N^{(1)}$ and $N^{(2)}$ by $\widehat{N}^{(1)}$ and $\widehat{N}^{(2)}$ respectively. We write $S^{(J,1)}$ and $S^{(J,2)}$ for the corresponding martingale measures. For the ‘large jumps’,

$$\begin{aligned}
&\mathbb{P}_\mu\left| \int_1^\infty e^{-\lambda s} s^\gamma \int_E \phi(x) S^{(J,2)}(ds, dx)\right| \\
&\leq 2\mathbb{P}_\mu \int_1^\infty e^{-\lambda s} s^\gamma ds \int_E X_s(dx) \int_{e^{\lambda s}s^{-\gamma}}^\infty r\pi^\phi(x, dr)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_1^\infty s^\gamma ds \int_E \nu(dx) \int_{e^{\lambda s}s^{-\gamma}}^\infty r\pi^\phi(x,dr)\\
&= \int_1^{R\vee 1} s^\gamma ds \int_E \nu(dx) \int_{e^{\lambda s}s^{-\gamma}}^\infty r\pi^\phi(x,dr) + \int_{R\vee 1}^\infty s^\gamma ds \int_E \nu(dx) \int_{e^{\lambda s}s^{-\gamma}}^\infty r\pi^\phi(x,dr)\\
&:= I + II,
\end{aligned}
$$

where $R > \gamma/\lambda$ is a number such that (3.49) holds for $r > R$. It is obvious that $I < \infty$, we only need to investigate the finiteness of II. Recall that f is defined by (3.48) and g is the inverse of f on $(\gamma/\lambda, \infty)$. Applying Fubini's Theorem,

$$
II \le \int_E \nu(dx) \int_{f(\gamma/\lambda)}^\infty r\pi^\phi(x,dr) \int_0^{g(r)} s^\gamma ds
$$

It follows from (3.50) that

$$
\int_0^{g(r)} s^\gamma ds = \frac{g(r)^{\gamma+1}}{\gamma+1} \asymp (\ln r)^{\gamma+1} \qquad \text{for } r > R.
$$

Thus when (1.18) holds, we have

$$
\sup_{t>1} \mathbb{P}_\mu \left| \int_1^t e^{-\lambda s} s^\gamma \int_E \phi(x) S^{(J,2)}(ds,dx) \right| < \infty.
$$

Therefore the process $\int_1^t e^{-\lambda s} s^\gamma \int_E \phi(x) S^{(J,2)}(ds,dx)$ converges $\mathbb{P}_\mu$-a.s. and in $L^1(\mathbb{P}_\mu)$. Now let us analyze the 'small jumps' part.

$$
\begin{aligned}
&\mathbb{P}_\mu \left[\left(\sup_{t>1} \int_1^t e^{-\lambda s} s^\gamma \int_E \phi(x) S^{(J,1)}(ds,dx) \right)^2 \right]\\
&= \int_1^\infty e^{-2\lambda s} s^{2\gamma} ds \int_E P_s^\beta \Big(\int_0^{f(s)} r^2 \pi^\phi(\cdot, dr) \Big)(y) \mu(dy)\\
&\lesssim \int_1^\infty s^\gamma / f(s) ds \int_E \nu(dx) \int_0^{f(s)} r^2 \pi^\phi(x,dr)\\
&\le \int_1^\infty s^\gamma / f(s) ds \int_E \nu(dx) \int_0^1 r^2 \pi^\phi(x,dr)\\
&\quad + \int_1^{1\vee R} s^\gamma / f(s) ds \int_E \nu(dx) \int_1^{1\vee f(s)} r^2 \pi^\phi(x,dr)\\
&\quad + \int_{1\vee R}^\infty s^\gamma / f(s) ds \int_E \nu(dx) \int_1^{1\vee f(s)} r^2 \pi^\phi(x,dr)\\
&:= III + IV + V.
\end{aligned}
$$

It is easy to check that both III and IV are finite. Applying Fubini's theorem in V, we get

$$
V \le \int_E \nu(dx) \int_1^\infty r^2 \pi^\phi(x,dr) \int_{g(r)}^\infty s^\gamma / f(s) ds.
$$

Let $H(r) = \int_{g(r)}^\infty s^\gamma / f(s) ds$, then $\lim_{r\to\infty} H(r) = 0$. Note that as $r \to \infty$,

$$
H'(r) = \frac{g(r)^\gamma g'(r)}{f(g(r))} \asymp \frac{(\ln r)^\gamma}{r^2}.
$$

Thus $H(r) \asymp \dfrac{(\ln r)^\gamma}{r}$ as $r \to \infty$. Therefore, $V < \infty$ when (1.18) holds. Hence it follows that the martingale $\int_1^t e^{-\lambda s} s^\gamma \int_E \phi(x) S^{(J,1)}(ds,dx)$ converges $\mathbb{P}_\mu$-a.s. and in $L^2(\mathbb{P}_\mu)$ as $t \to \infty$. In conclusion, when the moment condition (1.18) holds, the martingale $\widetilde{C}_t(\gamma)$ converges $\mathbb{P}_\mu$-almost surely and in $L^1(\mathbb{P}_\mu)$ as $t \to \infty$. It follows from Lemma 3.2 that

$$\int_0^t s^{\gamma-1}(M_\infty(\phi) - M_s(\phi))ds \quad \text{converges} \quad \mathbb{P}_\mu\text{-a.s.}$$

and $M_\infty(\phi) - M_t(\phi) = o(t^{-\gamma})$, $\mathbb{P}_\mu$-a.s. as $t \to \infty$. In particular, when $\gamma \geq 1$,

$$\int_0^\infty (M_\infty(\phi) - M_t(\phi))dt < \infty, \qquad \mathbb{P}_\mu\text{-a.s.}$$

(2) Now let us consider the case that $\int_E \nu(dx) \int_1^\infty r(\ln r)^{1+\gamma}\pi^\phi(x, dr) = \infty$. Without loss of generality, we may assume that

$$\int_E \nu(dx) \int_1^\infty r(\ln r)^\gamma \pi^\phi(x, dr) < \infty. \tag{3.52}$$

In fact, if

$$\int_E \nu(dx) \int_1^\infty r(\ln r)^\gamma \pi^\phi(x, dr) = \infty,$$

then by assumption (1.13), $\gamma > 1$. Therefore there is some $\tilde{\gamma} > 0$ and some integer $n > 0$ such that $\gamma = n + \tilde{\gamma}$,

$$\int_E \nu(dx) \int_1^\infty r(\ln r)^{1+\tilde{\gamma}} \pi^\phi(x, dr) = \infty \tag{3.53}$$

and

$$\int_E \nu(dx) \int_1^\infty r(\ln r)^{\tilde{\gamma}} \pi^\phi(x, dr) < \infty.$$

If we can prove that $C_t(\widetilde{\gamma})$ does not converge as $t \to \infty$, then $C_t(\gamma)$ does not converge either.

Let $\widehat{N}^{(2)}$ be the compensator of $N^{(2)}$. Then for any non-negative Borel function F on $\mathbb{R}_+ \times \mathcal{M}(E_\partial)$,

$$\int_0^\infty \int_{\mathcal{M}(E_\partial)} F(s, \upsilon)\widehat{N}^{(2)}(ds, d\upsilon) = \int_0^\infty ds \int_E X_s(dx) \int_{f(s)}^\infty F(s, r\phi(x)^{-1}\delta_x)\pi^\phi(x, dr).$$

Define a measure $L(ds, dx)$ on $[0, \infty) \times E$ such that for any non-negative Borel function g on $\mathbb{R}_+ \times E$,

$$\int_0^\infty \int_E g(s, x)L(ds, dx) = \int_0^\infty \int_{\mathcal{M}(E_\partial)} F_g(s, \upsilon)\widehat{N}^{(2)}(ds, d\upsilon),$$

which is equivalent to

$$\int_0^\infty \int_E g(s, x)L(ds, dx) = \int_0^\infty ds \int_E \phi^{-1}(x)X_s(dx) \int_{f(s)}^\infty rg(s, x)\pi^\phi(x, dr).$$

Suppose $\mu \in \mathcal{M}^0(E)$. We claim that as $t \to \infty$,

$$K_t(\gamma) := \int_1^t e^{-\lambda s}s^\gamma \int_E \phi(x)\,[M(ds, dx) + L(ds, dx)] \text{ converges } \mathbb{P}_\mu\text{-a.s.} \tag{3.54}$$

In fact, for the continuous part of M, by (3.51), $\int_1^t e^{-\lambda s}s^\gamma \int_E \phi(x)S^C(ds, dx)$ converges $\mathbb{P}_\mu$-almost surely as $t \to \infty$. For the 'small jump' part, using the arguments for the 'small jumps' in (1), assumption (3.52) is enough to guarantee that $\int_1^t e^{-\lambda s}s^\gamma \int_E \phi(x)S^{(J,1)}(ds, dx)$ converges

$\mathbb{P}_\mu$-almost surely as $t \to \infty$. We are left to analyze the 'big jumps' part. Thanks to assumption (3.52),

$$\mathbb{P}_\mu\left(\sum_{\substack{\Delta\overline{X}_s(\phi)\geq f(s)\\ s>1}} 1\right) = \mathbb{P}_\mu\left(\int_1^\infty ds \int_E X_s(dx)\int_{f(s)}^\infty \pi^\phi(x,dr)\right)$$
$$= \int_E \mu(dy)\int_1^\infty ds\, P_s^\beta\Big(\int_{f(s)}^\infty \pi^\phi(\cdot,dr)\Big)(y)$$
$$\lesssim \mu(\phi)\int_E \nu(dx)\int_1^\infty e^{\lambda s}ds\int_{f(s)}^\infty \pi^\phi(x,dr)$$
$$\leq \int_E \nu(dx)\int_{f(\frac{\gamma}{\lambda})}^\infty \pi^\phi(x,dr)\int_0^{g(r)} e^{\lambda s}ds$$
$$\lesssim \int_E \nu(dx)\int_1^\infty r(\ln^+ r)^\gamma \pi^\phi(x,dr) < \infty,$$

where the second to last inequality comes from (3.50) and the fact that

$$\int_0^{g(r)} e^{\lambda s}ds = \frac{1}{\lambda}f(s)s^\gamma\Big|_0^{g(r)} = \frac{1}{\lambda}(rg(r)^\gamma - 1).$$

Thus $N^{(2)}$ is a finite measure. Consequently we have as $t \to \infty$,

$$\int_1^t\int_{\mathcal{M}(E)} F_{e^{-\lambda s}s^\gamma\phi(x)}(s,\upsilon)N^{(2)}(ds,d\upsilon) \to \sum_{\substack{\Delta\overline{X}_s(\phi)\geq f(s)\\ s>1}} e^{-\lambda s}s^\gamma\Delta\overline{X}_s(\phi) < \infty, \tag{3.55}$$

since the sum is a finite sum. Now (3.55) implies our claim (3.54).

Set $L_t = \int_0^t e^{-\lambda s}\int_E \phi(x)L(ds,dx)$ and let L_∞ denote its increasing limit. Then

$$L_\infty = \int_0^\infty e^{-\lambda s}\int_E \phi(x)L(ds,dx).$$

We first claim that $L_\infty < \infty$, $\mathbb{P}_\mu$-a.s. In fact, by the definition (3.48) of f, $f(s) \geq f(\gamma/\lambda)$ for any $s > 0$. Thus it follows from (1.11) that for any $s > 0$,

$$\int_{f(s)}^\infty r\pi^\phi(x,dr) \leq \int_{f(\gamma/\lambda)}^\infty r\pi^\phi(x,dr) \lesssim \phi(x).$$

Thus

$$\int_0^{\gamma/\lambda} e^{-\lambda s}\int_E \phi(x)L(ds,dx) = \int_0^{\gamma/\lambda} e^{-\lambda s}ds\int_E \overline{X}_{s-}(dx)\int_{f(s)}^\infty r\pi^\phi(x,dr)$$
$$\lesssim \int_0^{\gamma/\lambda} e^{-\lambda s}ds\int_E \phi(x)X_s(dx) = \int_0^{\gamma/\lambda} M_s(\phi)ds < \infty, \quad \mathbb{P}_\mu\text{-a.s.}$$

By Assumption 2,

$$\mathbb{P}_\mu\left(\int_{\gamma/\lambda}^\infty e^{-\lambda s}\int_E \phi(x)L(ds,dx)\right) = \mathbb{P}_\mu\left(\int_{\gamma/\lambda}^\infty e^{-\lambda s}ds\int_E \overline{X}_{s-}(dx)\int_{f(s)}^\infty r\pi^\phi(x,dr)\right)$$
$$= \int_{\gamma/\lambda}^\infty e^{-\lambda s}ds\int_E \mu(dy)P_s^\beta\left(\int_{f(s)}^\infty r\pi^\phi(\cdot,dr)\right)(y)$$

$$\begin{aligned}
&\lesssim \int_{\gamma/\lambda}^{\infty} ds \int_E \nu(dx) \int_{f(s)}^{\infty} r\pi^{\phi}(x, dr) \\
&= \int_E \nu(dx) \int_{f(\gamma/\lambda)}^{\infty} r\pi^{\phi}(x, dr) \int_{\gamma/\lambda}^{g(r)} ds \\
&\le \int_E \nu(dx) \int_{f(\lambda/\gamma)}^{\infty} rg(r)\pi^{\phi}(x, dr) < \infty,
\end{aligned}$$

which implies our claim.

Now using Lemma 3.2 and Remark 3.3, (3.54) implies that

$$\int_0^t s^{\gamma-1} \left(M_\infty(\phi) - M_s(\phi) + L_\infty - L_s\right) ds$$

converges and $(M_\infty(\phi)-M_t(\phi))+(L_\infty-L_t) = o(t^{-\gamma})$ $\mathbb{P}_\mu$-a.s. Thus the $\mathbb{P}_\mu$-almost sure convergence of $\int_0^t s^{\gamma-1} (M_\infty(\phi) - M_s(\phi))\, ds$ as $t \to \infty$ is equivalent to that $\int_0^t s^{\gamma-1} (L_\infty - L_s)\, ds$ converges $\mathbb{P}_\mu$-almost surely to a finite random variable as $t \to \infty$, and $M_\infty(\phi) - M_t(\phi) = o(t^{-\gamma})$ if and only if $(L_\infty - L_t)$ does. Since the integrand is non-negative, we always have limit $\int_1^\infty s^{\gamma-1} (L_\infty - L_s)\, ds \le \infty$. Suppose we can prove that, under the assumption (1.19),

$$\mathbb{P}_\mu \left(\int_1^\infty s^{\gamma-1} (L_\infty - L_s)\, ds = \infty \right) > 0. \tag{3.56}$$

Then $\int_0^t s^{\gamma-1} (M_\infty(\phi) - M_s(\phi))\, ds$ does not converge $\mathbb{P}_\mu$-almost surely as $t \to \infty$. Now we are left to prove (3.56). Note that by (1.19), there exists $T_2 > \max(T_1, \gamma/\lambda)$ such that for $t \ge T_2$,

$$\begin{aligned}
L_\infty - L_t &= \int_t^\infty e^{-\lambda s} ds \int_E \overline{X}_{s-}(dx) \int_{f(s)}^\infty r\pi^{\phi}(x, dr) \\
&\gtrsim \int_t^\infty e^{-\lambda s} ds \int_F \phi(x)\overline{X}_{s-}(dx) \int_{f(s)}^\infty r \int_E \nu(dy)\pi^{\phi}(y, dr).
\end{aligned}$$

Put $\rho(dr) = \int_E \nu(dy)\pi^{\phi}(y, dr)$. Then for $t \ge T_2$,

$$\begin{aligned}
L_\infty - L_t &\gtrsim \int_t^\infty e^{-\lambda s} X_s(\phi 1_F) ds \int_{f(s)}^\infty r\rho(dr) \\
&\gtrsim \sum_{n=1+[t]}^\infty \int_n^{n+1} ds e^{-\lambda s} X_s(\phi 1_F) \int_{f(s)}^\infty r\rho(dr) \\
&\gtrsim \sum_{n=1+[t]}^\infty \int_{f(n+1)}^\infty r\rho(dr) \int_n^{n+1} e^{-\lambda s} X_s(\phi 1_F) ds.
\end{aligned} \tag{3.57}$$

The third inequality comes from the fact that $f(s)$ is an increasing function for $s > \gamma/\lambda$. It is easy to check that $\int_0^{1+T_2} t^{\gamma-1} dt \sum_{n=1+[t]}^\infty \int_{f(n+1)}^\infty r\rho(dr) \int_n^{n+1} e^{-\lambda s} X_s(\phi I_F) ds < \infty$ $\mathbb{P}_\mu$-almost surely. So $\int_{1+T_2}^\infty t^{\gamma-1} dt \sum_{n=1+[t]}^\infty \int_{f(n+1)}^\infty r\rho(dr) \int_n^{n+1} e^{-\lambda s} X_s(\phi 1_F) ds$ have the same convergence property as $\int_0^\infty t^{\gamma-1} dt \sum_{n=1+[t]}^\infty \int_{f(n+1)}^\infty r\rho(dr) \int_n^{n+1} e^{-\lambda s} X_s(\phi 1_F) ds$. By Theorem A.1 in the Appendix, the two integrals above converge almost surely on $\{M_\infty(\phi) > 0\}$ if and only if the following integral

$$\int_0^\infty t^{\gamma-1} dt \sum_{n=1+[t]}^\infty \int_{f(n+1)}^\infty r\rho(dr)$$

is finite. Exchanging the order of integration, we obtain

$$\begin{aligned}
&\int_0^\infty t^{\gamma-1}dt \sum_{n=1+[t]}^\infty \int_{f(n+1)}^\infty r\rho(dr) \geq \int_0^\infty t^{\gamma-1}dt \int_{t+1}^\infty ds \int_{f(s)}^\infty r\rho(dr)\\
&= \int_1^\infty ds \int_0^{s-1} dt \int_{f(s)}^\infty r\rho(dr) = \frac{1}{\gamma}\int_1^\infty (s-1)^\gamma ds \int_{f(s)}^\infty r\rho(dr)\\
&\geq \frac{1}{\gamma}\int_{f(\gamma/\lambda)}^\infty r \int_{(\gamma/\lambda)\vee 1}^{g(r)\vee 1} (s-1)^\gamma ds\\
&= \frac{1}{\gamma(\gamma+1)}\int_{f(\gamma/\lambda)}^\infty r[(g(r)-1)^{\gamma+1} - (\gamma/\lambda \vee 1 - 1)^{\gamma+1}]\rho(dr)\\
&= \infty.
\end{aligned}$$

The last equality is due to that $g(r) \asymp \ln r$ as $r \to \infty$ and (1.20). Thus on the event $\{M_\infty(\phi) > 0\}$, which has positive probability,

$$\int_0^\infty s^{\gamma-1}(L_\infty - L_s)\, ds = \infty,$$

almost surely. Thus (3.56) is valid and the proof is complete.

Now we analyze the convergence rate of $L_\infty - L_t$. It follows from (3.57), Theorem A.1 and the monotonicity of f that, on $\{M_\infty(\phi) > 0\}$, almost surely when $t \geq T_2$,

$$L_\infty - L_t \gtrsim \int_t^\infty ds \int_{f(s)}^\infty r\rho(dr) = \int_{f(t)}^\infty r[g(r) - t]\rho(dr).$$

If $L_\infty - L_t = o(t^{-\gamma})$, then $\int_{f(t)}^\infty r[g(r) - t]\rho(dr) = o(t^{-\gamma})$, or equivalently, $\int_t^\infty r[g(r) - g(t)]\rho(dr) = o(g(t)^{-\gamma})$ as $t \to \infty$. From (3.49),

$$\int_t^\infty r[g(r) - g(t)]\rho(dr) = \int_t^\infty r\rho(dr)\int_t^r g'(u)du \asymp \int_t^\infty r[\ln r - \ln t]\rho(dr).$$

By (3.50), $\int_t^\infty r[g(r) - g(t)]\rho(dr) = o(g(t)^{-\gamma})$ is equivalent to $\int_t^\infty r[\ln r - \ln t]\rho(dr) = o((\ln t)^{-\gamma})$. Conversely, when $\int_t^\infty r[\ln r - \ln t]\rho(dr) = o((\ln t)^{-\gamma})$ as $t \to \infty$ does not hold, $L_\infty - L_t = o(t^{-\gamma})$ does not hold almost surely. Consequently $M_\infty(\phi) - M_t(\phi) = o(t^{-\gamma})$ does not hold almost surely. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

In this Appendix, we prove the following result used in the proof of Theorem 1.6.

Theorem A.1. *For any Borel subset F of E and $\mu \in \mathcal{M}(E)$,*

$$\lim_{n\to\infty} \int_n^{n+1} e^{-\lambda s} \langle \phi 1_F, X_s \rangle ds = \langle \phi 1_F, \nu \rangle M_\infty(\phi), \quad \mathbb{P}_\mu\text{-a.s.}$$

The proof of this theorem is based on the following five results. The idea of the proof is mainly from [34]. For any $n \in \mathbb{N}, u > 0$, and $h \in \mathcal{B}_b^+(E)$, define

$$H_{n+u}(h) := e^{-\lambda(n+u)} \int_0^{n+u} \int_E P^\beta_{(n+u)-s}(\phi h)(x) S^{(1,1)}(ds, dx),$$

$$L_{n+u}(h) := e^{-\lambda(n+u)} \int_0^{n+u} \int_E P^\beta_{(n+u)-s}(\phi h)(x) S^{(2,1)}(ds, dx),$$

and

$$C_{n+u}(h) := e^{-\lambda(n+u)} \int_0^{n+u} \int_E (P^\beta_{(n+u)-s} \phi h)(x) S^C(ds, dx).$$

Lemma A.2. *If $\int_E l(x)\nu(dx) < \infty$, then for any $u > 0$, $\mu \in \mathcal{M}(E)$ and $h \in \mathcal{B}_b^+(E)$,*

$$\sum_{n=1}^\infty \mathbb{P}_\mu \left[H_{n+u}(h) - \mathbb{P}_\mu(H_{n+u}(h) \big| \mathcal{F}_n) \right]^2 < \infty \tag{A.1}$$

and

$$\lim_{n\to\infty} \left(H_{n+u}(h) - \mathbb{P}_\mu[H_{n+u}(h) \big| \mathcal{F}_n] \right) = 0, \quad \text{in } L^2(\mathbb{P}_\mu) \text{ and } \mathbb{P}_\mu\text{-a.s.} \tag{A.2}$$

Moreover,

$$\lim_{n\to\infty} \int_0^1 \left(H_{n+u}(h) - \mathbb{P}_\mu[H_{n+u}(h) \big| \mathcal{F}_n] \right) du = 0, \quad \mathbb{P}_\mu\text{-a.s.} \tag{A.3}$$

Lemma A.3. *If $\int_E l(x)\nu(dx) < \infty$, then for any $u > 0$, $\mu \in \mathcal{M}(E)$ and $h \in \mathcal{B}_b^+(E)$ we have*

$$\lim_{n\to\infty} \left(L_{n+u}(h) - \mathbb{P}_\mu \left[L_{n+u}(h) \big| \mathcal{F}_n \right] \right) = 0, \quad \text{in } L^1(\mathbb{P}_\mu) \text{ and } \mathbb{P}_\mu\text{-a.s.} \tag{A.4}$$

and

$$\lim_{n\to\infty} \int_0^1 \left(L_{n+u}(h) - \mathbb{P}_\mu \left[L_{n+u}(h) \big| \mathcal{F}_n \right] \right) du = 0, \quad \mathbb{P}_\mu\text{-a.s.} \tag{A.5}$$

Lemma A.4. *For any $u > 0$, $\mu \in \mathcal{M}(E)$ and $h \in \mathcal{B}_b^+(E)$ we have*

$$\lim_{n\to\infty} \left(C_{n+u}(h) - \mathbb{P}_\mu[C_{n+u}(h) \big| \mathcal{F}_n] \right) = 0, \quad \text{in } L^2(\mathbb{P}_\mu) \text{ and } \mathbb{P}_\mu\text{-a.s.} \tag{A.6}$$

and

$$\lim_{n\to\infty} \int_0^1 \left(C_{n+u}(h) - \mathbb{P}_\mu[C_{n+u}(h) \big| \mathcal{F}_n] \right) du = 0, \quad \mathbb{P}_\mu\text{-a.s.} \tag{A.7}$$

The proofs of the above three lemmas are similar to those of corresponding results in [33, Section 3]. We omit the details here. Combining the three lemmas above, we have

Lemma A.5. *If $\int_E l(x)\nu(dx) < \infty$, then for any $u > 0$, $\mu \in \mathcal{M}(E)$ and $h \in \mathcal{B}_b^+(E)$ we have*

$$\lim_{n\to\infty} \left(e^{-\lambda(n+u)}\langle \phi h, X_{n+u}\rangle - \mathbb{P}_\mu\left[e^{-\lambda(n+u)}\langle \phi h, X_{n+u}\rangle \middle| \mathcal{F}_n\right]\right) = 0, \quad \text{in } L^1(\mathbb{P}_\mu) \text{ and } \mathbb{P}_\mu\text{-a.s.} \tag{A.8}$$

Using the arguments similar to that of [33, Theorem 3.5], we have

Theorem A.6. *If $\int_E l(x)\nu(dx) < \infty$, then for any $\mu \in \mathcal{M}(E)$ and $h \in \mathcal{B}_b^+(E)$ we have*

$$\lim_{n\to\infty} e^{-\lambda n}\langle \phi h, X_n\rangle = M_\infty(\phi)\int_E \phi(z)h(z)\nu(dz), \quad \text{in } L^1(\mathbb{P}_\mu) \text{ and } \mathbb{P}_\mu\text{-a.s.}$$

Proof of Theorem A.1. For any $s > n$

$$\begin{aligned}
&e^{-\lambda s}\langle \phi 1_F, X_s\rangle \\
&= e^{-\lambda s}\langle P^\beta_{s-n}(\phi 1_F), X_n\rangle + e^{-\lambda s}\int_n^s P^\beta_{s-u}(\phi 1_F)(x)M(du,dx) \\
&= e^{-\lambda s}\langle P^\beta_{s-n}(\phi 1_F), X_n\rangle + \left(H_s(\phi 1_F) - \mathbb{P}_\mu[H_s(\phi 1_F)\big|\mathcal{F}_n]\right) \\
&\quad + \left(L_s(\phi 1_F) - \mathbb{P}_\mu\left[L_s(\phi 1_F)|\mathcal{F}_n\right]\right) \\
&\quad + C_s(\phi 1_F) - \mathbb{P}_\mu\left[C_s(\phi 1_F)|\mathcal{F}_n\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_n^{n+1} e^{-\lambda s}\langle \phi 1_F, X_s\rangle ds \\
&= \int_n^{n+1} e^{-\lambda s}\langle P^\beta_{s-n}(\phi 1_F), X_n\rangle ds + \int_n^{n+1}\left(H_s(\phi 1_F) - \mathbb{P}_\mu[H_s(\phi 1_F)\big|\mathcal{F}_n]\right)ds \\
&\quad + \int_n^{n+1}\left(L_s(\phi 1_F) - \mathbb{P}_\mu\left[L_s(\phi 1_F)|\mathcal{F}_n\right]\right)ds + \int_n^{n+1} C_s(\phi 1_F) - \mathbb{P}_\mu\left[C_s(\phi 1_F)|\mathcal{F}_n\right]ds \\
&= e^{-\lambda n}\langle\left(\int_0^1 e^{-\lambda s}P^\beta_s(\phi 1_F)ds\right), X_n\rangle + \int_0^1\left(H_{n+s}(\phi 1_F) - \mathbb{P}_\mu[H_{n+s}(\phi 1_F)\big|\mathcal{F}_n]\right)ds \\
&\quad + \int_0^1\left(L_{s+n}(\phi 1_F) - \mathbb{P}_\mu\left[L_{s+n}(\phi 1_F)|\mathcal{F}_n\right]\right)ds \\
&\quad + \int_0^1 C_{n+s}(\phi 1_F) - \mathbb{P}_\mu\left[C_{n+s}(\phi 1_F)|\mathcal{F}_n\right]ds. \\
&= I_n + II_n + III_n + IV_n.
\end{aligned}$$

It has been shown in Lemmas A.2–A.4 that

$$\lim_{n\to\infty} II_n + III_n + IV_n = 0.$$

Since $\int_0^1 e^{-\lambda s}P^\beta_s(\phi 1_F)(x)ds \le \phi(x)$, by Theorem A.6,

$$\begin{aligned}
\lim_{n\to\infty} e^{-\lambda n}\langle\left(\int_0^1 e^{-\lambda s}P^\beta_s(\phi 1_F)ds\right), X_n\rangle &= M_\infty(\phi)\langle\int_0^1 e^{-\lambda s}P^\beta_s(\phi 1_F)(x)ds, \nu\rangle \\
&= M_\infty(\phi)\langle \phi 1_F, \nu\rangle. \quad \square
\end{aligned}$$

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