SCIENCE CHINA
Mathematics

Special Topic on Probability and Stochastic Analysis
• ARTICLES •

August 2019 Vol. 62 No. 8: 1519–1552
https://doi.org/10.1007/s11425-018-9402-4

Supercritical superprocesses: Proper normalization and non-degenerate strong limit

In Memory of Professor Kai Lai Chung on the 100th Anniversary of His Birth

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Received January 26, 2018; accepted September 19, 2018; published online July 9, 2019

Abstract Suppose that $X = \{X_t, t \geqslant 0; \mathbb{P}_\mu\}$ is a supercritical superprocess in a locally compact separable metric space E. Let ϕ_0 be a positive eigenfunction corresponding to the first eigenvalue λ_0 of the generator of the mean semigroup of X. Then $M_t := \mathrm{e}^{-\lambda_0 t}\langle \phi_0, X_t\rangle$ is a positive martingale. Let M_∞ be the limit of M_t. It is known (see Liu et al. (2009)) that M_∞ is non-degenerate if and only if the $L\log L$ condition is satisfied. In this paper we are mainly interested in the case when the $L\log L$ condition is not satisfied. We prove that, under some conditions, there exist a positive function γ_t on $[0,\infty)$ and a non-degenerate random variable W such that for any finite nonzero Borel measure μ on E,

$$\lim_{t\to\infty} \gamma_t \langle \phi_0, X_t\rangle = W, \quad \text{a.s.-}\mathbb{P}_\mu.$$

We also give the almost sure limit of $\gamma_t\langle f, X_t\rangle$ for a class of general test functions f.

Keywords superprocesses, Seneta-Heyde norming, non-degenerate strong limit, martingales

MSC(2010) 60J68, 60F15

Citation: Ren Y-X, Song R M, Zhang R. Supercritical superprocesses: Proper normalization and non-degenerate strong limit. Sci China Math, 2019, 62: 1519–1552, https://doi.org/10.1007/s11425-018-9402-4

1 Introduction

Suppose $\{Z_n, n \geqslant 0\}$ is a supercritical Galton-Watson process with offspring number L. Let $m := \mathrm{E}L \in (1,\infty)$ be the mean of L. Then $M_n := \frac{Z_n}{m^n}$ is a non-negative martingale and thus has a finite limit M_∞. The well-known Kesten-Stigum theorem says that the following three statements are equivalent: (i) $\mathrm{P}(M_\infty = 0) = \mathrm{P}(Z_n = 0 \text{ for large } n)$, i.e., the events $\{M_\infty = 0\}$ and $\{Z_n = 0 \text{ for large } n\}$ are almost the same; (ii) $\mathrm{E}M_\infty = 1$; (iii) $\mathrm{E}(L\log L) < \infty$. For a classical proof of this result, the reader is referred to the book [1] of Athreya and Ney. In 1995, Lyons et al. [19] gave a probabilistic proof of the above $L\log L$

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criterion of Kesten and Stigum [12]. If $\mathrm{E}(L\log L)=\infty$, then $\lim_{n\to\infty}\frac{Z_n}{m^n}=0$ almost surely, which says that m^n does not give the right growth rate of Z_n conditional on non-extinction. It is natural to ask what the right growth rate of Z_n is. In 1968, Seneta [27] proved that there is a sequence of positive numbers c_n such that c_nZ_n converges to a non-degenerate random variable W in distribution. Heyde [10] strengthened the convergence in distribution to almost sure convergence. Later the problem of finding c_n such that c_nZ_n converges to a non-degenerate limit is called the Seneta-Heyde norming problem.

Hoppe [11] generalized the result of Heyde [10] to supercritical multitype branching processes, Grey [8] proved a similar result for continuous state branching processes and Hering [9] obtained a similar result for supercritical branching diffusions. In this paper we are going to consider the Seneta-Heyde norming problem for general superprocesses under some conditions which are easy to check and satisfied by many superprocesses, including superdiffusions in a bounded domain and also superprocesses with discontinuous spatial motions. We emphasize that we are mainly interested in the case when the $L\log L$ condition fails, since the norming problem is already solved in [17] for superdiffusions and [21] for more general superprocesses when the $L\log L$ condition holds.

1.1 Superprocesses and assumptions

In this subsection, we describe the setup of this paper and formulate our assumptions.

Suppose that E is a locally compact separable metric space. We will use $\mathcal{B}(E)$ ($\mathcal{B}^+(E)$) to denote the family of (non-negative) Borel functions on E, $\mathcal{B}_b(E)$ ($\mathcal{B}_b^+(E)$) to denote the family of (non-negative) bounded Borel functions on E, and $C(E)$ ($C_0(E)$, respectively) to denote the family of continuous functions (vanishing at infinity, respectively) on E.

Suppose that ∂ is a separate point not contained in E. We will use E_∂ to denote $E\cup\{\partial\}$. Every function f on E is automatically extended to E_∂ by setting $f(\partial)=0$. We will assume that $\xi=\{\xi_t,\Pi_x\}$ is a Hunt process on E and $\zeta:=\inf\{t>0:\xi_t=\partial\}$ is the lifetime of ξ. We will use $\{P_t:t\geqslant 0\}$ to denote the semigroup of ξ. Suppose that m is a σ-finite Borel measure on E with full support. We will assume below that $\{P_t:t\geqslant 0\}$ has a dual with respect to the measure m and the dual semigroup is sub-Markovian.

The superprocess $X=\{X_t:t\geqslant 0\}$ we are going to work with is determined by two parameters: a spatial motion $\xi=\{\xi_t,\Pi_x\}$ on E which is a Hunt process, and a branching mechanism φ of the form

$$\varphi(x,s)=-\alpha(x)s+\beta(x)s^2+\int_{(0,+\infty)}(\mathrm{e}^{-s\theta}-1+s\theta)n(x,d\theta),\quad x\in E,\quad s\geqslant 0,\tag{1.1}$$

where $\alpha\in\mathcal{B}_b(E)$, $\beta\in\mathcal{B}_b^+(E)$ and n is a kernel from E to $(0,\infty)$ satisfying

$$\sup_{x\in E}\int_{(0,+\infty)}(\theta\wedge\theta^2)n(x,d\theta)<\infty.\tag{1.2}$$

Then there exists $M>0$ such that

$$|\alpha(x)|+\beta(x)+\int_{(0,+\infty)}(\theta\wedge\theta^2)n(x,d\theta)\leqslant M.$$

In this paper, we will exclude the case when $\beta(\cdot)+n(\cdot,(0,\infty))=0$ m-almost everywhere.

Let $\mathcal{M}_F(E)$ be the space of finite measures on E, equipped with the topology of weak convergence. The superprocess X with spatial motion ξ and branching mechanism φ is a Markov process taking values in $\mathcal{M}_F(E)$. The existence of such superprocesses is well known (see, for example, [5,6,16]). For any $\mu\in\mathcal{M}_F(E)$, we denote the law of X with initial configuration μ by $\mathbb{P}_\mu$. As usual, $\langle f,\mu\rangle:=\int_E f(x)\mu(dx)$ and $\|\mu\|:=\langle 1,\mu\rangle$. Throughout this paper, a real-valued function $u(t,x)$ on $[0,\infty)\times E_\partial$ is said to be locally bounded if, for any $t>0$, $\sup_{s\in[0,t],x\in E_\partial}|u(s,x)|<\infty$. According to [16, Theorem 5.12], there is a Hunt process $X=\{\Omega,\mathcal{G},\mathcal{G}_t,X_t,\mathbb{P}_\mu\}$ taking values in $\mathcal{M}_F(E)$ such that for every $f\in\mathcal{B}_b^+(E)$ and $\mu\in\mathcal{M}_F(E)$,

$$-\log\mathbb{P}_\mu(\mathrm{e}^{-\langle f,X_t\rangle})=\langle V_tf,\mu\rangle,\tag{1.3}$$

where $V_tf(x)$ is the unique locally bounded non-negative solution to the equation

$$V_tf(x)+\Pi_x\int_0^t\varphi(\xi_s,V_{t-s}f(\xi_s))ds=\Pi_xf(\xi_t),\quad x\in E_\partial, \tag{1.4}$$

where we use the convention that $\varphi(\partial,s)=0$ for all $s\geqslant 0$. Since $f(\partial)=0$, we have $V_tf(\partial)=0$ for any $t\geqslant 0$. In this paper, the superprocess we deal with is always this Hunt realization.

For any $f\in\mathcal{B}_b(E)$ and $(t,x)\in(0,\infty)\times E$, we define

$$T_tf(x):=\Pi_x[\mathrm{e}^{\int_0^t\alpha(\xi_s)ds}f(\xi_t)]. \tag{1.5}$$

It is well known that $T_tf(x)=\mathbb{P}_{\delta_x}\langle f,X_t\rangle$ for every $x\in E$.

We will always assume that there exists a family of continuous and strictly positive functions $\{p(t,x,y):t>0\}$ on $E\times E$ such that for any $t>0$ and the non-negative function f on E,

$$P_tf(x)=\int_E p(t,x,y)f(y)m(dy).$$

Define

$$a_t(x):=\int_E p(t,x,y)^2\,m(dy),\quad \hat{a}_t(x):=\int_E p(t,y,x)^2\,m(dy).$$

Our first main assumption is as follows.

Assumption 1.1. (i) *For any $t>0$, $\int_E p(t,x,y)\,m(dx)\leqslant 1$.*

(ii) *For any $t>0$, we have*

$$\int_E a_t(x)\,m(dx)=\int_E\hat{a}_t(x)\,m(dx)=\int_E\int_E p(t,x,y)^2\,m(dy)\,m(dx)<\infty. \tag{1.6}$$

Moreover, the functions $x\to a_t(x)$ and $x\to\hat{a}_t(x)$ are continuous on E.

Note that, in Assumption 1.1(i), the integration is with respect to the first space variable. It implies that the dual semigroup $\{\widehat{P}_t:t\geqslant 0\}$ of $\{P_t:t\geqslant 0\}$ with respect to m defined by

$$\widehat{P}_tf(x)=\int_E p(t,y,x)f(y)m(dy)$$

is sub-Markovian. Assumption 1.1(ii) is a pretty weak L^2 condition and it allows us to apply results on operator semigroups in Hilbert spaces. By Hölder's inequality, we have

$$p(t+s,x,y)=\int_E p(t,x,z)p(s,z,y)\,m(dz)\leqslant(a_t(x))^{1/2}(\hat{a}_s(y))^{1/2}. \tag{1.7}$$

It is well known and easy to check that $\{P_t:t\geqslant 0\}$ and $\{\widehat{P}_t:t\geqslant 0\}$ are strongly continuous contraction semigroups on $L^2(E,m)$ (see [23] for a proof). We will use $\langle\cdot,\cdot\rangle_m$ to denote the inner product in $L^2(E,m)$. Since $p(t,x,y)$ is continuous in (x,y), by (1.7), Assumption 1.1(ii) and the dominated convergence theorem, we have that, for any $f\in L^2(E,m)$, P_tf and $\widehat{P}_tf$ are continuous.

It follows from Assumption 1.1(ii) that, for each $t>0$, P_t and $\widehat{P}_t$ are compact operators on $L^2(E,m)$. Let $\widetilde{L}$ and $\widehat{\widetilde{L}}$ be the infinitesimal generators of the semigroups $\{P_t\}$ and $\{\widehat{P}_t\}$ in $L^2(E,m)$, respectively. Define $\widetilde{\lambda}_0:=\sup\Re(\sigma(\widetilde{L}))=\sup\Re(\sigma(\widehat{\widetilde{L}}))$. By Jentzsch's theorem (see [26, Theorem V.6.6, p. 337]), $\widetilde{\lambda}_0$ is an eigenvalue of multiplicity 1 for both $\widetilde{L}$ and $\widehat{\widetilde{L}}$, and an eigenfunction $\widetilde{\phi}_0$ of $\widetilde{L}$ corresponding to $\widetilde{\lambda}_0$ can be chosen to be strictly positive m-almost everywhere with $\|\widetilde{\phi}_0\|_2=1$ and an eigenfunction $\widetilde{\psi}_0$ of $\widehat{\widetilde{L}}$ corresponding to $\widetilde{\lambda}_0$ can be chosen to be strictly positive m-almost everywhere with $\langle\widetilde{\phi}_0,\widetilde{\psi}_0\rangle_m=1$. Thus for m-almost every $x\in E$,

$$\mathrm{e}^{\widetilde{\lambda}_0}\widetilde{\phi}_0(x)=P_1\widetilde{\phi}_0(x),\quad \mathrm{e}^{\widetilde{\lambda}_0}\widetilde{\psi}_0(x)=\widehat{P}_1\widetilde{\psi}_0(x).$$

Hence $\widetilde{\phi}_0$ and $\widetilde{\psi}_0$ can be chosen to be continuous and strictly positive everywhere on E.

Our second assumption is as follows.

Assumption 1.2. (i) $\widetilde{\phi}_0$ *is bounded.*

(ii) *The semigroup $\{P_t, t \geqslant 0\}$ is intrinsically ultracontractive, i.e., there exists $c_t > 0$ such that*

$$p(t,x,y) \leqslant c_t \widetilde{\phi}_0(x) \widetilde{\psi}_0(y). \tag{1.8}$$

Assumption 1.2 is a pretty strong assumption on the semigroup $\{P_t : t \geqslant 0\}$. However, this assumption is satisfied in a lot of cases. See [22, Subsection 1.4] for examples of Markov processes satisfying the assumption above. The concept of intrinsic ultracontractivity was introduced by Davies and Simon [4] in the setting of symmetric semigroups. This concept was extended to the non-symmetric setting in [13–15]. Intrinsic ultracontractivity has been studied intensively in the last 30 years and there are many results on the intrinsic ultracontractivity of semigroups (see [13–15] and the references therein).

We have proved in [22, Lemma 2.1] that there exists a function $q(t,x,y)$ on $(0,\infty) \times E \times E$ which is continuous in (x,y) for each $t > 0$ such that

$$\mathrm{e}^{-Mt} p(t,x,y) \leqslant q(t,x,y) \leqslant \mathrm{e}^{Mt} p(t,x,y), \quad (t,x,y) \in (0,\infty) \times E \times E, \tag{1.9}$$

and that for any bounded Borel function f and any $(t,x) \in (0,\infty) \times E$,

$$T_t f(x) = \int_E q(t,x,y) f(y)\, m(dy).$$

It follows immediately that

$$\|T_t f\|_2 \leqslant \mathrm{e}^{Mt} \|P_t f\|_2 \leqslant \mathrm{e}^{Mt} \|f\|_2. \tag{1.10}$$

In [23], we have proved that $\{T_t : t \geqslant 0\}$ is a strongly continuous semigroup on $L^2(E,m)$. Let $\{\widehat{T}_t, t > 0\}$ be the adjoint semigroup on $L^2(E,m)$ of $\{T_t, t > 0\}$, i.e., for $f \in L^2(E,m)$,

$$\widehat{T}_t f(x) = \int_E q(t,y,x) f(y)\, m(dy).$$

We have proved in [23] that $\{\widehat{T}_t : t \geqslant 0\}$ is also a strongly continuous semigroup on $L^2(E,m)$. We claim that, for all $t > 0$ and $f \in L^2(E,m)$, $T_t f$ and $\widehat{T}_t f$ are continuous. In fact, since $q(t,x,y)$ is continuous in (x,y), by (1.7), (1.9), Assumption 1.1(ii) and the dominated convergence theorem, we have that, for any $f \in L^2(E,m)$, $T_t f$ and $\widehat{T}_t f$ are continuous.

By Assumption 1.1(ii) and (1.9), we get that

$$\int_E \int_E q^2(t,x,y)\, m(dx)\, m(dy) \leqslant \mathrm{e}^{2Mt} \int_E \int_E p^2(t,x,y)\, m(dx)\, m(dy) < \infty.$$

Thus, for each $t > 0$, T_t and $\widehat{T}_t$ are compact operators on $L^2(E,m)$. Let L and $\widehat{L}$ be the infinitesimal generators of the semigroups $\{T_t\}$ and $\{\widehat{T}_t\}$ in $L^2(E,m)$, respectively. Define $\lambda_0 := \sup \Re(\sigma(L)) = \sup \Re(\sigma(\widehat{L}))$. By Jentzsch's theorem, λ_0 is an eigenvalue of multiplicity 1 for both L and $\widehat{L}$, and an eigenfunction ϕ_0 of L corresponding to λ_0 can be chosen to be strictly positive m-almost everywhere with $\|\phi_0\|_2 = 1$ and an eigenfunction ψ_0 of $\widehat{L}$ corresponding to λ_0 can be chosen to be strictly positive m-almost everywhere with $\langle \phi_0, \psi_0 \rangle_m = 1$. Thus for m-almost every $x \in E$,

$$\mathrm{e}^{\lambda_0} \phi_0(x) = T_1 \phi_0(x), \quad \mathrm{e}^{\lambda_0} \psi_0(x) = \widehat{T}_1 \psi_0(x).$$

Hence ψ_0 and ϕ_0 can be chosen to be continuous and strictly positive everywhere on E.

Using Assumption 1.2, the boundedness of α and an argument similar to that used in the proof of [4, Theorem 3.4], one can show the following:

(i) ϕ_0 is bounded.

(ii) The semigroup $\{T_t, t \geqslant 0\}$ is intrinsically ultracontractive, i.e., there exists $c_t > 0$ such that

$$q(t,x,y) \leqslant c_t \phi_0(x) \psi_0(y). \tag{1.11}$$

The main interest of this paper is on supercritical superprocesses, so we assume as follows.

Assumption 1.3. *It holds that* $\lambda_0 > 0$.

Define $q_t(x) := \mathbb{P}_{\delta_x}(\|X_t\| = 0)$. Since $\mathbb{P}_{\delta_x}\|X_t\| = T_t 1(x) > 0$, we have $q_t(x) < 1$. Note that $q_t(x)$ is non-decreasing in t. Hence the limit $q(x) := \lim_{t\to\infty} q_t(x)$ exists. Then $q(x) = \mathbb{P}_{\delta_x}\{\|X_t\| = 0 \text{ for some } t > 0\}$ is the extinction probability. In this paper, we also assume as follows.

Assumption 1.4. *There exists* $t_0 > 0$ *such that*

$$\inf_{x\in E} q_{t_0}(x) > 0. \tag{1.12}$$

In [22, Subsection 2.2], we gave a sufficient condition (in term of the branching mechnism φ) for Assumption 1.4. In particularly, if $\inf_{x\in E}\beta(x) > 0$, then Assumption 1.4 holds. In Lemma 3.1, we will show that, under our assumptions, $q(x) < 1$, for all $x \in E$.

1.2 Main results

Define $M_t := \mathrm{e}^{-\lambda_0 t}\langle \phi_0, X_t\rangle$, $t \geqslant 0$. It is easy to prove that (see, for example, [21, Theorem 3.2]), for every $\mu \in \mathcal{M}_F(E)$, $\{M_t, t \geqslant 0\}$ is a non-negative $\mathbb{P}_\mu$-martingale with respect to the filtration $\{\mathcal{G}_t : t \geqslant 0\}$. Thus $\{M_t, t \geqslant 0\}$ has a $\mathbb{P}_\mu$-a.s. finite limit denoted as M_∞.

Let $n^{\phi_0}(x, d\theta)$ be the kernel from E to $(0,\infty)$ defined by

$$\int_0^\infty f(\theta) n^{\phi_0}(x, d\theta) = \int_0^\infty f(\theta\phi_0(x)) n(x, d\theta).$$

By the boundedness of ϕ_0 and the assumption (1.2), we get that there exists $\widetilde{M} > 0$ such that

$$\sup_{x\in E}\int_0^\infty (\theta \wedge \theta^2) n^{\phi_0}(x, d\theta) \leqslant \widetilde{M}. \tag{1.13}$$

Let $l(x) := \int_1^\infty \theta\log\theta\, n^{\phi_0}(x, d\theta)$. The following $L\log L$ criterion was proved for superdiffusions in [17].

***L* log *L* criterion.** M_∞ is non-degenerate under $\mathbb{P}_\mu$ for all nonzero finite measures μ on E if and only if

$$\int_E \psi_0(x) l(x) m(dx) < \infty. \tag{1.14}$$

At first glance, the roles of ϕ_0 and ψ_0 are not symmetric in (1.14). This is not the case. In fact, (1.14) is equivalent to

$$\int_E \phi_0(x)\psi_0(x) m(dx) \int_1^\infty (\theta\log\theta) n(x, d\theta) < \infty, \tag{1.15}$$

which says that the spatial average of the "$\theta\log\theta$" moment $\int_1^\infty(\theta\log\theta)n(x, d\theta)$ with respect to the probability measure $\phi_0(x)\psi_0(x)m(dx)$ is finite. Note that

$$\begin{aligned} l(x) = \int_1^\infty \theta\log\theta\, n^{\phi_0}(x, d\theta) &= \phi_0(x)\int_{\phi_0(x)^{-1}}^\infty \theta(\log\theta + \log\phi_0(x))\, n(x, d\theta) \\ &= \phi_0(x)\int_1^\infty \theta\log\theta\, n(x, d\theta) + \phi_0(x)\int_{\phi_0(x)^{-1}}^1 \theta\log\theta\, n(x, d\theta) \\ &\quad + \phi_0(x)\log\phi_0(x)\int_{\phi_0(x)^{-1}}^\infty \theta\, n(x, d\theta). \end{aligned}$$

Since ϕ_0 is bounded, $\phi_0(x)|\log\phi_0(x)|$ is bounded above, say by C. Thus,

$$\sup_{x\in E}\phi_0(x)|\log\phi_0(x)|\int_{\phi_0(x)^{-1}}^\infty \theta\, n(x, d\theta) \leqslant C\sup_{x\in E}\int_{\|\phi_0\|_\infty^{-1}}^\infty \theta\, n(x, d\theta) < \infty.$$

Note also that

$$\sup_{x\in E}\phi_0(x)\left|\int_{\phi_0(x)^{-1}}^1 \theta\log\theta\, n(x, d\theta)\right|$$

$$\leqslant \sup_{x\in E}\phi_0(x)|\log\phi_0(x)|\left|\int_{\phi_0(x)^{-1}}^1 \theta\, n(x,d\theta)\right|$$
$$\leqslant C\sup_{x\in E}\left(\mathbf{1}_{\phi_0(x)>1}\int_{\|\phi_0\|_\infty^{-1}}^1 \theta\, n(x,d\theta)+\mathbf{1}_{\phi_0(x)\leqslant 1}\int_1^\infty \theta\, n(x,d\theta)\right)<\infty.$$

In Section 2, we will show that $\psi_0\in L^1(E,m)$ (see the first paragraph of Section 2). Thus (1.14) is equivalent to (1.15).

Recently, the $L\log L$ criterion above was extended to more general superprocesses with possible non-local branching mechanisms in [21].

The $L\log L$ criterion above says that, under the condition (1.14), $\mathrm{e}^{\lambda_0 t}$ gives the growth rate of $\langle\phi_0,X_t\rangle$ as $t\to\infty$ conditioned on non-extinction. However, when the condition (1.14) is not satisfied, the theorem above does not provide much information about the growth rate of $\langle\phi_0,X_t\rangle$.

The first objective of this paper is to solve the Seneta-Heyde norming problem for the martingale M_t, i.e., to find a positive function γ_t on $[0,\infty)$ such that $\gamma_t\langle\phi_0,X_t\rangle$ has a non-degenerate limit as $t\to\infty$. Although our results (see Theorems 1.5, 6.10 and 6.12) also cover the case when (1.14) holds, only the results in the case when (1.14) fails are new (see Theorem 1.6 below). It is easy to find examples such that (1.14) fails. For example, if $n(x,d\theta)=c(x)[I_{(0,2)}(\theta)+I_{[2,\infty)}(\theta)\theta^{-2}(\log\theta)^{-\beta}]d\theta$ with $\beta\in(1,2]$ and $c(x)$ being a strictly positive bounded measurable function on E, then (1.2) is satisfied, but (1.15), which is equivalent to (1.14), fails.

Let $v(x):=-\log q(x)$. By the branching property of X, we have

$$\mathbb{P}_\mu(\|X_t\|=0\text{ for some }t>0)=\mathrm{e}^{-\langle v,\mu\rangle}.$$

Theorem 1.5. *There exist a positive function γ_t on $[0,\infty)$ and a non-degenerate random variable W such that*

$$\lim_{t\to\infty}\frac{\gamma_t}{\gamma_{t+s}}=\mathrm{e}^{\lambda_0 s},\quad \forall\, s\geqslant 0,$$

and that for any nonzero $\mu\in\mathcal{M}_F(E)$,

$$\lim_{t\to\infty}\gamma_t\langle\phi_0,X_t\rangle=W,\quad a.s.\text{-}\mathbb{P}_\mu$$

and

$$\mathbb{P}_\mu(W=0)=\mathrm{e}^{-\langle v,\mu\rangle},\quad \mathbb{P}_\mu(W<\infty)=1.$$

Moreover, we have the following $L\log L$ criterion.

Theorem 1.6. *The following conditions are equivalent*:

(1) M_∞ *is non-degenerate for some nonzero* $\mu\in\mathcal{M}_F(E)$;
(2) M_∞ *is non-degenerate for all nonzero* $\mu\in\mathcal{M}_F(E)$;
(3) $l_0:=\lim_{t\to\infty}\mathrm{e}^{\lambda_0 t}\gamma_t\in(0,\infty)$;
(4) $\int_E\psi_0(x)l(x)m(dx)<\infty$;
(5) $\mathbb{P}_\mu W<\infty$ *for some nonzero* $\mu\in\mathcal{M}_F(E)$;
(6) $\mathbb{P}_\mu W<\infty$ *for all* $\mu\in\mathcal{M}_F(E)$.

Further properties of the limit random variable W, such as absolute continuity and tail probabilities, are studied in [24], a sequel to the present paper.

The second objective of this paper is to study the almost sure limit behavior of $\gamma_t\langle f,X_t\rangle$ as $t\to\infty$ for a class of bounded continuous functions f. It turns out that, for f belonging to this class, $\lim_{t\to\infty}\gamma_t\langle f,X_t\rangle=\langle f,\psi_0\rangle_m W$, $\mathbb{P}_\mu$-a.s. for any nonzero $\mu\in\mathcal{M}_F(E)$ (see Theorems 6.10 and 6.12).

The rest of the paper is organized as follows. Section 2 contains our basic estimates and Section 3 deals with some properties of the extinction probability. In Section 4, we will define and investigate backward iterates, which is needed in the proof of Theorem 1.5. The proofs of Theorems 1.5 and 1.6 are given in Section 5. We remark that we will prove Theorem 1.6 without using the $L\log L$ criterion in [17]. The strong limit behavior of $\gamma_t\langle f,X_t\rangle$ as $t\to\infty$ for a class of general bounded continuous functions f is given in Section 6. In the last section, we give some concluding remarks.

In the remainder of this paper, C will stand for a constant whose value might change from one appearance to the next.

2 Some estimates

According to [13, Thorem 2.7], under Assumptions 1.1–1.2, for any $\delta > 0$, there exist constants $\gamma = \gamma(\delta) > 0$ and $c = c(\delta) > 0$ such that, for any $(t, x, y) \in [\delta, \infty) \times E \times E$, we have

$$|\mathrm{e}^{-\lambda_0 t} q(t, x, y) - \phi_0(x)\psi_0(y)| \leqslant c\mathrm{e}^{-\gamma t}\phi_0(x)\psi_0(y). \tag{2.1}$$

Take t large enough so that $c\mathrm{e}^{-\gamma t} < \frac{1}{2}$. Then we have

$$\mathrm{e}^{-\lambda_0 t} q(t, x, y) \geqslant \frac{1}{2}\phi_0(x)\psi_0(y).$$

Since $q(t, x, \cdot) \in L^1(E, m)$, we have $\psi_0 \in L^1(E, m)$.

It follows from (2.1) that, if $f \in \mathcal{B}_b^+(E)$ then $\langle f, \psi_0\rangle_m < \infty$ and for any $(t, x) \in [\delta, \infty) \times E$,

$$|\mathrm{e}^{-\lambda_0 t} T_t f(x) - \langle f, \psi_0\rangle_m \phi_0(x)| \leqslant c\mathrm{e}^{-\gamma t}\langle |f|, \psi_0\rangle_m \phi_0(x) \tag{2.2}$$

and

$$(1 - c\mathrm{e}^{-\gamma t})\langle |f|, \psi_0\rangle_m \phi_0(x) \leqslant \mathrm{e}^{-\lambda_0 t} T_t |f|(x) \leqslant (1 + c)\langle |f|, \psi_0\rangle_m \phi_0(x). \tag{2.3}$$

For $x \in E$ and $s > 0$, we define

$$r(x, s) = \varphi(x, s) + \alpha(x)s.$$

Lemma 2.1. *For any $H \geqslant 1$,*

$$0 \leqslant r(x, s) \leqslant (3/2 + H/2)Ms^2 + \int_H^\infty \theta\, n(x, d\theta)s. \tag{2.4}$$

Proof. By definition,

$$r(x, s) = \beta(x)s^2 + \int_0^\infty (\mathrm{e}^{-\theta s} - 1 + \theta s)\, n(x, d\theta). \tag{2.5}$$

Note that for any $\theta > 0$,

$$0 < \mathrm{e}^{-\theta} - 1 + \theta \leqslant \left(\frac{\theta}{2} \wedge 1\right)\theta. \tag{2.6}$$

Thus for any $H \geqslant 1$,

$$\begin{aligned} r(x, s) &\leqslant Ms^2 + \frac{1}{2}s^2 \int_0^H \theta^2\, n(x, d\theta) + s\int_H^\infty \theta\, n(x, d\theta) \\ &\leqslant Ms^2 + \frac{1}{2}s^2\left[\int_0^1 \theta^2\, n(x, d\theta) + H\int_1^H \theta\, n(x, d\theta)\right] + s\int_H^\infty \theta\, n(x, d\theta) \\ &\leqslant (3/2 + H/2)Ms^2 + \int_H^\infty \theta\, n(x, d\theta)s. \end{aligned}$$

This completes the proof. □

For any $f \in \mathcal{B}_b^+(E)$ satisfying $m(f > 0) > 0$, define, for any $(t, x) \in [0, \infty) \times E$,

$$R_f(t, x) := T_t f(x) - V_t f(x), \tag{2.7}$$

and

$$g_f(t, x) := \frac{R_f(t, x)}{\mathrm{e}^{\lambda_0 t}\langle f, \psi_0\rangle_m \phi_0(x)}. \tag{2.8}$$

Lemma 2.2. *Assume that $f \in \mathcal{B}_b^+(E)$ and $m(f>0)>0$. Then*

$$R_f(t,x) \geqslant 0, \quad (t,x) \in [0,\infty) \times E, \tag{2.9}$$

and

$$\lim_{\|f\|_\infty \to 0} \|g_f(t)\|_\infty = 0, \quad t \geqslant 0. \tag{2.10}$$

Proof. It follows from [16, Theorem 2.23] and (1.4) that $V_t f(x)$ also satisfies

$$V_t f(x) = -\int_0^t T_s[r(\cdot, V_{t-s}f(\cdot))](x)ds + T_t f(x), \quad t \geqslant 0, \quad x \in E, \tag{2.11}$$

which implies that

$$R_f(t,x) = \int_0^t T_s[r(\cdot, V_{t-s}f(\cdot))](x)ds, \quad t \geqslant 0, \quad x \in E. \tag{2.12}$$

Since $r(x,s) \geqslant 0$, we have $R_f(t,x) \geqslant 0$, i.e., (2.9) holds.

Since $T_0 f(x) = V_0 f(x) = f(x)$, we have $R_f(0,x) = 0$, which implies that $g_f(0,x) = 0$. Thus, it suffices to prove that, for any $\delta > 0$, (2.10) is true for $t > \delta$. It follows from Lemma 2.1 that for any $H \geqslant 1$,

$$\begin{aligned} r(x, V_{t-s}f(x)) &\leqslant (2+H)M(V_{t-s}f(x))^2 + V_{t-s}f(x) \int_H^\infty \theta\, n(x, d\theta) \\ &\leqslant (2+H)M(T_{t-s}f(x))^2 + T_{t-s}f(x) \int_H^\infty \theta\, n(x, d\theta). \end{aligned}$$

By (2.12), we have

$$\begin{aligned} R_f(t,x) &\leqslant (2+H)M \int_0^t T_s[(T_{t-s}f)^2](x)ds + \int_0^t T_s\left[T_{t-s}f \int_H^\infty \theta\, n(\cdot, d\theta)\right](x)ds \\ &=: \text{(I)} + \text{(II)}. \end{aligned}$$

For Part (I), since $T_{t-s}f(x) \leqslant \|f\|_\infty \mathrm{e}^{Mt}$, we have $T_s[(T_{t-s}f)^2](x) \leqslant \mathrm{e}^{Mt}\|f\|_\infty T_t f(x)$. Thus, by (2.3), we have that, for any $t > \delta$,

$$\begin{aligned} \text{(I)} &\leqslant (2+H)M\mathrm{e}^{Mt} t \|f\|_\infty T_t f(x) \\ &\leqslant (2+H)M\mathrm{e}^{Mt} t (1 + c(\delta)) \mathrm{e}^{\lambda_0 t} \langle f, \psi_0 \rangle_m \phi_0(x) \|f\|_\infty. \end{aligned}$$

Hence we have that, for any $t > \delta$,

$$\lim_{\|f\|_\infty \to 0} \sup_{x \in E} \frac{\text{(I)}}{\mathrm{e}^{\lambda_0 t} \langle f, \psi_0 \rangle_m \phi_0(x)} = 0. \tag{2.13}$$

Now we deal with Part (II). For any $H > 1$, $t > \delta$ and $0 < \epsilon < t$,

$$\left(\int_0^\epsilon + \int_{t-\epsilon}^t\right) T_s\left[T_{t-s}f \int_H^\infty \theta\, n(\cdot, d\theta)\right](x)ds \leqslant 2\epsilon M T_t f(x) \leqslant 2\epsilon M (1 + c(\delta)) \mathrm{e}^{\lambda_0 t} \langle f, \psi_0 \rangle_m \phi_0(x). \tag{2.14}$$

For any $\epsilon < s < t - \epsilon$, by (2.3),

$$T_{t-s} f(x) \leqslant (1 + c(\epsilon)) \mathrm{e}^{\lambda_0 t} \langle f, \psi_0 \rangle_m \phi_0(x).$$

Hence, we have

$$\begin{aligned} &\int_\epsilon^{t-\epsilon} T_s\left[T_{t-s}f \int_H^\infty \theta\, n(\cdot, d\theta)\right](x)ds \\ &\quad \leqslant (1 + c(\epsilon)) \mathrm{e}^{\lambda_0 t} \langle f, \psi_0 \rangle_m \int_\epsilon^{t-\epsilon} T_s\left[\int_H^\infty \theta\, n(\cdot, d\theta) \phi_0\right](x)ds \end{aligned}$$

$$\leqslant (1+c(\epsilon))^2 \mathrm{e}^{2\lambda_0 t} t \left\langle \int_H^\infty \theta\, n(\cdot, d\theta)\phi_0, \psi_0 \right\rangle_m \langle f, \psi_0\rangle_m \phi_0(x). \tag{2.15}$$

Thus, combining (2.14) and (2.15), we get that

$$\limsup_{\|f\|_\infty \to 0} \sup_{x\in E} \frac{\text{(II)}}{\mathrm{e}^{\lambda_0 t}\langle f, \psi_0\rangle_m \phi_0(x)} \leqslant 2\epsilon M(1+c(\delta)) + (1+c(\epsilon))^2 \mathrm{e}^{\lambda_0 t} t \left\langle \int_H^\infty \theta\, n(\cdot, d\theta)\phi_0, \psi_0 \right\rangle_m.$$

Now, first letting $H \to \infty$ and then $\epsilon \to 0$, applying the monotone convergence theorem, we get that

$$\lim_{H\to\infty} \limsup_{\|f\|_\infty \to 0} \sup_{x\in E} \frac{\text{(II)}}{\mathrm{e}^{\lambda_0 t}\langle f, \psi_0\rangle_m \phi_0(x)} = 0. \tag{2.16}$$

Combining (2.13) and (2.16), we get that

$$\lim_{\|f\|_\infty \to 0} \sup_{x\in E} \frac{R_f(t,x)}{\mathrm{e}^{\lambda_0 t}\langle f, \psi_0\rangle_m \phi_0(x)} = 0.$$

This completes the proof. □

3 Extinction probability

Recall that, for any $t > 0$ and $x \in E$,

$$q_t(x) = \mathbb{P}_{\delta_x}(\|X_t\| = 0) \quad \text{and} \quad q(x) = \lim_{t\to\infty} q_t(x).$$

Lemma 3.1. *For any $x \in E$, $q(x) < 1$.*

Proof. Let $v_t(x) := -\log q_t(x)$. Recall that $v(x) = -\log q(x)$. Since $q_t(x) < 1$, we have $v_t(x) > 0$. By Assumption 1.4, we have for $s > t_0$,

$$\|v\|_\infty \leqslant \|v_s\|_\infty \leqslant \|v_{t_0}\|_\infty = -\log\left(\inf_{x\in E} q_{t_0}(x)\right) < \infty.$$

Recall that, for $\theta > 0$,

$$V_t\theta(x) = -\log \mathbb{P}_{\delta_x} \mathrm{e}^{-\langle \theta, X_t\rangle}.$$

By the Markov property of X,

$$q_{t+s}(x) = \lim_{\theta\to\infty} \mathbb{P}_{\delta_x}(\mathrm{e}^{-\theta\|X_{t+s}\|}) = \lim_{\theta\to\infty} \mathbb{P}_{\delta_x}(\mathrm{e}^{-\langle V_s\theta, X_t\rangle}) = \mathbb{P}_{\delta_x}(\mathrm{e}^{-\langle v_s, X_t\rangle}). \tag{3.1}$$

It follows from (2.8) that, for any $s > t_0$,

$$v_{t+s}(x) = V_t(v_s)(x) = T_t(v_s)(x) - R_{v_s}(t,x) = T_t(v_s)(x) - g_{v_s}(t,x)\mathrm{e}^{\lambda_0 t}\langle v_s, \psi_0\rangle_m \phi_0(x). \tag{3.2}$$

Thus, for $s > t_0$, we have

$$\langle v_{t+s}, \psi_0\rangle_m \geqslant (1 - \|g_{v_s}(t)\|_\infty)\mathrm{e}^{\lambda_0 t}\langle v_s, \psi_0\rangle_m.$$

Since $v_t(x)$ is positive and non-increasing in t, we have that for all $t > 0$ and $s > t_0$,

$$(1 - \|g_{v_s}(t)\|_\infty)\mathrm{e}^{\lambda_0 t} \leqslant 1. \tag{3.3}$$

We claim that $\langle v, \psi_0\rangle_m > 0$. Otherwise, $\langle v, \psi_0\rangle_m = 0$. By (3.2), we have

$$\|v_{1+s}\|_\infty \leqslant \|T_1(v_s)\|_\infty \leqslant (1+c)\mathrm{e}^{\lambda_0}\langle v_s, \psi_0\rangle_m \|\phi_0\|_\infty \to 0$$

as $s \to \infty$. Thus $\lim_{s\to\infty} \|v_s\|_\infty = 0$. Hence by (2.10),

$$\lim_{s\to\infty} \|g_{v_s}(t)\|_\infty = 0.$$

It follows from (3.3) that, for all $t > 0$, $\mathrm{e}^{\lambda_0 t} \leqslant 1$, which leads to a contradiction to the assumption that $\lambda_0 > 0$. Hence the claim above is valid.

By letting $s \to \infty$ in (3.1), we get that

$$\exp\{-v(x)\} = q(x) = \mathbb{P}_{\delta_x} \exp\{-\langle v, X_t\rangle\}. \tag{3.4}$$

Let c and γ be the constants in (2.2) with $\delta = 1$. For t large enough, we have $1 - c\mathrm{e}^{-\gamma t} > 0$. Thus for t large enough, we have

$$T_t v(x) \geqslant (1 - c\mathrm{e}^{-\gamma t})\mathrm{e}^{\lambda_0 t}\langle v, \psi_0\rangle_m \phi_0(x) > 0.$$

Hence for all $x \in E$ and t large enough,

$$\mathbb{P}_{\delta_x}(\langle v, X_t\rangle > 0) > 0,$$

which implies that

$$q(x) = \mathbb{P}_{\delta_x}(\mathrm{e}^{-\langle v, X_t\rangle}) < 1.$$

The proof is now completed. □

Lemma 3.2. $V := \lim_{t\to\infty}\langle v, X_t\rangle \in [0, \infty]$ *exists, and satisfies that, for all* $x \in E$,

$$\mathbb{P}_{\delta_x}(V = 0) = \exp\{-v(x)\} = q(x)$$

and

$$\mathbb{P}_{\delta_x}(V = \infty) = 1 - \exp\{-v(x)\} = 1 - q(x).$$

Moreover, for any $\theta > 0$*, we have*

$$\lim_{t\to\infty} V_t(\theta v)(x) = v(x), \quad x \in E. \tag{3.5}$$

Proof. By (3.4) and the Markov property of X, $\{\mathrm{e}^{-\langle v, X_t\rangle}, t \geqslant 0\}$ is a bounded martingale. Thus $\lim_{t\to\infty}\exp\{-\langle v, X_t\rangle\}$ exists and is in $[0, 1]$, which implies that $V := \lim_{t\to\infty}\langle v, X_t\rangle \in [0, \infty]$ exists.

Since $\exp\{-\langle v, X_t\rangle\} \leqslant 1$, we have

$$\mathbb{P}_{\delta_x} \exp\{-V\} = \exp\{-v(x)\}. \tag{3.6}$$

On the other hand,

$$\mathbb{P}_{\delta_x}(V = 0) \geqslant \lim_{t\to\infty} \mathbb{P}_{\delta_x}(\|X_t\| = 0) = \exp\{-v(x)\},$$

which implies that

$$\mathbb{P}_{\delta_x} \exp\{-V\} \geqslant \mathbb{P}_{\delta_x}(V = 0) \geqslant \exp\{-v(x)\}. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\mathbb{P}_{\delta_x}(V = 0) = \exp\{-v(x)\} \quad \text{and} \quad \mathbb{P}_{\delta_x}(V = \infty) = 1 - \exp\{-v(x)\}.$$

By the dominated convergence theorem, we get that, for any $\theta > 0$,

$$\lim_{t\to\infty} V_t(\theta v)(x) = -\log \mathbb{P}_{\delta_x}(\mathrm{e}^{-\theta V}) = -\log \mathbb{P}_{\delta_x}(V = 0) = v(x).$$

This completes the proof. □

4 Backward iterates

It is clear that $V_t 0 = 0$. It follows from (3.4) that $V_t v = v$. Thus v and 0 are two fixed points of V_t.

Definition 4.1. A family $(\eta_t, t \geqslant 0) \subset \mathcal{B}^+(E)$ is called a family of backward iterates if $\eta_t \leqslant v$ for all $t \geqslant 0$ and

$$\eta_t(x) = V_s(\eta_{t+s})(x), \quad t, s \geqslant 0, \quad x \in E.$$

A family $(\eta_t, t \geqslant 0)$ of backward iterates is said to be non-trivial if, for some $t \geqslant 0$, neither $\eta_t = 0, m$-a.e. nor $\eta_t = v, m$-a.e.

It is well known that (see, for example, [25, (1.44)]), there exists a metric d on $C(E)$ such that $(C(E), d)$ is a complete metric space, and convergence in $(C(E), d)$ is equivalent to uniform convergence on each compact subset K of E.

For any $a > 0$, let $\mathcal{D}_a(E) := \{f \in \mathcal{B}^+(E) : \|f\|_\infty \leqslant a\}$.

Lemma 4.2. *For any $t > 0$, $V_t(\mathcal{D}_a(E))$ is a relatively compact subset of $C(E)$.*

Proof. Without loss of generality, we only prove the lemma for $t = 1$ and $a \geqslant 1$. We first show that, for any compact subset $K \subset E$, $\{V_1 f : f \in \mathcal{D}_a(E)\}$ are equicontinuous on K.

Recall that

$$V_1 f(x) = T_1 f(x) - \int_0^1 T_s(r(\cdot, V_{1-s} f))(x) ds.$$

It is clear that for $f \in \mathcal{D}_a(E)$,

$$|T_1 f(x) - T_1 f(y)| \leqslant a \int_E |q(1, x, z) - q(1, y, z)| \, m(dz).$$

Note that by (1.11), we have

$$q(1, x, z) \leqslant c_1 \|\phi_0\|_\infty \psi_0(z). \tag{4.1}$$

Since $\psi_0 \in L^1(E, m)$, for any $\epsilon > 0$, we can choose a compact set $\widetilde{K} \subset E$ such that

$$2c_1 \|\phi_0\|_\infty \int_{\widetilde{K}^c} \psi_0(z) m(dz) < \frac{\epsilon}{2}.$$

Using the continuity of $q(1, \cdot, \cdot)$ on $K \times \widetilde{K}$, (4.1) and the fact that $\psi_0 \in L^1(E, m)$, we can find a $\delta > 0$ such that for any $x, y \in K$ with $|x - y| < \delta$,

$$\int_{\widetilde{K}} |q(1, x, z) - q(1, y, z)| \, m(dz) < \frac{\epsilon}{2}.$$

Thus $\{T_1 f : f \in \mathcal{D}_a(E)\}$ are equicontinuous on K. By (2.4) and the fact that $V_t f(x) \leqslant T_t f(x) \leqslant \mathrm{e}^{Mt} \|f\|_\infty$, we have that for any $f \in \mathcal{D}_a(E)$, $r(x, V_{1-s} f(x)) \leqslant 2M(\mathrm{e}^{2M} a^2 + \mathrm{e}^M a)$. Thus

$$\begin{aligned}
&\left| \int_0^1 T_s(r(\cdot V_{1-s} f)(x) ds - \int_0^1 T_s(r(\cdot, V_{1-s} f)(y) ds \right| \\
&\quad \leqslant [2M(\mathrm{e}^{2M} a^2 + \mathrm{e}^M a)] \int_0^1 \int_E |q(s, x, z) - q(s, y, z)| \, m(dz) ds.
\end{aligned}$$

Note that, for $\eta \in (0, 1)$,

$$\begin{aligned}
&\int_0^1 \int_E |q(s, x, z) - q(s, y, z)| \, m(dz) ds \\
&\quad \leqslant \int_0^\eta \int_E q(s, x, z) + q(s, y, z) \, m(dz) ds + \int_\eta^1 \int_E |q(s, x, z) - q(s, y, z)| \, m(dz) ds \\
&\quad \leqslant 2\mathrm{e}^M \eta + \int_\eta^1 \int_E |q(s, x, z) - q(s, y, z)| \, m(dz) ds
\end{aligned}$$

$$\leqslant 2\mathrm{e}^M\eta + \int_\eta^1 \int_E \int_E \left|q\left(\frac{\eta}{2}, x, w\right) - q\left(\frac{\eta}{2}, y, w\right)\right| q\left(s - \frac{\eta}{2}, w, z\right) m(dw)\, m(dz) ds.$$

For any $\epsilon > 0$, we choose $\eta \in (0,1)$ so that $2\mathrm{e}^M\eta < \frac{\epsilon}{3}$. It follows from (2.1) that, for any $s \in [\eta/2, 1)$,

$$q(s, x, z) \leqslant \left(1 + c\left(\frac{\eta}{2}\right)\right) \mathrm{e}^{\lambda_0} \|\phi_0\|_\infty \psi_0(z). \tag{4.2}$$

Hence

$$\begin{aligned}&\int_\eta^1 \int_E \int_E \left|q\left(\frac{\eta}{2}, x, w\right) - q\left(\frac{\eta}{2}, y, w\right)\right| q\left(s - \frac{\eta}{2}, w, z\right) m(dw)\, m(dz) ds \\ &\quad\leqslant \left(1 + c\left(\frac{\eta}{2}\right)\right) \mathrm{e}^{\lambda_0} \|\phi_0\|_\infty \int_\eta^1 \int_E \int_E \left|q\left(\frac{\eta}{2}, x, w\right) - q\left(\frac{\eta}{2}, y, w\right)\right| m(dw) \psi_0(z)\, m(dz) ds.\end{aligned}$$

Applying (4.2) again, we can find a compact $\widetilde{K} \subset E$ such that

$$\left(1 + c\left(\frac{\eta}{2}\right)\right) \mathrm{e}^{\lambda_0} \|\phi_0\|_\infty \int_\eta^1 \int_E \int_{\widetilde{K}^c} \left|q\left(\frac{\eta}{2}, x, w\right) - q\left(\frac{\eta}{2}, y, w\right)\right| m(dw) \psi_0(z)\, m(dz) ds < \frac{\epsilon}{3}.$$

Using the continuity of $q(\frac{\eta}{2}, \cdot, \cdot)$ on $K \times \widetilde{K}$, (4.2) and the fact that $\psi_0 \in L^1(E, m)$, we can find a $\delta > 0$ such that for any $x, y \in K$ with $|x - y| < \delta$,

$$\left(1 + c\left(\frac{\eta}{2}\right)\right) \mathrm{e}^{\lambda_0} \|\phi_0\|_\infty \int_\eta^1 \int_E \int_{\widetilde{K}} \left|q\left(\frac{\eta}{2}, x, w\right) - q\left(\frac{\eta}{2}, y, w\right)\right| m(dw) \psi_0(z)\, m(dz) ds < \frac{\epsilon}{3}.$$

Thus $\{\int_0^1 T_s(r(\cdot, V_{1-s}f))(x) ds : f \in \mathcal{D}_a(E)\}$ are equicontinuous on K. It follows that $\{V_1 f : f \in \mathcal{D}_a(E)\}$ are equicontinuous on K. In particular, $V_1(\mathcal{D}_a(E)) \subset C(E)$.

Let K_n be an increasing sequence of compact subsets of E. Using the equicontinuity of $\{V_1 f : f \in \mathcal{D}_a(E)\}$ on each K_n and a routine diagonalization argument, we can easily show that any sequence of functions in $V_1(\mathcal{D}_a(E))$ contains a subsequence which converges in $C(E)$. □

Proposition 4.3. *There exists a non-trivial family of backward iterates.*

Proof. Let $\mathcal{D} = \{f \in \mathcal{B}^+(E) : f \leqslant v\}$. For any $f \in \mathcal{D}$, $0 \leqslant V_s f \leqslant V_s v \leqslant v$. Thus $V_{t+s} f = V_t(V_s f) \in V_t(\mathcal{D})$, which implies that $t \to V_t(\mathcal{D})$ is decreasing.

For any $g, h \in \mathcal{D}$ and $t > 0$, it is easy to see that $\lambda \mapsto V_t(\lambda g + (1-\lambda)h)$, $\lambda \in [0,1]$, is a continuous curve in $V_t(\mathcal{D})$ connecting $V_t g$ to $V_t h$. Thus $V_t(\mathcal{D})$ is connected. Hence $\mathcal{D}_\infty := \bigcap_{n\in\mathbb{N}} V_n(\mathcal{D})$ is connected. Since $0, v \in \mathcal{D}_\infty$ and $v > 0$, there exists an $\eta \in \mathcal{D}_\infty$, such that $0 < \eta < v$ on a set of positive measure. Thus, for every $n \in \mathbb{N}$, there exists $\eta_{n,n} \in \mathcal{D}$ such that $V_n(\eta_{n,n}) = \eta$. Define $\eta_{n,j} = V_{n-j}(\eta_{n,n})$, $j = 0, 1, 2, \ldots, n-1$. Note that $\eta_{n,0} = \eta$.

Since $0 \leqslant \eta_{n,j} \leqslant v$ and $\eta_{n,j} = V_1(\eta_{n,j+1})$, it follows from Lemma 4.2 that $\{\eta_{n,j}, n \geqslant 0, 0 \leqslant j < n\}$ is relatively compact in $C(E)$. Thus there exists a sequence $(\eta_{n_l,j})$ such that $\eta_j := \lim_{l\to\infty} \eta_{n_l,j} \in C(E) \cap \mathcal{D}$ exists. Since $\eta_{n_l,j} = V_1(\eta_{n_l,j+1})$, letting $l \to \infty$, we get that $\eta_j = V_1(\eta_{j+1})$.

Define $\eta_t := V_{[t+1]-t}(\eta_{[t+1]})$, for $t \geqslant 0$. Then

$$V_s(\eta_{t+s}) = V_s V_{[t+s+1]-t-s}(\eta_{[t+s+1]}) = V_{[t+s+1]-t}(\eta_{[t+s+1]}) = V_{[t+1]-t}(\eta_{[t+1]}) = \eta_t.$$

It follows from $\eta_{n,0} = \eta$ that $\eta_0 = \eta$, which implies that the family $\{\eta_t, t \geqslant 0\}$ of backward iterates is nontrivial. □

Lemma 4.4. *If $0 \leqslant f \leqslant v$, then*

$$V_t f(x) \geqslant \mathrm{e}^{-at} T_t f(x), \quad t \geqslant 0, \quad x \in E,$$

where $a = 2M(1 + \|v\|_\infty)$.

Proof. Using (2.4) with $H=1$ and the fact that $V_{t-s}f(x)\leqslant V_{t-s}v(x)=v(x)\leqslant\|v\|_\infty$ for any $0\leqslant s\leqslant t$ and any $x\in E$, we get

$$r(x,V_{t-s}f(x))\leqslant 2M(1+\|v\|_\infty)V_{t-s}f(x)=aV_{t-s}f(x),\quad 0\leqslant s\leqslant t,\quad x\in E.$$

Recall that

$$V_tf(x)=T_tf(x)-\int_0^t T_s(r(\cdot,V_{t-s}f(\cdot)))(x)ds.$$

Thus we have

$$V_tf(x)=\mathrm{e}^{-at}T_tf(x)-\int_0^t \mathrm{e}^{-as}T_s(r(\cdot,V_{t-s}f(\cdot)))(x)ds+a\int_0^t \mathrm{e}^{-as}T_s(V_{t-s}f)(x)ds.$$

Consequently we have

$$V_tf(x)\geqslant \mathrm{e}^{-at}T_tf(x),\quad t\geqslant 0,\quad x\in E.$$

This completes the proof. □

Lemma 4.5. *If $(\eta_t,t\geqslant 0)$ is a non-trivial family of backward iterates, then*

$$\lim_{t\to\infty}\|\eta_t\|_\infty=0.$$

Proof. Without loss of generality, we may and will assume that $m(\eta_0>0)>0$ and $m(\eta_0<v)>0$. We claim that $\langle\eta_t,\psi_0\rangle_m\to 0$ as $t\to\infty$. Otherwise, there exist a sequence $t_j\uparrow\infty$ and a constant $c_0>0$ such that

$$\langle\eta_{t_j},\psi_0\rangle_m>c_0.$$

Fix $s>1$ large enough such that $1-c\mathrm{e}^{-\gamma s}>0$, where c and γ are the constants in (2.2) with $\delta=1$. Then, for j large enough so that $t_j>s$, we have

$$\eta_0(x)=V_{t_j}(\eta_{t_j})(x)=V_{t_j-s}(V_s(\eta_{t_j}))(x),\quad x\in E. \tag{4.3}$$

By Lemma 4.4, we have

$$V_s(\eta_{t_j})(x)\geqslant \mathrm{e}^{-as}T_s(\eta_{t_j})(x)\geqslant \mathrm{e}^{-as}(1-c\mathrm{e}^{-\gamma s})\mathrm{e}^{\lambda_0 s}\langle\eta_{t_j},\psi_0\rangle_m\phi_0(x)\geqslant c_0\mathrm{e}^{-as}(1-c\mathrm{e}^{-\gamma s})\mathrm{e}^{\lambda_0 s}\phi_0(x).$$

It follows from (2.7) and (2.8) that

$$v(x)=V_1v(x)\leqslant T_1v(x)\leqslant (1+c)\mathrm{e}^{\lambda_0}\langle v,\psi_0\rangle_m\phi_0(x). \tag{4.4}$$

Thus

$$V_s(\eta_{t_j})(x)\geqslant c_0\mathrm{e}^{-as}(1-c\mathrm{e}^{-\gamma s})\mathrm{e}^{\lambda_0 s}(1+c)^{-1}\mathrm{e}^{-\lambda_0}\langle v,\psi_0\rangle_m^{-1}v(x)=:C_sv(x). \tag{4.5}$$

Combining (4.5), (4.3) and (3.5), we get

$$\eta_0(x)\geqslant \lim_{j\to\infty}V_{t_j-s}(C_sv)(x)=v(x),$$

which contradicts the definition of η_0. Thus the claim above is true.

Note that, for any $s\geqslant 1$ and $t>0$,

$$\eta_t(x)=V_s(\eta_{t+s})(x)\leqslant T_s(\eta_{t+s})(x)\leqslant (1+c\mathrm{e}^{-\gamma s})\mathrm{e}^{\lambda_0 s}\langle\eta_{t+s},\psi_0\rangle_m\phi_0(x). \tag{4.6}$$

Thus

$$\|\eta_t\|_\infty\leqslant (1+c)\mathrm{e}^{\lambda_0 s}\langle\eta_{t+s},\psi_0\rangle_m\|\phi_0\|_\infty\to 0,$$

as $t\to\infty$. The proof is now completed. □

Lemma 4.6. *If $(\eta_t, t \geqslant 0)$ is a non-trivial family of backward iterates, then there exist $\{h_t : t \geqslant 0\} \subset \mathcal{B}_b(E)$ such that*

$$\eta_t(x) = (1 + h_t(x))\langle \eta_t, \psi_0\rangle_m \phi_0(x), \quad t \geqslant 0, \quad x \in E,$$

and

$$\lim_{t\to\infty} \|h_t\|_\infty = 0.$$

Moreover,

$$\frac{\langle \eta_t, \psi_0\rangle_m}{\langle \eta_{t+s}, \psi_0\rangle_m} \geqslant \mathrm{e}^{\lambda_0 s}, \quad \forall\, s, t \geqslant 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{\langle \eta_t, \psi_0\rangle_m}{\langle \eta_{t+s}, \psi_0\rangle_m} = \mathrm{e}^{\lambda_0 s}, \quad \forall\, s \geqslant 0. \tag{4.7}$$

Proof. Since $\langle \eta_t, \psi_0\rangle_m \phi_0(x) > 0$, we can define

$$h_t(x) := \frac{\eta_t(x)}{\langle \eta_t, \psi_0\rangle_m \phi_0(x)} - 1, \quad t \geqslant 0, \quad x \in E.$$

It follows from (2.2) and Lemma 2.2 that for $s > 1$,

$$\eta_t(x) = V_s(\eta_{t+s})(x) = T_s(\eta_{t+s})(x) - R_{\eta_{t+s}}(s, x) \tag{4.8a}$$

$$\geqslant \mathrm{e}^{\lambda_0 s}(1 - c\mathrm{e}^{-\gamma s} - \|g_{\eta_{t+s}}(s)\|_\infty)\langle \eta_{t+s}, \psi_0\rangle_m \phi_0(x), \tag{4.8b}$$

where c and γ are the constants in (2.2) with $\delta = 1$. In the remainder of this proof, we assume that s is large enough such that $1 - c\mathrm{e}^{-\gamma s} > 0$. By Lemma 4.5 and (2.10), $\lim_{t\to\infty} \|g_{\eta_{t+s}}(s)\|_\infty = 0$. Thus, for large enough t, $1 - c\mathrm{e}^{-\gamma s} - \|g_{\eta_{t+s}}(s)\|_\infty > 0$. It follows from (4.6) and (4.8b) that

$$-\frac{2c\mathrm{e}^{-\gamma s} + \|g_{\eta_{t+s}}(s)\|_\infty}{1 + c\mathrm{e}^{-\gamma s}} \leqslant h_t(x) = \frac{\eta_t(x)}{\langle \eta_t, \psi_0\rangle_m \phi_0(x)} - 1 \leqslant \frac{2c\mathrm{e}^{-\gamma s} + \|g_{\eta_{t+s}}(s)\|_\infty}{1 - c\mathrm{e}^{-\gamma s} - \|g_{\eta_{t+s}}(s)\|_\infty}.$$

Letting $t \to \infty$ and then $s \to \infty$, we get that

$$\lim_{t\to\infty} \|h_t\|_\infty = 0.$$

It follows from (4.8a) that

$$\begin{aligned}\langle \eta_t, \psi_0\rangle_m &= \langle T_s(\eta_{t+s}), \psi_0\rangle_m - \langle R_{\eta_{t+s}}(s, x), \psi_0\rangle_m \\ &= \mathrm{e}^{\lambda_0 s}\langle \eta_{t+s}, \psi_0\rangle_m - \mathrm{e}^{\lambda_0 s}\langle \eta_{t+s}, \psi_0\rangle_m \langle g_{\eta_{t+s}}(s)\phi_0, \psi_0\rangle_m,\end{aligned}$$

which implies that

$$0 \leqslant \mathrm{e}^{\lambda_0 s} - \frac{\langle \eta_t, \psi_0\rangle_m}{\langle \eta_{t+s}, \psi_0\rangle_m} = \mathrm{e}^{\lambda_0 s}\langle g_{\eta_{t+s}}(s)\phi_0, \psi_0\rangle_m \leqslant \mathrm{e}^{\lambda_0 s}\|g_{\eta_{t+s}}(s)\|_\infty \to 0,$$

as $t \to \infty$, where the last limit follows from (2.10) and Lemma 4.5. The proof is now completed. □

Remark 4.7. Let $L(t) := \mathrm{e}^{\lambda_0 t}\langle \eta_t, \psi_0\rangle_m$. It follows from (4.7) that $t \to L(t)$ is nondecreasing and, for any $s \geqslant 0$,

$$\lim_{t\to\infty} \frac{L(t+s)}{L(t)} = 1.$$

Therefore, $l_0 := \lim_{t\to\infty} L(t) \in (0, \infty]$ exists.

5 Seneta-Heyde norming for M

Lemma 5.1. *If $f \in \mathcal{B}^+(E)$ satisfies $f(x) \leqslant v(x)$ and $f(x) = V_t f(x)$ for all $t > 0$ and $x \in E$, then either $f(x) = v(x)$ for all $x \in E$ or $f(x) = 0$ for all $x \in E$.*

Proof. Let $s_0 > 1$ be such that $1 - c\mathrm{e}^{-\gamma s_0} > 0$, where c and γ are the constants from (2.2) with $\delta = 1$. By Lemma 4.4 and (4.4), we have

$$f(x) = V_{s_0} f(x) \geqslant \mathrm{e}^{-as_0} T_{s_0} f(x) \geqslant \mathrm{e}^{-as_0}(1 - c\mathrm{e}^{-\gamma s_0})\mathrm{e}^{\lambda_0 s_0}\langle f, \psi_0\rangle_m \phi_0(x) \geqslant c_0\langle f, \psi_0\rangle_m v(x)$$

for some constant $c_0 > 0$. If $\langle f, \psi_0\rangle_m > 0$, then by (3.5) we have

$$f(x) = \lim_{t\to\infty} V_t f(x) \geqslant \lim_{t\to\infty} V_t(c_0\langle f, \psi_0\rangle_m v)(x) = v(x),$$

which implies that $f(x) = v(x)$ for all $x \in E$. If $\langle f, \psi_0\rangle_m = 0$, then

$$f(x) \leqslant T_1 f(x) \leqslant (1 + c)\mathrm{e}^{\lambda_0}\langle f, \psi_0\rangle_m \phi_0(x) = 0,$$

which implies that $f(x) = 0$ for all $x \in E$. The proof is now completed. □

Theorem 5.2. *Let $(\eta_t, t \geqslant 0)$ be a non-trivial family of backward iterates and define $\gamma_t := \langle \eta_t, \psi_0\rangle_m$. Then there is a non-degenerate random variable W such that for any nonzero $\mu \in \mathcal{M}_F(E)$,*

$$\lim_{t\to\infty} \gamma_t\langle \phi_0, X_t\rangle = W, \quad a.s.\text{-}\mathbb{P}_\mu$$

and

$$\mathbb{P}_\mu(W = 0) = \exp\{-\langle v, \mu\rangle\}, \quad \mathbb{P}_\mu(W < \infty) = 1. \tag{5.1}$$

Proof. By the definition of η_t, for any nonzero $\mu \in \mathcal{M}_F(E)$,

$$\mathbb{P}_\mu(\exp\{-\eta_{t+s}, X_{t+s}\} \,|\, \mathcal{G}_t) = \mathbb{P}_{X_t}(\exp\{-\eta_{t+s}, X_s\}) = \exp\{-\langle \eta_t, X_t\rangle\},$$

which implies that $\{\exp\{-\langle \eta_t, X_t\rangle\}, t \geqslant 0\}$ is a non-negative martingale. Thus, by the martingale convergence theorem, $\lim_{t\to\infty} \exp\{-\langle \eta_t, X_t\rangle\}$ exists $\mathbb{P}_\mu$ almost surely and hence $W := \lim_{t\to\infty}\langle \eta_t, X_t\rangle \in [0, \infty]$ exists $\mathbb{P}_\mu$ almost surely.

It follows from Lemma 4.6 that

$$(1 - \|h_t\|_\infty)\gamma_t\langle \phi_0, X_t\rangle \leqslant \langle \eta_t, X_t\rangle \leqslant (1 + \|h_t\|_\infty)\gamma_t\langle \phi_0, X_t\rangle.$$

Since $\lim_{t\to\infty} \|h_t\|_\infty = 0$, we have $1 - \|h_t\|_\infty > 0$ for t large enough. Thus for large t,

$$(1 + \|h_t\|_\infty)^{-1}\langle \eta_t, X_t\rangle \leqslant \gamma_t\langle \phi_0, X_t\rangle \leqslant (1 - \|h_t\|_\infty)^{-1}\langle \eta_t, X_t\rangle.$$

Letting $t \to \infty$, we get that

$$\lim_{t\to\infty} \gamma_t\langle \phi_0, X_t\rangle = W, \quad \text{a.s.-}\mathbb{P}_\mu.$$

Define

$$\Phi(s, x) := -\log \mathbb{P}_{\delta_x} \exp\{-sW\}. \tag{5.2}$$

Then

$$\begin{aligned} -\log \mathbb{P}_\mu \exp\{-sW\} &= \lim_{t\to\infty} -\log \mathbb{P}_\mu \exp\{-s\gamma_t\langle \phi_0, X_t\rangle\} \\ &= \lim_{t\to\infty}\langle -\log \mathbb{P}_{\delta_x} \exp\{-s\gamma_t\langle \phi_0, X_t\rangle\}, \mu\rangle = \langle \Phi(s, \cdot), \mu\rangle. \end{aligned}$$

Put $\Phi_\infty(x) := \lim_{s\to\infty} \Phi(s, x)$ and $\Phi_0(x) = \lim_{s\to 0} \Phi(s, x)$. Then

$$\mathbb{P}_\mu(W = 0) = \lim_{s\to\infty} \mathbb{P}_\mu \exp\{-sW\} = \exp\{-\langle \Phi_\infty, \mu\rangle\}$$

and

$$\mathbb{P}_\mu(W < \infty) = \lim_{s\to 0} \mathbb{P}_\mu \exp\{-sW\} = \exp\{-\langle \Phi_0, \mu\rangle\}.$$

For any $s, t > 0$, we have

$$\exp\{-\Phi(s, x)\} = \lim_{u\to\infty} \mathbb{P}_{\delta_x}(\exp\{-s\gamma_{t+u}\langle \phi_0, X_{t+u}\rangle\})$$

$$= \lim_{u\to\infty} \mathbb{P}_{\delta_x}\mathbb{P}_{X_t}(\exp\{-s\gamma_{t+u}\gamma_u^{-1}\gamma_u\langle\phi_0, X_u\rangle\})$$
$$= \mathbb{P}_{\delta_x}\mathbb{P}_{X_t}\exp\{-s\mathrm{e}^{-\lambda_0 t}W\} = \mathbb{P}_{\delta_x}\exp\{-\langle\Phi(s\mathrm{e}^{-\lambda_0 t},\cdot), X_t\rangle\},$$

which implies that

$$\Phi(s,x) = V_t(\Phi(s\mathrm{e}^{-\lambda_0 t},\cdot))(x). \tag{5.3}$$

Thus, letting $s\to\infty$ and $s\to 0$, we get respectively

$$\Phi_\infty(x) = V_t(\Phi_\infty)(x), \quad \Phi_0(x) = V_t(\Phi_0)(x).$$

Since $s\to\Phi(s,x)$ is increasing, we have $\Phi_0(x)\leqslant\Phi(1,x)\leqslant\Phi_\infty(x)$. Note that

$$\Phi(1,x) = -\log\mathbb{P}_{\delta_x}\exp\{-W\} = \lim_{t\to\infty} -\log\mathbb{P}_{\delta_x}\exp\{-\langle\eta_t, X_t\rangle\} = \eta_0(x).$$

On the other hand,

$$\mathbb{P}_{\delta_x}(W=0) \geqslant \mathbb{P}_{\delta_x}(\exists\, t>0, \|X_t\|=0) = \exp\{-v(x)\},$$

which implies that, for all $x\in E$, $\Phi_\infty(x)\leqslant v(x)$. Thus, $\Phi_0(x)\leqslant\eta_0(x)\leqslant\Phi_\infty(x)\leqslant v(x)$. It follows from Lemma 5.1 that $\Phi_0=0$, $\Phi_\infty=v$. The proof is now completed. □

Now Theorem 1.5 follows immediately from Lemma 4.6 and Theorem 5.2.

Proposition 5.3. *Let $(\eta_t, t\geqslant 0)$ be a non-trivial family of backward iterates and W be the limit in Theorem* 5.2 *corresponding to $(\eta_t, t\geqslant 0)$. Then a family $(\eta_t^*, t\geqslant 0)$ is a non-trivial family of backward iterates if and only if there exists a positive number a such that*

$$\eta_t^*(x) = \Phi(a\mathrm{e}^{-\lambda_0 t}, x), \quad t\geqslant 0, \quad x\in E,$$

where Φ is defined in (5.2). *In particular,*

$$\eta_t(x) = \Phi(\mathrm{e}^{-\lambda_0 t}, x), \quad t\geqslant 0, \quad x\in E. \tag{5.4}$$

Proof. For any $a>0$, by (5.3), we have

$$V_s(\Phi(a\mathrm{e}^{-\lambda_0(t+s)},\cdot))(x) = \Phi(a\mathrm{e}^{-\lambda_0 t}, x).$$

Thus $(\Phi(a\mathrm{e}^{-\lambda_0 t}, x), t\geqslant 0)$ is a non-trivial family of backward iterates.

If $(\eta_t^*, t\geqslant 0)$ is a non-trivial family of backward iterates, then it follows from Lemma 4.6 that, for any $s\geqslant 0$, $(\eta_{t+s}^*, t\geqslant 0)$ is also a non-trivial family of backward iterates. Let $W^{(s)}$ be the limit in Theorem 5.2 corresponding to $(\eta_{t+s}^*, t\geqslant 0)$. By (5.1), we get that, for any $s\geqslant 0$,

$$\{W>0\} = \{W^{(s)}>0\} = \{\forall\, t>0, \|X_t\|>0\}, \quad \text{a.s.-}\mathbb{P}_\mu.$$

Thus, we have that, for any $\omega\in\{W>0\}\cap\{W^{(s)}>0\}$,

$$\frac{W^{(s)}(\omega)}{W(\omega)} = \lim_{t\to\infty}\frac{\langle\eta_{t+s}^*, \psi_0\rangle_m}{\langle\eta_t, \psi_0\rangle_m} =: e(s) > 0,$$

where $e(s)$ is deterministic. Therefore,

$$W^{(s)} = e(s)W, \quad \text{a.s.-}\mathbb{P}_\mu. \tag{5.5}$$

By (4.7), we have that, for any $s,r\geqslant 0$,

$$W^{(s)} = \lim_{t\to\infty}\frac{\langle\eta_{t+s}^*, \psi_0\rangle_m}{\langle\eta_{t+r}^*, \psi_0\rangle_m}W^{(r)} = \mathrm{e}^{-\lambda_0(s-r)}W^{(r)}, \quad \text{a.s.-}\mathbb{P}_\mu.$$

It follows that $e(s) = \mathrm{e}^{-\lambda_0(s-r)}e(r)$, which implies that there exists a constant $a>0$ such that

$$e(s) = a\mathrm{e}^{-\lambda_0 s}.$$

Note that, for any $s \geqslant 0$ and $x \in E$,

$$\begin{aligned}\eta_s^*(x) &= \lim_{t\to\infty} -\log \mathbb{P}_{\delta_x} \exp\{-\langle \eta_{t+s}^*, X_t\rangle\} = -\log \mathbb{P}_{\delta_x} \exp\{-W^{(s)}\} \\ &= -\log \mathbb{P}_{\delta_x} \exp\{-a\mathrm{e}^{-\lambda_0 s} W\} = \Phi(a\mathrm{e}^{-\lambda_0 s}, x).\end{aligned}$$

The proof is now completed. □

Recall from Remark 4.7 that $l_0 = \lim_{t\to\infty} \mathrm{e}^{\lambda_0 t}\gamma_t \in (0,\infty]$.

Proposition 5.4. *Let $(\eta_t, t \geqslant 0)$ be a non-trivial family of backward iterates and W be the limit in Theorem 5.2 corresponding to $(\eta_t, t \geqslant 0)$.*

(1) *If $l_0 < \infty$, then $\mathbb{P}_\mu W < \infty$ for any $\mu \in \mathcal{M}_F(E)$.*

(2) *If $\mathbb{P}_\mu W < \infty$ for some nonzero $\mu \in \mathcal{M}_F(E)$, then $l_0 < \infty$.*

Moreover,

$$\mathbb{P}_\mu W = \lim_{t\to\infty} \gamma_t \mathbb{P}_\mu \langle \phi_0, X_t\rangle = l_0 \langle \phi_0, \mu\rangle. \tag{5.6}$$

Proof. (1) Since $W = \lim_{t\to\infty} \gamma_t\langle \phi_0, X_t\rangle$, we have by Fatou's lemma that

$$\mathbb{P}_\mu W \leqslant \lim_{t\to\infty} \mathbb{P}_\mu \gamma_t \langle \phi_0, X_t\rangle = \lim_{t\to\infty} \gamma_t \mathrm{e}^{\lambda_0 t}\langle \phi_0, \mu\rangle = l_0\langle \phi_0, \mu\rangle.$$

Thus $l_0 < \infty$ implies $\mathbb{P}_\mu W < \infty$ for any $\mu \in \mathcal{M}_F(E)$.

(2) It follows from (5.4) and Lemma 4.6 that

$$\begin{aligned}\lim_{s\to 0} \frac{\langle \Phi(s,\cdot), \mu\rangle}{s} &= \lim_{t\to\infty} \mathrm{e}^{\lambda_0 t}\langle \Phi(\mathrm{e}^{-\lambda_0 t}, \cdot), \mu\rangle = \lim_{t\to\infty} \mathrm{e}^{\lambda_0 t}\langle \eta_t, \mu\rangle \\ &= \lim_{t\to\infty} \mathrm{e}^{\lambda_0 t}\gamma_t\langle (1+h_t)\phi_0, \mu\rangle = l_0\langle \phi_0, \mu\rangle.\end{aligned} \tag{5.7}$$

If $\mathbb{P}_\mu W < \infty$ for some nonzero $\mu \in \mathcal{M}_F(E)$, then

$$\mathbb{P}_\mu W = \lim_{s\to 0} \frac{1 - \mathbb{P}_\mu(\exp\{-sW\})}{s} = \lim_{s\to 0} \frac{\langle \Phi(s,\cdot), \mu\rangle}{s} = l_0\langle \phi_0, \mu\rangle,$$

which implies $l_0 < \infty$. The proof is now completed. □

It was shown in [1, Subsection 1.10, Lemma 1] that, if Y is a non-negative random variable with $\mathrm{E}Y < \infty$, then

$$\mathrm{E}Y \log^+ Y < \infty$$

if and only if, for some $a_0 > 0$,

$$\int_0^{a_0} s^{-2}\mathrm{E}(\mathrm{e}^{-sY} - 1 + sY)ds < \infty.$$

Recall that $l(x) := \int_1^\infty \theta \log\theta\, n^{\phi_0}(x, d\theta)$.

Lemma 5.5. *$\int_E \psi_0(x) l(x) m(dx) < \infty$ if and only if, for any $t > 0$,*

$$\int_E \psi_0(x)\mathbb{P}_{\delta_x}[\langle \phi_0, X_t\rangle \log^+\langle \phi_0, X_t\rangle] m(dx) < \infty.$$

Proof. Without loss of generality, we only prove the result for $t = 1$. Note that

$$\int_E \psi_0(x)\mathbb{P}_{\delta_x}[\langle \phi_0, X_1\rangle \log^+\langle \phi_0, X_1\rangle] m(dx) < \infty$$

if and only if, for some $a_0 > 0$,

$$\int_E \psi_0(x) m(dx) \int_0^{a_0} s^{-2}\mathbb{P}_{\delta_x}(\mathrm{e}^{-s\langle \phi_0, X_1\rangle} - 1 + s\langle \phi_0, X_1\rangle)\, ds < \infty. \tag{5.8}$$

Put $R(f)(x) := R_f(1,x) = T_1 f(x) - V_1 f(x)$. Then, we have

$$\begin{aligned}\mathbb{P}_{\delta_x}(\mathrm{e}^{-s\langle\phi_0,X_1\rangle} - 1 + s\langle\phi_0,X_1\rangle) &= \exp\{-V_1(s\phi_0)(x)\} - 1 + V_1(s\phi_0)(x) + R(s\phi_0)(x)\\ &\leqslant \frac{1}{2}V_1(s\phi_0)(x)^2 + R(s\phi_0)(x)\\ &\leqslant \frac{1}{2}T_1(s\phi_0)(x)^2 + R(s\phi_0)(x)\\ &= \frac{1}{2}\mathrm{e}^{2\lambda_0}s^2\phi_0(x)^2 + R(s\phi_0)(x).\end{aligned}$$

On the other hand,

$$\mathbb{P}_{\delta_x}(\mathrm{e}^{-s\langle\phi_0,X_1\rangle} - 1 + s\langle\phi_0,X_1\rangle) = \exp\{-V_1(s\phi_0)(x)\} - 1 + T_1(s\phi_0)(x) \geqslant R(s\phi_0)(x).$$

Thus, (5.8) holds if and only if, for some $a_0 > 0$,

$$\int_0^{a_0} s^{-2}\langle R(s\phi_0),\psi_0\rangle_m ds < \infty. \tag{5.9}$$

By (2.5), we have

$$\begin{aligned}&\langle r(\cdot, V_{1-t}(s\phi_0)),\psi_0\rangle_m\\ &\quad= \langle \beta V_{1-t}(s\phi_0)^2,\psi_0\rangle_m + \left\langle\left(\int_0^{\phi_0^{-1}} + \int_{\phi_0^{-1}}^\infty\right)(\mathrm{e}^{-\theta V_{1-t}(s\phi_0)} - 1 + \theta V_{1-t}(s\phi_0))n(\cdot,d\theta),\psi_0\right\rangle_m\\ &\quad=: J_1(s,t) + J_2(s,t) + J_3(s,t).\end{aligned}$$

Thus, by (2.12), we have

$$\begin{aligned}&\int_0^{a_0} s^{-2}\langle R(s\phi_0),\psi_0\rangle_m ds\\ &\quad= \int_0^{a_0} s^{-2}ds\int_0^1 \mathrm{e}^{\lambda_0 t}\langle r(\cdot,V_{1-t}(s\phi_0)),\psi_0\rangle_m\,dt\\ &\quad= \int_0^{a_0} s^{-2}ds\int_0^1 \mathrm{e}^{\lambda_0 t}J_1(s,t)\,dt + \int_0^{a_0} s^{-2}ds\int_0^1 \mathrm{e}^{\lambda_0 t}J_2(s,t)\,dt\\ &\qquad+ \int_0^{a_0} s^{-2}ds\int_0^1 \mathrm{e}^{\lambda_0 t}J_3(s,t)\,dt.\end{aligned}$$

Since

$$V_{1-t}(s\phi_0)(x) \leqslant T_{1-t}(s\phi_0)(x) \leqslant s\mathrm{e}^{\lambda_0}\phi_0(x), \tag{5.10}$$

we have that $J_1(s,t) \leqslant M\mathrm{e}^{2\lambda_0}s^2\|\phi_0\|_\infty$. Thus

$$\int_0^{a_0} s^{-2}ds\int_0^1 \mathrm{e}^{\lambda_0 t}J_1(s,t)dt < \infty. \tag{5.11}$$

Note that

$$J_2(s,t) \leqslant \left\langle\int_0^{\phi_0^{-1}}\theta^2 V_{1-t}(s\phi_0)^2 n(\cdot,d\theta),\psi_0\right\rangle_m \leqslant \mathrm{e}^{2\lambda_0}s^2\left\langle\int_0^1\theta^2 n^{\phi_0}(\cdot,d\theta),\psi_0\right\rangle_m \leqslant Cs^2,$$

which implies that

$$\int_0^{a_0} s^{-2}ds\int_0^1 \mathrm{e}^{\lambda_0 t}J_2(s,t)dt < \infty. \tag{5.12}$$

Now we deal with J_3. By Lemma 4.4 and (2.3) with $s_0 > 1$ large enough such that $c(1)\mathrm{e}^{-\gamma(1)s_0} < \frac{1}{2}$,

$$v(x) = V_{s_0}v(x) \geqslant \mathrm{e}^{-as_0}T_{s_0}v(x) \geqslant \frac{1}{2}\mathrm{e}^{(\lambda_0-a)s_0}\phi_0(x).$$

We put $A_0 = \frac{1}{2}\mathrm{e}^{(\lambda_0 - a)s_0}$. Hence by Lemma 4.4, for any $s \leqslant A_0$,

$$V_{1-t}(s\phi_0)(x) \geqslant \mathrm{e}^{-a} T_{1-t}(s\phi_0)(x) \geqslant \mathrm{e}^{-a} s\phi_0(x). \tag{5.13}$$

Thus, combining (5.10) and (5.13), there exist $C_1, C_2 > 0$ such that for any $s \leqslant A_0$,

$$\begin{aligned}&\left\langle \int_{\phi_0^{-1}}^{\infty} (\mathrm{e}^{-C_1\theta s\phi_0} - 1 + C_1\theta s\phi_0) n(\cdot, d\theta), \psi_0 \right\rangle_m \\ &\quad \leqslant J_3(s,t) \leqslant \left\langle \int_{\phi_0^{-1}}^{\infty} (\mathrm{e}^{-C_2\theta s\phi_0} - 1 + C_2\theta s\phi_0) n(\cdot, d\theta), \psi_0 \right\rangle_m.\end{aligned}$$

Note that for any $C > 0$, we have

$$\begin{aligned}&\int_0^{A_0} s^{-2} ds \int_0^1 \mathrm{e}^{\lambda_0 t} \left\langle \int_{\phi_0^{-1}}^{\infty} (\mathrm{e}^{-C\theta s\phi_0} - 1 + C\theta s\phi_0) n(\cdot, d\theta), \psi_0 \right\rangle_m dt \\ &= \int_0^1 \mathrm{e}^{\lambda_0 t} dt \int_E \psi_0(x) m(dx) \int_1^{\infty} n^{\phi_0}(x, d\theta) \int_0^{A_0} s^{-2}(\mathrm{e}^{-C\theta s} - 1 + C\theta s) ds \\ &= \int_0^1 \mathrm{e}^{\lambda_0 t} dt \int_E \psi_0(x) m(dx) \int_1^{\infty} \theta n^{\phi_0}(x, d\theta) \int_0^{A_0\theta} s^{-2}(\mathrm{e}^{-Cs} - 1 + Cs) ds.\end{aligned}$$

Since

$$\lim_{\theta\to\infty} \frac{\int_0^{A_0\theta} s^{-2}(\mathrm{e}^{-Cs} - 1 + Cs) ds}{\log\theta} = C,$$

we have

$$\int_0^{A_0} s^{-2} ds \int_0^1 \mathrm{e}^{\lambda_0 t} J_3(s,t) dt < \infty \Leftrightarrow \int_E l(x)\psi_0(x) m(dx) < \infty.$$

Now the conclusion follows immediately. □

Recall that

$$M_\infty = \lim_{t\to\infty} M_t = \lim_{t\to\infty} \mathrm{e}^{-\lambda_0 t}\langle \phi_0, X_t\rangle.$$

Proposition 5.6. *If $\int_E \psi_0(x) l(x) m(dx) < \infty$, then for any nonzero $\mu \in \mathcal{M}_F(E)$, M_∞ is non-degenerate under $\mathbb{P}_\mu$ and $\mathbb{P}_\mu M_\infty = \langle \phi_0, \mu\rangle$.*

Proof. Suppose $\mu \in \mathcal{M}_F(E)$ is nonzero and fixed. For any $\theta > 0$, put

$$\Psi_t(\theta, x) := -\log \mathbb{P}_{\delta_x}(\exp\{-\theta M_t\}) \quad \text{and} \quad \Psi(\theta, x) := -\log \mathbb{P}_{\delta_x}(\exp\{-\theta M_\infty\}).$$

Then for any $x \in E$, $\Psi_t(\theta, x)$ is non-increasing in t. By the dominated convergence theorem and monotone convergence theorem, we have

$$-\log \mathbb{P}_\mu(\exp\{-\theta M_\infty\}) = \lim_{n\to\infty} \langle \Psi_n(\theta, \cdot), \mu\rangle = \langle \Psi(\theta, \cdot), \mu\rangle. \tag{5.14}$$

We claim that there exists some $\theta > 0$ such that

$$\langle \Psi(\theta, \cdot), \psi_0\rangle_m > 0. \tag{5.15}$$

By the Markov property of X, we have

$$\begin{aligned}\Psi_{t+s}(\theta, x) &= -\log \mathbb{P}_{\delta_x} \mathbb{P}_{X_t}(\exp\{-\theta \mathrm{e}^{-\lambda_0 t} M_s\}) \\ &= -\log \mathbb{P}_{\delta_x} \exp\{-\langle \Psi_s(\theta \mathrm{e}^{-\lambda_0 t}), X_t\rangle\} = V_t(\Psi_s(\theta \mathrm{e}^{-\lambda_0 t}))(x).\end{aligned} \tag{5.16}$$

Letting $s \to \infty$, we get

$$\Psi(\theta \mathrm{e}^{\lambda_0 t}, x) = V_t(\Psi(\theta))(x).$$

If (5.15) holds, then by (2.3), we get that, for $t > 1$ large enough,

$$\mathbb{P}_{\delta_x}\langle \Psi(\theta), X_t\rangle = T_t \Psi(\theta)(x) > 0,$$

which implies that $\mathbb{P}_{\delta_x}(\langle\Psi(\theta), X_t\rangle > 0) > 0$. Hence we have that, for any $x \in E$,

$$\Psi(\theta e^{\lambda_0 t}, x) = V_t(\Psi(\theta))(x) > 0.$$

Thus, by (5.14),

$$\mathbb{P}_\mu \exp\{-\theta e^{\lambda_0 t} M_\infty\} = \exp\{-\langle\Psi(\theta e^{\lambda_0 t}), \mu\rangle\} < 1,$$

which implies that $\mathbb{P}_\mu(M_\infty = 0) < 1$.

Now we prove claim (5.15). Put $R(f)(x) := R_f(1, x)$. It follows from (5.16) and (2.7) that

$$\begin{aligned}\langle\Psi_n(\theta,\cdot),\psi_0\rangle_m &= \langle V_1(\Psi_{n-1}(\theta e^{-\lambda_0})),\psi_0\rangle_m\\ &= \langle T_1(\Psi_{n-1}(\theta e^{-\lambda_0})),\psi_0\rangle_m - \langle R(\Psi_{n-1}(\theta e^{-\lambda_0})),\psi_0\rangle_m\\ &= e^{\lambda_0}\langle \Psi_{n-1}(\theta e^{-\lambda_0}),\psi_0\rangle_m - \langle R(\Psi_{n-1}(\theta e^{-\lambda_0})),\psi_0\rangle_m\\ &= e^{\lambda_0(n-1)}\langle\Psi_1(\theta e^{-(n-1)\lambda_0}),\psi_0\rangle_m - \sum_{k=1}^{n-1} e^{\lambda_0(k-1)}\langle R(\Psi_{n-k}(\theta e^{-\lambda_0 k})),\psi_0\rangle_m. \end{aligned} \tag{5.17}$$

Note that, by Jensen's inequality, we have

$$\Psi_{n-k}(\theta e^{-\lambda_0 k})(x) = -\log \mathbb{P}_{\delta_x}\exp\{-\theta e^{-\lambda_0 k} M_{n-k}\} \leqslant \mathbb{P}_{\delta_x}\theta e^{-\lambda_0 k} M_{n-k} = \theta e^{-\lambda_0 k}\phi_0(x).$$

By the dominated convergence theorem, we get

$$\lim_{n\to\infty} e^{\lambda_0(n-1)}\langle\Psi_1(\theta e^{-(n-1)\lambda_0}),\psi_0\rangle_m = \theta\langle \mathbb{P}_{\delta_\cdot} M_1,\psi_0\rangle_m = \theta.$$

Thus we have

$$\langle\Psi(\theta,\cdot),\psi_0\rangle_m = \lim_{n\to\infty}\langle\Psi_n(\theta,\cdot),\psi_0\rangle_m \geqslant \theta - \lim_{n\to\infty}\sum_{k=1}^{n-1} e^{\lambda_0(k-1)}\langle R(\theta e^{-\lambda_0 k}\phi_0),\psi_0\rangle_m. \tag{5.18}$$

Since $t \to \langle R(\theta e^{-\lambda_0 t}\phi_0),\psi_0\rangle_m$ is decreasing, we have

$$\begin{aligned}&\sum_{k=1}^{\infty} e^{\lambda_0(k-1)}\langle R(\theta e^{-\lambda_0 k}\phi_0),\psi_0\rangle_m\\ &\quad\leqslant \int_0^\infty e^{\lambda_0 t}\langle R(\theta e^{-\lambda_0 t}\phi_0),\psi_0\rangle_m dt\\ &\quad= \lambda_0^{-1}\int_0^1 s^{-2}\langle R(s\theta\phi_0),\psi_0\rangle_m ds\\ &\quad\leqslant \lambda_0^{-1}\int_0^1 s^{-2}\langle T_1(s\theta\phi_0) - (1-\exp\{-V_1(s\theta\phi_0)\}),\psi_0\rangle_m ds\\ &\quad= \lambda_0^{-1}\int_E \psi_0(x)m(dx)\mathbb{P}_{\delta_x}\int_0^1 s^{-2}(s\theta\langle\phi_0, X_1\rangle - 1 + \exp\{-s\theta\langle\phi_0, X_1\rangle\})ds.\end{aligned}$$

Since $e^{-s} - 1 + s \leqslant s\wedge(s^2/2)$, there exists $C > 0$ such that for any $r \geqslant 0$,

$$\begin{aligned}\int_0^1 s^{-2}(rs - 1 + \exp\{-rs\})\,ds &= r\int_0^r s^{-2}(s - 1 + \exp\{-s\})\,ds\\ &\leqslant \frac{1}{2}r^2 \mathrm{I}_{r\leqslant 2} + r\left(1 + \int_2^r s^{-1}\,ds\right)\mathrm{I}_{r>2}\\ &\leqslant \frac{1}{2}r^2\mathrm{I}_{r\leqslant 2} + Cr\log r\,\mathrm{I}_{r>2}.\end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty} e^{\lambda_0(k-1)}\langle R(\theta e^{-\lambda_0 k}\phi_0),\psi_0\rangle_m$$

$$\leqslant \theta\lambda_0^{-1}/2\bigg(\int_E \mathbb{P}_{\delta_x}(\theta\langle\phi_0, X_1\rangle^2; \theta\langle\phi_0, X_1\rangle \leqslant 2)\psi_0(x)m(dx)$$
$$+ 2C\int_E \mathbb{P}_{\delta_x}(\langle\phi_0, X_1\rangle \log(\theta\langle\phi_0, X_1\rangle); \theta\langle\phi_0, X_1\rangle > 2)\psi_0(x)m(dx)\bigg).$$

Using the dominated convergence theorem, we get

$$\lim_{\theta\to 0}\int_E \mathbb{P}_{\delta_x}(\theta\langle\phi_0, X_1\rangle^2; \theta\langle\phi_0, X_1\rangle \leqslant 2)\psi_0(x)m(dx) = 0.$$

By Lemma 5.5, we have

$$\int_E \mathbb{P}_{\delta_x}(\langle\phi_0, X_1\rangle \log(\langle\phi_0, X_1\rangle); \langle\phi_0, X_1\rangle > 2)\psi_0(x)m(dx) < \infty.$$

Applying the dominated convergence theorem again, we get

$$\lim_{\theta\to 0}\int_E \mathbb{P}_{\delta_x}(\langle\phi_0, X_1\rangle \log(\theta\langle\phi_0, X_1\rangle); \theta\langle\phi_0, X_1\rangle > 2)\psi_0(x)m(dx) = 0.$$

Therefore, there exists $\theta_0 > 0$ such that for any $\theta \in (0, \theta_0]$,

$$\sum_{k=1}^{\infty} \mathrm{e}^{\lambda_0(k-1)}\langle R(\theta \mathrm{e}^{-\lambda_0 k}\phi_0), \psi_0\rangle_m \leqslant \theta/2.$$

It follows from (5.18) that, for $0 < \theta \leqslant \theta_0$,

$$\langle \Psi(\theta, \cdot), \psi_0\rangle_m \geqslant \theta/2 > 0.$$

Now the claim (5.15) is proved, and hence M_∞ is non-degenerate.

It is easy to see that

$$M_\infty = \lim_{t\to\infty} \mathrm{e}^{-\lambda_0 t}\langle\phi_0, X_t\rangle = \lim_{t\to\infty} \mathrm{e}^{-\lambda_0 t}(\gamma_t)^{-1}\gamma_t\langle\phi_0, X_t\rangle = l_0^{-1}W. \tag{5.19}$$

Since M_∞ is non-degenerate, we have $l_0 < \infty$. Thus by (5.6), $\mathbb{P}_\mu M_\infty = \langle\phi_0, \mu\rangle$. The proof is now completed. □

Proof of Theorem 1.6. By (5.19), (1) ⇔ (2) ⇔ (3). By Propositions 5.4 and 5.6, (3) ⇔ (5) ⇔ (6) and (4) ⇒ (2). Thus, we only need to show that (3) ⇒ (4).

By (2.11) and the fact that $\eta_s = V_t(\eta_{t+s})$, we have

$$\eta_0(x) = V_t(\eta_t)(x) = T_t(\eta_t)(x) - \int_0^t T_s\left[r(\cdot, \eta_s(\cdot))\right](x)ds.$$

Thus,

$$\mathrm{e}^{\lambda_0 t}\gamma_t = \langle T_t(\eta_t), \psi_0\rangle_m = \gamma_0 + \int_E \psi_0(x)m(dx)\int_0^t \mathrm{e}^{\lambda_0 s} r(x, \eta_s(x))ds,$$

which implies that

$$l_0 = \gamma_0 + \int_E \psi_0(x)m(dx)\int_0^\infty \mathrm{e}^{\lambda_0 s} r(x, \eta_s(x))ds. \tag{5.20}$$

Recall that,

$$r(x, s) = \beta(x)s^2 + \int_0^\infty (\mathrm{e}^{-s\theta} - 1 + s\theta)\, n(x, d\theta).$$

Hence,

$$\int_0^\infty \mathrm{e}^{\lambda_0 s} r(x, \eta_s(x))ds \geqslant \int_0^\infty \mathrm{e}^{\lambda_0 s}ds \int_{\mathrm{e}^{\lambda_0 s}\phi_0(x)^{-1}}^\infty (\mathrm{e}^{-\theta\eta_s(x)} - 1 + \theta\eta_s(x))\, n(x, d\theta)$$

$$= \int_{\phi_0(x)^{-1}}^{\infty} n(x, d\theta) \int_0^{(\log(\theta\phi_0(x)))/\lambda_0} \mathrm{e}^{\lambda_0 s}(\mathrm{e}^{-\theta\eta_s(x)} - 1 + \theta\eta_s(x))ds. \tag{5.21}$$

Choose $s_0 > 1$ large enough such that $(1 - c\mathrm{e}^{-\gamma s_0}) > 0$, where $c = c(1)$ and $\gamma = \gamma(1)$ are the constants in (2.3). By (2.3) and Lemma 4.4, we get that,

$$\eta_s(x) = V_{s_0}(\eta_{s+s_0})(x) \geqslant \mathrm{e}^{-as_0} T_{s_0}(\eta_{s+s_0})(x) \geqslant \mathrm{e}^{-as_0}(1 - c\mathrm{e}^{-\gamma s_0})\mathrm{e}^{\lambda_0 s_0}\gamma_{s+s_0}\phi_0(x) \geqslant C\gamma_{s+s_0}\phi_0(x).$$

Thus, by Remark 4.7, for any $s \leqslant (\log(\theta\phi_0(x)))/\lambda_0$, we have

$$\theta\eta_s(x) \geqslant C\theta\gamma_{s+s_0}\phi_0(x) \geqslant C\mathrm{e}^{\lambda_0 s}\gamma_{s+s_0} \geqslant CL(s + s_0) \geqslant CL(0) > 0,$$

where $L(t) = \mathrm{e}^{\lambda_0 t}\gamma_t$. Therefore,

$$\inf_{s\leqslant(\log(\theta\phi_0(x)))/\lambda_0} \frac{\mathrm{e}^{-\theta\eta_s(x)} - 1 + \theta\eta_s(x)}{\theta\eta_s(x)} \geqslant \inf_{r\geqslant CL(0)} \frac{\mathrm{e}^{-r} - 1 + r}{r} \geqslant c,$$

for some constant $c > 0$. It follows that

$$\begin{aligned}
&\int_{\phi_0(x)^{-1}}^{\infty} n(x, d\theta) \int_0^{(\log(\theta\phi_0(x)))/\lambda_0} \mathrm{e}^{\lambda_0 s}(\mathrm{e}^{-\theta\eta_s(x)} - 1 + \theta\eta_s(x))ds \\
&\quad\geqslant c \int_{\phi_0(x)^{-1}}^{\infty} \theta n(x, d\theta) \int_0^{(\log(\theta\phi_0(x)))/\lambda_0} \mathrm{e}^{\lambda_0 s}\eta_s(x)ds \\
&\quad\geqslant C \int_{\phi_0(x)^{-1}}^{\infty} \theta n(x, d\theta) \int_0^{(\log(\theta\phi_0(x)))/\lambda_0} \mathrm{e}^{\lambda_0 s}\gamma_{s+s_0} ds\phi_0(x) \\
&\quad\geqslant C \int_1^{\infty} \theta n^{\phi_0}(x, d\theta) \int_0^{\log\theta/\lambda_0} L(s + s_0)ds \\
&\quad\geqslant CL(0) \int_1^{\infty} \theta\log\theta n^{\phi_0}(x, d\theta) \geqslant Cl(x).
\end{aligned} \tag{5.22}$$

Combining (5.20)–(5.22), we get that $l_0 \geqslant C\langle l, \psi_0\rangle_m$. Thus (3) ⇒ (4).

The proof is now completed. □

6 Strong convergence with general test functions

In this section, we fix a non-trivial family $(\eta_t : t \geqslant 0)$ of backward iterates and let $\gamma_t := \langle \eta_t, \psi_0\rangle_m$. The goal of this section is to determine the almost sure limit of $\gamma_t\langle f, X_t\rangle$ for general test functions f.

6.1 The martingale problems of superprocesses

In this subsection, we recall the martingale problem for the superprocess X. Let J denote the set of jump times of X, i.e.,

$$J := \{s \geqslant 0 : \Delta X_s = X_s - X_{s-} \neq 0\}.$$

Since X is a càdlàg process in $\mathcal{M}_F(E)$, J is a countable set. Let $N(ds, d\nu)$ be the optional random measure on $[0, \infty) \times \mathcal{M}_F(E)$ defined by

$$N(ds, d\nu) := \sum_{s\in J} \delta_{(s,\Delta X_s)}(ds, d\nu),$$

and $\widehat{N}(ds, d\nu)$ be the predictable compensator of $N(ds, d\nu)$ which satisfies that for any non-negative predictable function F on $\mathbb{R}_+ \times \mathcal{M}_F(E) \times \Omega$,

$$\int_0^t \int_{\mathcal{M}_F(E)} F(s, \nu)\,\widehat{N}(ds, d\nu) = \int_0^t ds \int_E X_s(dx) \int_0^{\infty} F(s, \theta\delta_x)n(x, d\theta), \tag{6.1}$$

where n is the kernel in the branching mechanism φ. Define

$$\widetilde{N}(ds,d\nu) := N(ds,d\nu) - \widehat{N}(ds,d\nu).$$

Then $\widetilde{N}(ds,d\nu)$ is a martingale measure. The "stochastic integral"

$$\int_0^t \int_{\mathcal{M}_F(E)} F(s,\nu)\widetilde{N}(ds,d\nu)$$

can be defined for a wide class of Borel functions F on $[0,t]\times\mathcal{M}_F(E)$. In particular, if f is a bounded Borel function on $[0,t]\times E$ and $F_f(s,\nu) := \int_E f(s,x)\nu(dx)$, then the integral $\int_0^t\int_{\mathcal{M}_F(E)} F_f(s,\nu)\widetilde{N}(ds,d\nu)$ is well defined. Let $\mathcal{L}_N^2$ be the space of predictable processes $(F(s,\nu): s>0, \nu\in\mathcal{M}_F(E))$ satisfying, for all $\mu\in\mathcal{M}_F(E)$,

$$\mathbb{P}_\mu \int_0^t ds \int_E X_s(dx)\int_0^\infty F(s,\theta\delta_x)^2 n(x,d\theta) < \infty.$$

For any $F\in\mathcal{L}_N^2$,

$$M_t^d(F) := \int_0^t \int_{\mathcal{M}_F(E)} F(s,\nu)\,\widetilde{N}(ds,d\nu), \quad t\geqslant 0,$$

is a square integrable martingale such that

$$\mathbb{P}_\mu(M_t^d(F)^2) = \mathbb{P}_\mu\bigg[\int_0^t ds\int_E X_s(dx)\int_0^\infty F(s,\theta\delta_x)^2 n(x,d\theta)\bigg]. \tag{6.2}$$

Note that $C_0(E)$ is a Banach space under the supremum norm. In the remainder of this paper, we assume as follows.

Assumption 6.1. (i) $\{P_t, t\geqslant 0\}$ *is a Feller semigroup, i.e.,* $\{P_t,t\geqslant 0\}$ *preserves* $C_0(E)$ *and* $[0,\infty)\ni t\to P_tf\in C_0(E)$ *is continuous for every* $f\in C_0(E)$.

(ii)

$$\lim_{a\to\infty}\sup_{x\in E}\int_a^\infty \theta n(x,d\theta) = 0. \tag{6.3}$$

In the reminder of this subsection, we will (also) use $\widetilde{L}$ to denote the infinitesimal generator of $\{P_t,t\geqslant 0\}$ in the space $C_0(E)$ and use $\mathrm{Dom}(\widetilde{L})$ to denote its domain. It is known (see, for example, [5, Subsection 6.1]) that, for any $f\in\mathrm{Dom}(\widetilde{L})$, we have that

$$\langle f,X_t\rangle = \langle f,X_0\rangle + \int_0^t \langle \widetilde{L}f-\alpha f, X_s\rangle ds + M_t^c(f) + M_t^d(f),$$

where $M_t^c(f)$ is a continuous local martingale with the quadratic variation $\int_0^t 2\langle\beta f^2,X_s\rangle ds$ and

$$M_t^d(f) = \int_0^t\int_{\mathcal{M}_F(E)} \langle f,\nu\rangle \widetilde{N}(ds,d\nu)$$

is a purely discontinuous local martingale. Here we remark that if we assume that $\alpha,\beta\in C(E)$ and that $x\to(\theta\wedge\theta^2)n(x,d\theta)$ is continuous in the topology of weak convergence, then the above result follows from [16, Theorem 7.25]. $M_t^c(f)$ induces a worthy $(\mathcal{G}_t)$-martingale measure $S^C(ds,dx)$ (see [16, Subsection 7.3] for the definition of worthy martingale measure) satisfying

$$M_t^c(f) = \int_0^t\int_E f(x)S^C(ds,dx).$$

Standard arguments then show that the "stochastic integral"

$$\int_0^t\int_E f(s,x)S^C(ds,dx)$$

can be defined for a wide class of integrands f on $[0,t]\times E$ (see, for example, [16, Theorem 7.25] or [7] for more details). Thus, one can show that (see [7, Proposition 2.13] or [20, Exercise II.5.2] for the case when the branching mechanism has finite variance) for any bounded function g on E,

$$\langle g, X_t\rangle = \langle T_t g, X_0\rangle + \int_0^t \int_{\mathcal{M}_F(E)} \langle T_{t-s}g, \nu\rangle \widetilde{N}(ds, d\nu) + \int_0^t \int_E T_{t-s}g(x) S^C(ds, dx). \tag{6.4}$$

6.2 Discrete times

In this subsection, we show that for any $\delta > 0$ and $f \in \mathcal{B}_b^+(E)$, $\gamma_{n\delta}\langle f\phi_0, X_{n\delta}\rangle$ has an almost sure limit as $n \to \infty$. We will extend this result to continuous times in two different scenarios in the next two subsections.

Theorem 6.2. *For any $\delta > 0$, $\mu \in \mathcal{M}_F(E)$ and $f \in \mathcal{B}_b^+(E)$, we have*

$$\lim_{n\to\infty} \gamma_{n\delta}\langle \phi_0 f, X_{n\delta}\rangle = \langle f\phi_0, \psi_0\rangle_m W, \quad a.s.\text{-}\mathbb{P}_\mu.$$

To prove Theorem 6.2, we first make some preparations. For any $s > 0$, we define

$$\mathcal{D}_{<1}(s) := \{\nu \in \mathcal{M}_F(E) : 0 < \gamma_s\langle \phi_0, \nu\rangle < 1\} \tag{6.5}$$

and

$$\mathcal{D}_{\geqslant 1}(s) := \{\nu \in \mathcal{M}_F(E) : \gamma_s\langle \phi_0, \nu\rangle \geqslant 1\}. \tag{6.6}$$

For any $m \in \mathbb{N}$, $\delta > 0$, $\mu \in \mathcal{M}_F(E)$ and $f \in \mathcal{B}_b^+(E)$, by (6.4), we have

$$\begin{aligned}
&\gamma_{(n+m)\delta}\langle \phi_0 f, X_{(n+m)\delta}\rangle \\
&\quad= \gamma_{(n+m)\delta}\langle T_{(n+m)\delta}(\phi_0 f), \mu\rangle + \gamma_{(n+m)\delta}\int_0^{(n+m)\delta}\int_E T_{(n+m)\delta-s}(\phi_0 f)(x) S^C(ds, dx) \\
&\qquad+ \gamma_{(n+m)\delta}\int_0^{(n+m)\delta}\int_{\mathcal{D}_{<1}(s)} \langle T_{(n+m)\delta-s}(\phi_0 f), \nu\rangle \widetilde{N}(ds, d\nu) \\
&\qquad+ \gamma_{(n+m)\delta}\int_0^{(n+m)\delta}\int_{\mathcal{D}_{\geqslant 1}(s)} \langle T_{(n+m)\delta-s}(\phi_0 f), \nu\rangle \widetilde{N}(ds, d\nu) \\
&\quad=: \gamma_{(n+m)\delta}\langle T_{(n+m)\delta}(\phi_0 f), \mu\rangle + C_{(n+m)\delta}(f) + H_{(n+m)\delta}(f) + L_{(n+m)\delta}(f).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\gamma_{(n+m)\delta}\langle \phi_0 f, X_{(n+m)\delta}\rangle - \mathbb{P}_\mu[\gamma_{(n+m)\delta}\langle \phi_0 f, X_{(n+m)\delta}\rangle \,|\, \mathcal{G}_{n\delta}] \\
&\quad= (H_{(n+m)\delta}(f) - \mathbb{P}_\mu(H_{(n+m)\delta}(f) \,|\, \mathcal{G}_{n\delta})) + (L_{(n+m)\delta}(f) - \mathbb{P}_\mu(L_{(n+m)\delta}(f) \,|\, \mathcal{G}_{n\delta})) \\
&\qquad+ (C_{(n+m)\delta}(f) - \mathbb{P}_\mu(C_{(n+m)\delta}(f) \,|\, \mathcal{G}_{n\delta})).
\end{aligned}$$

We now deal with the three parts separately. Before doing this, we prove a lemma first.

Lemma 6.3. *If $\{a_n : n \geqslant 1\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} a_{n+1}/a_n = a > 1$, then*

$$\sup_{x\in E}\sum_{n=1}^\infty a_n^{-1}\int_0^{a_n} \theta^2 n^{\phi_0}(x, d\theta) < \infty \tag{6.7}$$

and

$$\sup_{x\in E}\sum_{n=1}^\infty a_n\int_{a_n}^\infty n^{\phi_0}(x, d\theta) < \infty. \tag{6.8}$$

Proof. Since $\lim_{n\to\infty} a_{n+1}/a_n = a > 1$, for any $a^* \in (1, a)$, there exists $K > 0$ such that for any $n \geqslant K$,

$$\frac{a_{n+1}}{a_n} > a^*.$$

Without loss of generality, we assume that $a_n \uparrow \infty$. For convenience, we put $a_0 = 0$. Then we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n^{-1} \int_0^{a_n} \theta^2 n^{\phi_0}(x, d\theta) &= \sum_{n=1}^{\infty} a_n^{-1} \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} \theta^2 n^{\phi_0}(x, d\theta) \\ &= \sum_{k=1}^{\infty} \int_{a_{k-1}}^{a_k} \theta^2 n^{\phi_0}(x, d\theta) \sum_{n=k}^{\infty} a_n^{-1} \\ &\leqslant \sum_{k=1}^{K} \int_{a_{k-1}}^{a_k} \theta^2 n^{\phi_0}(x, d\theta) \sum_{n=k}^{\infty} a_n^{-1} + \sum_{k=K+1}^{\infty} \int_{a_{k-1}}^{a_k} \theta n^{\phi_0}(x, d\theta) a_k \sum_{n=k}^{\infty} a_n^{-1} \\ &\leqslant \int_0^{a_K} \theta^2 n^{\phi_0}(x, d\theta) \sum_{n=1}^{\infty} a_n^{-1} + \sum_{k=K+1}^{\infty} \int_{a_{k-1}}^{a_k} \theta n^{\phi_0}(x, d\theta) a_k \sum_{n=k}^{\infty} a_n^{-1}.\end{aligned}$$

For any $k > K$, we have

$$\sum_{n=k}^{\infty} a_k a_n^{-1} \leqslant \sum_{n=k}^{\infty} (a^*)^{-(n-k)} = \sum_{n=1}^{\infty} (a^*)^{-n} < \infty.$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n^{-1} \int_0^{a_n} \theta^2 n^{\phi_0}(x, d\theta) &\leqslant C\bigg[\int_0^{a_K} \theta^2 n^{\phi_0}(x, d\theta) + \int_{a_K}^{\infty} \theta n^{\phi_0}(x, d\theta)\bigg] \\ &\leqslant C \sup_{x\in E} \int_0^{\infty} (\theta \wedge \theta^2) n^{\phi_0}(x, d\theta) < \infty.\end{aligned}$$

Note that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \int_{a_n}^{\infty} n^{\phi_0}(x, d\theta) &= \sum_{n=1}^{\infty} a_n \sum_{k=n}^{\infty} \int_{a_k}^{a_{k+1}} n^{\phi_0}(x, d\theta) \\ &\leqslant \sum_{k=1}^{\infty} \int_{a_k}^{a_{k+1}} \theta n^{\phi_0}(x, d\theta) \bigg(a_k^{-1} \sum_{n=1}^{k} a_n\bigg).\end{aligned}$$

Using elementary calculus, one can easily show that

$$\lim_{k\to\infty} a_k^{-1} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} \frac{a_{k+1}}{a_{k+1} - a_k} = \frac{a}{a-1}.$$

Thus $\sup_{k\geqslant 1} a_k^{-1} \sum_{n=1}^{k} a_n < \infty$. It follows that

$$\sup_{x\in E} \sum_{n=1}^{\infty} a_n \int_{a_n}^{\infty} n^{\phi_0}(x, d\theta) \leqslant C \sup_{x\in E} \int_{a_1}^{\infty} \theta n^{\phi_0}(x, d\theta) < \infty.$$

The proof is now completed. □

Define

$$I(a, x) := \int_0^a \theta^2 n^{\phi_0}(x, d\theta).$$

Lemma 6.4. *For any $m \in \mathbb{N}$, $\delta > 0$, $\mu \in \mathcal{M}_F(E)$ and $f \in \mathcal{B}_b^+(E)$, we have*

$$\lim_{n\to\infty} H_{(n+m)\delta}(f) - \mathbb{P}_\mu(H_{(n+m)\delta}(f) \,|\, \mathcal{G}_{n\delta}) = 0, \quad a.s.\text{-}\mathbb{P}_\mu.$$

Proof. By the conditional Borel-Cantelli lemma, it suffices to prove that

$$\sum_{n=1}^{\infty} \mathbb{P}_\mu([H_{(n+m)\delta}(f) - \mathbb{P}_\mu(H_{(n+m)\delta}(f) \,|\, \mathcal{G}_{n\delta})]^2 \,|\, \mathcal{G}_{(n-1)\delta}) < \infty. \tag{6.9}$$

Recall from (6.5) that

$$\mathcal{D}_{<1}(s) := \{\nu \in \mathcal{M}_F(E) : 0 < \gamma_s \langle \phi_0, \nu \rangle < 1\}.$$

Since $\widetilde{N}(ds, d\nu)$ is a martingale measure, we have

$$\mathbb{P}_\mu(H_{(n+m)\delta}(f) \,|\, \mathcal{G}_{n\delta}) = \gamma_{(n+m)\delta} \int_0^{n\delta} \int_{\mathcal{D}_{<1}(s)} \langle T_{(n+m)\delta-s}(\phi_0 f), \nu \rangle \widetilde{N}(ds, d\nu),$$

which implies that

$$H_{(n+m)\delta}(f) - \mathbb{P}_\mu(H_{(n+m)\delta}(f) \,|\, \mathcal{G}_{n\delta}) = \gamma_{(n+m)\delta} \int_{n\delta}^{(n+m)\delta} \int_{\mathcal{D}_{<1}(s)} \langle T_{(n+m)\delta-s}(\phi_0 f), \nu \rangle \widetilde{N}(ds, d\nu). \quad (6.10)$$

By (5.4),

$$\gamma_t = \langle \Phi(\mathrm{e}^{-\lambda_0 t}, \cdot), \psi_0 \rangle_m,$$

which implies that $t \to \gamma_t$ is non-increasing. Thus by (6.2) and (2.3), we have

$$\begin{aligned}
&\mathbb{P}_\mu([H_{(n+m)\delta}(f) - \mathbb{P}_\mu(H_{(n+m)\delta}(f) \,|\, \mathcal{G}_{n\delta})]^2 \,|\, \mathcal{G}_{(n-1)\delta}) \\
&= \gamma^2_{(n+m)\delta} \mathbb{P}_{X_{(n-1)\delta}} \left[\int_\delta^{(1+m)\delta} ds \int_E X_s(dx) \int_0^{\gamma^{-1}_{s+(n-1)\delta} \phi_0(x)^{-1}} \theta^2 [T_{(1+m)\delta-s}(\phi_0 f)(x)]^2 n(x, d\theta) \right] \\
&\leqslant \|f\|^2_\infty \gamma^2_{(n+m)\delta} \mathbb{P}_{X_{(n-1)\delta}} \left[\int_\delta^{(1+m)\delta} \mathrm{e}^{2\lambda_0((1+m)\delta-s)} ds \int_E X_s(dx) \int_0^{\gamma^{-1}_{(n+m)\delta}} \theta^2 n^{\phi_0}(x, d\theta) \right] \\
&= \|f\|^2_\infty \gamma^2_{(n+m)\delta} \int_\delta^{(1+m)\delta} \mathrm{e}^{2\lambda_0((1+m)\delta-s)} \langle T_s[I(\gamma^{-1}_{(n+m)\delta})], X_{(n-1)\delta} \rangle ds \\
&\leqslant (1 + c(\delta)) \|f\|^2_\infty \mathrm{e}^{2\lambda_0(1+m)\delta} \int_\delta^{(1+m)\delta} \mathrm{e}^{-\lambda_0 s} ds \gamma^2_{(n+m)\delta} \langle I(\gamma^{-1}_{(n+m)\delta}), \psi_0 \rangle_m \langle \phi_0, X_{(n-1)\delta} \rangle \\
&\leqslant C[\gamma_{(n+m)\delta} \langle \phi_0, X_{(n-1)\delta} \rangle] \gamma_{(n+m)\delta} \langle I(\gamma^{-1}_{(m+n)\delta}), \psi_0 \rangle_m. \quad (6.11)
\end{aligned}$$

It follows from the fact that

$$\lim_{n\to\infty} \frac{\gamma^{-1}_{(n+m)\delta}}{\gamma^{-1}_{(n+m-1)\delta}} = \mathrm{e}^{\lambda_0 \delta}$$

and Lemma 6.3 that

$$\sum_{n=1}^\infty \gamma_{(n+m)\delta} \langle I(\gamma^{-1}_{(m+n)\delta}), \psi_0 \rangle_m < \infty. \quad (6.12)$$

Since

$$\lim_{n\to\infty} \gamma_{(n+m)\delta} \langle \phi_0, X_{(n-1)\delta} \rangle = \mathrm{e}^{-\lambda_0(m+1)\delta} W,$$

combining (6.11) and (6.12), (6.9) follows immediately. The proof is now completed. □

Lemma 6.5. *For any $m \in \mathbb{N}$, $\delta > 0$, $\mu \in \mathcal{M}_F(E)$ and $f \in \mathcal{B}_b^+(E)$, we have*

$$\lim_{n\to\infty} L_{(n+m)\delta}(f) - \mathbb{P}_\mu(L_{(n+m)\delta}(f) \,|\, \mathcal{G}_{n\delta}) = 0, \quad \text{a.s.-}\mathbb{P}_\mu.$$

Proof. Recall the definition of $\mathcal{D}_{\geqslant 1}(s)$ in (6.6):

$$\mathcal{D}_{\geqslant 1}(s) = \{\nu \in \mathcal{M}_F(E) : \gamma_s \langle \phi_0, \nu \rangle \geqslant 1\}.$$

Since $\widetilde{N}(ds, d\nu)$ is a martingale measure, we have

$$\mathbb{P}_\mu(L_{(n+m)\delta}(f) \,|\, \mathcal{G}_{n\delta}) = \gamma_{(n+m)\delta} \int_0^{n\delta} \int_{\mathcal{D}_{\geqslant 1}(s)} \langle T_{(n+m)\delta-s}(\phi_0 f), \nu \rangle \widetilde{N}(ds, d\nu),$$

which implies that

$$\begin{aligned}
&L_{(n+m)\delta}(f) - \mathbb{P}_\mu(L_{(n+m)\delta}(f) \,|\, \mathcal{G}_{n\delta}) \\
&\quad = \gamma_{(n+m)\delta} \int_{n\delta}^{(n+m)\delta} \int_{\mathcal{D}_{\geqslant 1}(s)} \langle T_{(n+m)\delta - s}(\phi_0 f), \nu\rangle (N(ds, d\nu) - \widehat{N}(ds, d\nu)).
\end{aligned}$$

We claim that

$$\mathbb{P}_\mu\bigg(\int_{n\delta}^{(n+m)\delta} \int_{\mathcal{D}_{\geqslant 1}(s)} N(ds, d\nu) > 0, \text{ i.o.} \bigg) = 0. \tag{6.13}$$

In fact, since $\int_{n\delta}^{(n+m)\delta} \int_{\mathcal{D}_{\geqslant 1}(s)} N(ds, d\nu)$ is a non-negative integer, by the Markov property of X,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{P}_\mu\bigg(\int_{n\delta}^{(n+m)\delta} \int_{\mathcal{D}_{\geqslant 1}(s)} N(ds, d\nu) > 0 \,\bigg|\, \mathcal{G}_{(n-1)\delta} \bigg) \\
&\quad \leqslant \sum_{n=1}^{\infty} \mathbb{P}_\mu\bigg(\int_{n\delta}^{(n+m)\delta} \int_{\mathcal{D}_{\geqslant 1}(s)} N(ds, d\nu) \,\bigg|\, \mathcal{G}_{(n-1)\delta} \bigg) \\
&\quad = \sum_{n=1}^{\infty} \mathbb{P}_{X_{(n-1)\delta}}\bigg(\int_{\delta}^{(1+m)\delta} \int_{\mathcal{D}_{\geqslant 1}(s+(n-1)\delta)} N(ds, d\nu) \bigg) \\
&\quad = \sum_{n=1}^{\infty} \mathbb{P}_{X_{(n-1)\delta}}\bigg[\int_{\delta}^{(1+m)\delta} ds \int_E X_s(dx) \int_{\phi_0(x)^{-1}\gamma_{s+(n-1)\delta}^{-1}}^{\infty} n(x, d\theta) \bigg] \\
&\quad \leqslant \sum_{n=1}^{\infty} \int_{\delta}^{(1+m)\delta} ds \bigg\langle T_s\bigg[\int_{\gamma_{n\delta}^{-1}}^{\infty} n^{\phi_0}(\cdot, d\theta) \bigg], X_{(n-1)\delta} \bigg\rangle \\
&\quad \leqslant (1 + c(\delta)) m\delta \mathrm{e}^{\lambda_0 (m+1)\delta} \sum_{n=1}^{\infty} \bigg\langle \int_{\gamma_{n\delta}^{-1}}^{\infty} n^{\phi_0}(\cdot, d\theta), \psi_0 \bigg\rangle_m \langle \phi_0, X_{(n-1)\delta}\rangle,
\end{aligned} \tag{6.14}$$

where in the second to the last inequality, we use the fact that $\gamma_{s+(n-1)\delta} \leqslant \gamma_{n\delta}$, and the last inequality follows from (2.3). It follows from (6.8) that

$$\sum_{n=1}^{\infty} \gamma_{n\delta}^{-1} \bigg\langle \int_{\gamma_{n\delta}^{-1}}^{\infty} n^{\phi_0}(\cdot, d\theta), \psi_0 \bigg\rangle_m < \infty. \tag{6.15}$$

By Theorem 5.2, $\gamma_{n\delta}\langle \phi_0, X_{(n-1)\delta}\rangle \to \mathrm{e}^{-\lambda_0\delta} W$ as $n \to \infty$. Therefore we have

$$\sum_{n=1}^{\infty} \bigg\langle \int_{\gamma_{n\delta}^{-1}}^{\infty} n^{\phi_0}(\cdot, d\theta), \psi_0 \bigg\rangle_m \langle \phi_0, X_{(n-1)\delta}\rangle < \infty,$$

which implies that

$$\sum_{n=1}^{\infty} \mathbb{P}_\mu\bigg(\int_{n\delta}^{(n+m)\delta} \int_{\mathcal{D}_{\geqslant 1}(s)} N(ds, d\nu) > 0 \,\bigg|\, \mathcal{G}_{(n-1)\delta} \bigg) < \infty.$$

Now using the the conditional Borel-Cantelli lemma, we immediately get the claim (6.13).

By (6.13), we get

$$\lim_{n\to\infty} \gamma_{(n+m)\delta} \int_{n\delta}^{(n+m)\delta} \int_{\mathcal{D}_{\geqslant 1}(s)} \langle T_{(n+m)\delta - s}(\phi_0 f), \nu\rangle N(ds, d\nu) = 0, \quad \mathbb{P}_\mu\text{-a.s.} \tag{6.16}$$

To complete the proof, we only need to show that

$$\lim_{n\to\infty} \gamma_{(n+m)\delta} \int_{n\delta}^{(n+m)\delta} \int_{\mathcal{D}_{\geqslant 1}(s)} \langle T_{(n+m)\delta - s}(\phi_0 f), \nu\rangle \widehat{N}(ds, d\nu) = 0, \quad \mathbb{P}_\mu\text{-a.s.} \tag{6.17}$$

By (6.1), we have

$$\begin{aligned}&\gamma_{(n+m)\delta}\int_{n\delta}^{(n+m)\delta}\int_{\mathcal{D}_{\geqslant 1}(s)}\langle T_{(n+m)\delta-s}(\phi_0 f),\nu\rangle\widehat{N}(ds,d\nu)\\&=\gamma_{(n+m)\delta}\int_{n\delta}^{(n+m)\delta}ds\int_E T_{(n+m)\delta-s}(\phi_0 f)(x)X_s(dx)\int_{\phi_0(x)^{-1}\gamma_s^{-1}}^{\infty}\theta n(x,d\theta)\\&\leqslant\|f\|_\infty \mathrm{e}^{\lambda_0 m\delta}\gamma_{(n+m)\delta}\int_{n\delta}^{(n+m)\delta}ds\int_E\phi_0(x)X_s(dx)\int_{\phi_0(x)^{-1}\gamma_s^{-1}}^{\infty}\theta n(x,d\theta)\\&\leqslant\|f\|_\infty \mathrm{e}^{\lambda_0 m\delta}\gamma_{(n+m)\delta}\int_{n\delta}^{(n+m)\delta}\langle\phi_0,X_s\rangle ds\sup_{x\in E}\left(\int_{\|\phi_0\|_\infty^{-1}\gamma_{n\delta}^{-1}}^{\infty}\theta n(x,d\theta)\right).\end{aligned}$$

Note that

$$\begin{aligned}\lim_{n\to\infty}\gamma_{(n+m)\delta}\int_{n\delta}^{(n+m)\delta}\langle\phi_0,X_s\rangle ds&=\lim_{n\to\infty}\int_0^{m\delta}\gamma_{(n+m)\delta}\langle\phi_0,X_{s+n\delta}\rangle ds\\&=\int_0^{m\delta}\mathrm{e}^{-\lambda_0(m\delta-s)}dsW,\end{aligned}$$

and by (6.3) we have

$$\lim_{n\to\infty}\sup_{x\in E}\left(\int_{\|\phi_0\|_\infty^{-1}\gamma_{n\delta}^{-1}}^{\infty}\theta n(x,d\theta)\right)=0.$$

Now we easily see that (6.17) holds. The proof is now completed. □

Lemma 6.6. *For any $m\in\mathbb{N}$, $\delta>0$, $\mu\in\mathcal{M}_F(E)$ and $f\in\mathcal{B}_b^+(E)$, we have*

$$\lim_{n\to\infty}C_{(n+m)\delta}(f)-\mathbb{P}_\mu(C_{(n+m)\delta}(f)\,|\,\mathcal{G}_{n\delta})=0,\quad a.s.\text{-}\mathbb{P}_\mu. \tag{6.18}$$

Proof. Let

$$\widetilde{M}_t:=\int_0^t\int_E T_{(n+m)\delta-s}(\phi_0 f)(x)S^C(ds,dx).$$

Then $\{\widetilde{M}_t,0\leqslant t\leqslant(n+m)\delta\}$ is a martingale with the quadratic variation

$$\langle\widetilde{M}\rangle_t=2\int_0^t\langle\beta(T_{(n+m)\delta-s}(\phi_0 f))^2,X_s\rangle ds.$$

Note that

$$\begin{aligned}&C_{(n+m)\delta}(f)-\mathbb{P}_\mu(C_{(n+m)\delta}(f)\,|\,\mathcal{G}_{n\delta})\\&\quad=\gamma_{(n+m)\delta}\int_{n\delta}^{(n+m)\delta}\int_E T_{(n+m)\delta-s}(\phi_0 f)(x)S^C(ds,dx)\\&\quad=\gamma_{(m+n)\delta}\,(\widetilde{M}_{(n+m)\delta}-\widetilde{M}_{n\delta}).\end{aligned} \tag{6.19}$$

Using this we get

$$\begin{aligned}&\mathbb{P}_\mu\left([C_{(n+m)\delta}(f)-\mathbb{P}_\mu(C_{(n+m)\delta}(f)\,|\,\mathcal{G}_{n\delta})]^2\,|\,\mathcal{G}_{(n-1)\delta}\right)\\&\quad=\gamma_{(m+n)\delta}^2\mathbb{P}_\mu(\langle\widetilde{M}\rangle_{(n+m)\delta}-\langle\widetilde{M}\rangle_{n\delta}\,|\,\mathcal{G}_{(n-1)\delta})\\&\quad=\gamma_{(m+n)\delta}^2\mathbb{P}_\mu\left(2\int_{n\delta}^{(n+m)\delta}\langle\beta(T_{(n+m)\delta-s}(\phi_0 f))^2,X_s\rangle ds\,|\,\mathcal{G}_{(n-1)\delta}\right)\\&\quad=\gamma_{(m+n)\delta}^2\mathbb{P}_{X_{(n-1)\delta}}\left(2\int_{\delta}^{(1+m)\delta}\langle\beta(T_{(1+m)\delta-s}(\phi_0 f))^2,X_s\rangle ds\right).\end{aligned}$$

Note that

$$\beta(x)(T_{(1+m)\delta-s}(\phi_0 f))^2(x) \leqslant \|f\|_\infty^2 \beta(x) \mathrm{e}^{2\lambda_0((1+m)\delta-s)}\phi_0(x)^2 \leqslant \|f\|_\infty^2 \|\beta\phi_0\|_\infty \mathrm{e}^{2\lambda_0 m\delta}\phi_0(x).$$

Thus we have

$$\begin{aligned}&\mathbb{P}_\mu([C_{(n+m)\delta}(f) - \mathbb{P}_\mu(C_{(n+m)\delta}(f)\,|\,\mathcal{G}_{n\delta})]^2\,|\,\mathcal{G}_{(n-1)\delta})\\ &\quad\leqslant C\gamma_{(m+n)\delta}^2 \mathbb{P}_{X_{(n-1)\delta}} \int_\delta^{(1+m)\delta} \langle \phi_0, X_s\rangle ds = C\gamma_{(m+n)\delta}^2 \int_\delta^{(1+m)\delta} \langle T_s\phi_0, X_{(n-1)\delta}\rangle ds\\ &\quad= C\gamma_{(m+n)\delta}^2 \int_\delta^{(1+m)\delta} \mathrm{e}^{\lambda_0 s} ds\, \langle \phi_0, X_{(n-1)\delta}\rangle \leqslant C\gamma_{(m+n)\delta}(\gamma_{(m+n)\delta}\langle \phi_0, X_{(n-1)\delta}\rangle).\end{aligned}$$

By (4.7), we have that

$$\lim_{n\to\infty} \frac{\gamma_{(m+n)\delta}}{\gamma_{(m+n-1)\delta}} = \mathrm{e}^{-\lambda_0\delta} < 1,$$

which implies that $\sum_{n=1}^\infty \gamma_{(m+n)\delta} < \infty$. By Theorem 5.2,

$$\lim_{n\to\infty} \gamma_{(m+n)\delta}\langle \phi_0, X_{(n-1)\delta}\rangle = \mathrm{e}^{-\lambda_0(m+1)\delta} W.$$

Thus we have $\sum_{n=1}^\infty \gamma_{(m+n)\delta}(\gamma_{(m+n)\delta}\langle \phi_0, X_{(n-1)\delta}\rangle) < \infty$, which implies that

$$\sum_{n=1}^\infty \mathbb{P}_\mu([C_{(n+m)\delta}(f) - \mathbb{P}_\mu(C_{(n+m)\delta}(f)\,|\,\mathcal{G}_{n\delta})]^2\,|\,\mathcal{G}_{(n-1)\delta}) < \infty.$$

Now using the conditional Bore-Cantelli lemma, we immediately get (6.18). □

Combining the three results above, we get the following lemma.

Lemma 6.7. *For any $m \in \mathbb{N}$, $\delta > 0$, $\mu \in \mathcal{M}_F(E)$ and $f \in \mathcal{B}_b^+(E)$, we have*

$$\lim_{n\to\infty} \gamma_{(n+m)\delta}\langle \phi_0 f, X_{(n+m)\delta}\rangle - \mathbb{P}_\mu[\gamma_{(n+m)\delta}\langle \phi_0 f, X_{(n+m)\delta}\rangle\,|\,\mathcal{G}_{n\delta}] = 0, \quad \textit{a.s.-}\mathbb{P}_\mu.$$

Proof of Theorem 6.2. By the Markov property of X, we have

$$\gamma_{(n+m)\delta}\mathbb{P}_\mu(\langle \phi_0 f, X_{(n+m)\delta}\rangle\,|\,\mathcal{G}_{n\delta}) = \gamma_{(n+m)\delta}\langle T_{m\delta}(\phi_0 f), X_{n\delta}\rangle.$$

It follows from (2.2) that there exist constants $c > 0$ and $\gamma > 0$ such that for any $m \geqslant 1$,

$$(1 - c\mathrm{e}^{-\gamma m\delta})\mathrm{e}^{\lambda_0 m\delta}\langle f\phi_0, \psi_0\rangle_m \phi_0(x) \leqslant T_{m\delta}(\phi_0 f)(x) \leqslant (1 + c\mathrm{e}^{-\gamma m\delta})\mathrm{e}^{\lambda_0 m\delta}\langle f\phi_0, \psi_0\rangle_m \phi_0(x).$$

Thus, by Lemma 6.7, we have that, for any $m \in \mathbb{N}$,

$$\begin{aligned}\limsup_{n\to\infty} \gamma_{n\delta}\langle \phi_0 f, X_{n\delta}\rangle &= \limsup_{n\to\infty} \gamma_{(n+m)\delta}\langle \phi_0 f, X_{(n+m)\delta}\rangle\\ &= \limsup_{n\to\infty} \gamma_{(n+m)\delta}\langle T_{m\delta}(\phi_0 f), X_{n\delta}\rangle\\ &\leqslant \limsup_{n\to\infty} \gamma_{(n+m)\delta}(1 + c\mathrm{e}^{-\gamma m\delta})\mathrm{e}^{\lambda_0 m\delta}\langle f\phi_0, \psi_0\rangle_m \langle \phi_0, X_{n\delta}\rangle\\ &= (1 + c\mathrm{e}^{-\gamma m\delta})\langle f\phi_0, \psi_0\rangle_m W.\end{aligned}$$

Letting $m \to \infty$, we get

$$\limsup_{n\to\infty} \gamma_{n\delta}\langle \phi_0 f, X_{n\delta}\rangle \leqslant \langle f\phi_0, \psi_0\rangle_m W. \tag{6.20}$$

Similarly, we have

$$\liminf_{n\to\infty} \gamma_{n\delta}\langle \phi_0 f, X_{n\delta}\rangle \geqslant (1 - c\mathrm{e}^{-\gamma m\delta})\langle f\phi_0, \psi_0\rangle_m W.$$

Letting $m \to \infty$, we get

$$\liminf_{n\to\infty} \gamma_{n\delta}\langle \phi_0 f, X_{n\delta}\rangle \geqslant \langle f\phi_0, \psi_0\rangle_m W. \tag{6.21}$$

Combining (6.20) and (6.21), the conclusion follows immediately. □

6.3 Continuous times: Case I

Define a new semigroup $(T_t^{\phi_0}, t \geqslant 0)$ by

$$T_t^{\phi_0} f(x) := \frac{e^{-\lambda_0 t} T_t(f\phi_0)(x)}{\phi_0(x)}, \quad f \in \mathcal{B}_b(E).$$

Then $(T_t^{\phi_0}, t \geqslant 0)$ is a conservative semigroup with transition density

$$q^{\phi_0}(t,x,y) = \frac{e^{-\lambda_0 t} q(t,x,y)\phi_0(y)}{\phi_0(x)}.$$

In this subsection, we also make the following assumption.

Assumption 6.8. *For any $f \in C_0(E)$,*

$$\lim_{t\to 0} \|T_t^{\phi_0} f - f\|_\infty = 0. \tag{6.22}$$

See [2, Examples 4.4, 4.5, 4.7 and Remark 4.6] for examples satisfying the assumption above, and Assumptions 1.1 and 1.2.

Theorem 6.9. *Under Assumptions 1.1–1.4, 6.1 and 6.8, we have that, for any $\mu \in \mathcal{M}_F(E)$ and $f \in C_0(E)$,*

$$\lim_{t\to\infty} \gamma_t \langle \phi_0 f, X_t\rangle = \langle f\phi_0, \psi_0\rangle_m W, \quad a.s.\text{-}\mathbb{P}_\mu. \tag{6.23}$$

Proof. First, we claim that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{t\in[n\delta,(n+1)\delta]} |\gamma_t\langle \phi_0 T^{\phi_0}_{(n+1)\delta-t} f, X_t\rangle - \gamma_t\langle \phi_0 f, X_t\rangle| = 0. \tag{6.24}$$

In fact, we have

$$\begin{aligned}
&\sup_{t\in[n\delta,(n+1)\delta]} |\gamma_t\langle \phi_0 T^{\phi_0}_{(n+1)\delta-t} f, X_t\rangle - \gamma_t\langle \phi_0 f, X_t\rangle| \\
&\quad \leqslant \sup_{r\in(0,\delta)} \|T_r^{\phi_0} f - f\|_\infty \sup_{t\in[n\delta,(n+1)\delta]} \gamma_t\langle \phi_0, X_t\rangle.
\end{aligned}$$

Letting $n\to\infty$ and then $\delta\to 0$, using Assumption 6.8 and Theorem 5.2, one immediately arrives at the claim (6.24). Thus, by (6.24), to obtain (6.23), we only need to prove that, for any $f \in C_0(E)$,

$$\lim_{\delta\to 0} \lim_{n\to\infty} \sup_{t\in[n\delta,(n+1)\delta]} |\gamma_t\langle \phi_0 T^{\phi_0}_{(n+1)\delta-t} f, X_t\rangle - \langle f\phi_0, \psi_0\rangle_m W| = 0. \tag{6.25}$$

Since $\phi_0 T^{\phi_0}_{(n+1)\delta-t} f = e^{-\lambda_0((n+1)\delta-t)} T_{(n+1)\delta-t}(\phi_0 f)$, we only need to show that, for any $f \in C_0(E)$,

$$\lim_{\delta\to 0} \lim_{n\to\infty} \sup_{t\in[n\delta,(n+1)\delta]} |\gamma_t\langle T_{(n+1)\delta-t} f, X_t\rangle - \langle f, \psi_0\rangle_m W| = 0. \tag{6.26}$$

By (6.4), we have that, for any $t \in [n\delta,(n+1)\delta]$,

$$\begin{aligned}
\langle T_{(n+1)\delta-t} f, X_t\rangle &= \langle T_\delta f, X_{n\delta}\rangle + \int_{n\delta}^t \int_{\mathcal{D}_{<1}(s)} \langle T_{(n+1)\delta-s} f, \nu\rangle \widetilde{N}(ds,d\nu) \\
&\quad + \int_{n\delta}^t \int_{\mathcal{D}_{\geqslant 1}(s)} \langle T_{(n+1)\delta-s} f, \nu\rangle \widetilde{N}(ds,d\nu) + \int_{n\delta}^t \int_E T_{(n+1)\delta-s} f(x) S^C(ds,dx) \\
&=: \langle T_\delta f, X_{n\delta}\rangle + H_t^{n,\delta}(f) + L_t^{n,\delta}(f) + C_t^{n,\delta}(f).
\end{aligned}$$

It follows from (2.2) that $\phi_0(x)^{-1} T_\delta f(x) \in \mathcal{B}_b^+(E)$. Thus, by Theorem 5.2, we have

$$\lim_{n\to\infty} \gamma_{n\delta}\langle T_\delta f, X_{n\delta}\rangle = \langle T_\delta f, \psi_0\rangle_m W = e^{\lambda_0\delta}\langle f, \psi_0\rangle_m W.$$

Note that $\gamma_{(n+1)\delta} \leqslant \gamma_t \leqslant \gamma_{n\delta}$. Thus,

$$\lim_{\delta\to 0}\lim_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta]}|\gamma_t\langle T_\delta f, X_{n\delta}\rangle - \langle f,\psi_0\rangle_m W| = 0.$$

To finish the proof, it suffices to show that

$$\lim_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta]}\gamma_{(n+1)\delta}|H_t^{n,\delta}(f)| = 0, \tag{6.27}$$

$$\lim_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta]}\gamma_{(n+1)\delta}|L_t^{n,\delta}(f)| = 0, \tag{6.28}$$

and

$$\lim_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta]}\gamma_{(n+1)\delta}|C_t^{n,\delta}(f)| = 0. \tag{6.29}$$

Using the Markov property of the superprocess X, we get that

$$\begin{aligned}
&\gamma^2_{(n+1)\delta}\mathbb{P}_\mu\Big(\sup_{t\in[n\delta,(n+1)\delta]}(H_t^{n,\delta}(f))^2\,\Big|\,\mathcal{G}_{(n-1)\delta}\Big)\\
&\quad= \gamma^2_{(n+1)\delta}\mathbb{P}_{X_{(n-1)\delta}}\left(\sup_{t\in[\delta,2\delta]}\left[\int_\delta^t\int_{\mathcal{D}_{<1}(s+(n-1)\delta)}\langle T_{2\delta-s}f,\nu\rangle\widetilde{N}(ds,d\nu)\right]^2 v\right)\\
&\quad\leqslant 4\gamma^2_{(n+1)\delta}\mathbb{P}_{X_{(n-1)\delta}}\left(\left[\int_\delta^{2\delta}\int_{\mathcal{D}_{<1}(s+(n-1)\delta)}\langle T_{2\delta-s}f,\nu\rangle\widetilde{N}(ds,d\nu)\right]^2\right)\\
&\quad= 4\mathbb{P}_\mu([H_{(n+1)\delta}(f) - \mathbb{P}_\mu(H_{(n+1)\delta}(f)\,|\,\mathcal{G}_{n\delta})]^2\,|\,\mathcal{G}_{(n-1)\delta}),
\end{aligned}$$

where the second to the last lines follow from the fact that

$$\left(\int_\delta^t\int_{\mathcal{D}_{<1}(s+(n-1)\delta)}\langle T_{2\delta-s}f,\nu\rangle\widetilde{N}(ds,d\nu), t\in[\delta,2\delta]\right)$$

is a martingale. Therefore, by (6.9), we have

$$\sum_{n=1}^\infty \gamma^2_{(n+1)\delta}\mathbb{P}_\mu\Big(\sup_{t\in[n\delta,(n+1)\delta]}(H_t^{n,\delta}(f))^2\,\Big|\,\mathcal{G}_{(n-1)\delta}\Big) < \infty.$$

Using the conditional Borel-Cantelli lemma, (6.27) follows immediately.

Similarly, we can prove that (6.29) holds. We omit the details here.

Note that

$$|L_t^{n,\delta}(f)| \leqslant \int_{n\delta}^{(n+1)\delta}\int_{\mathcal{D}_{\geqslant 1}(s)}\langle T_{(n+1)\delta-s}f,\nu\rangle(N(ds,d\nu)+\widehat{N}(ds,d\nu)).$$

Now using (6.16) and (6.17) with $m=1$, we immediately get (6.28). The proof is now completed. □

Theorem 6.10. *Suppose that Assumptions* 1.1–1.4, 6.1 *and* 6.8 *hold. There exists* $\Omega_0\subset\Omega$ *of probability one (i.e.,* $\mathbb{P}_\mu(\Omega_0)=1$ *for every* $\mu\in\mathcal{M}_F(E)$*) such that, for every* $\omega\in\Omega_0$ *and for every bounded Borel function* f *on* E *satisfying* (a) $|f|\leqslant c\phi_0$ *for some* $c>0$ *and* (b) *the set of discontinuous points of* f *has zero* m*-measure, we have*

$$\lim_{t\to\infty}\gamma_t\langle f,X_t\rangle(\omega) = \langle f,\psi_0\rangle_m W(\omega).$$

Proof. With the preparation above, the proof of this theorem is similar to that of [2, Theorem 1.4]. We omit the details here. □

6.4 Continuous times: Case II

In this subsection, we will consider the almost sure limit of $\gamma_t\langle f, X_t\rangle$ with f being a general bounded continuous function for some class of superdiffusions. The underlying process ξ is a diffusion satisfying the following conditions.

Suppose that E is a domain of finite Lebesgue measure in $\mathbb{R}^d$. Denote by $C_b^1(E)$ the family of bounded differentiable functions on E whose first order partial derivatives are all continuous. The underling process $\{\xi, \Pi_x\}$ is a killed diffusion process on E corresponding to the infinitesimal generator

$$L = \frac{1}{2}\nabla\cdot a\nabla + b\cdot\nabla, \tag{6.30}$$

where a and b satisfy the following conditions:

(a) $a_{ij} \in C_b^1(E)$, $i,j = 1,2,\dots,d$, and the matrix $a = (a_{ij})$ is symmetric satisfying, for all $x \in E$ and $v \in \mathbb{R}^d$,

$$c_0|v|^2 \leqslant \sum_{i,j} a_{ij}v_iv_j,$$

for some positive constant c_0.

(b) $b_j \in \mathcal{B}_b(E)$, $j = 1,\dots,d$.

Using an argument similar to that in [3, section 3.2], one can easily show that P_t has a bounded continuous and strictly positive density $p(t,x,y)$. Thus Assumption 1.1 holds immediately. Since $m(E) < \infty$ and the first eigenfunction $\widetilde{\phi}_0 \in L^2(E,m)$, we have that $\widetilde{\phi}_0 \in L^1(E,m)$. Then using the fact that $p(1,x,y)$ is bounded and $\widetilde{\phi}_0(x) = \mathrm{e}^{-\widetilde{\lambda}_0}P_1\widetilde{\phi}_0(x)$, we get that $\widetilde{\phi}_0$ is bounded on E. Similarly, $\widetilde{\psi}_0$ is also bounded, which shows that Assumption 1.2(i) holds. We assume that the semigroup P_t is intrinsically ultracontractive.

Let $f \in \mathcal{B}_b(E)$, and U^qf, $q > 0$, be the q-potential of f, i.e.,

$$U^qf(x) = \int_0^\infty \mathrm{e}^{-qs}T_s^{\phi_0}f(x)ds.$$

For any $t > 0$,

$$\mathrm{e}^{-qt}T_t^{\phi_0}(U^qf)(x) = \int_t^\infty \mathrm{e}^{-qs}T_s^{\phi_0}f(x)ds. \tag{6.31}$$

Theorem 6.11. *For any $q > 0$, $\mu \in \mathcal{M}_F(E)$ and $f \in \mathcal{B}_b^+(E)$, we have*

$$\lim_{t\to\infty}\gamma_t\langle\phi_0 qU^qf, X_t\rangle = \langle f\phi_0, \psi_0\rangle_m W, \quad a.s.\text{-}\mathbb{P}_\mu. \tag{6.32}$$

Proof. First, we claim that

$$\lim_{\delta\to 0}\lim_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta]}|\gamma_t\langle\phi_0T^{\phi_0}_{(n+1)\delta-t}(U^qf), X_t\rangle - \gamma_t\langle\phi_0U^qf, X_t\rangle| = 0. \tag{6.33}$$

For any $r \in [0,\delta]$, we have

$$\begin{aligned}
&q\left|T_r^{\phi_0}(U^qf)(x) - U^qf(x)\right| \\
&\quad= q\left|(\mathrm{e}^{qr}-1)\int_r^\infty \mathrm{e}^{-qs}T_s^{\phi_0}f(x)ds - \int_0^r \mathrm{e}^{-qs}T_s^{\phi_0}f(x)ds\right| \\
&\quad\leqslant \|f\|_\infty\left(q(\mathrm{e}^{qr}-1)\int_r^\infty \mathrm{e}^{-qs}ds + q\int_0^r \mathrm{e}^{-qs}ds\right) \\
&\quad= 2\|f\|_\infty(1-\mathrm{e}^{-qr}).
\end{aligned}$$

Thus,

$$\sup_{t\in[n\delta,(n+1)\delta]}|\gamma_t\langle\phi_0T^{\phi_0}_{(n+1)\delta-t}(qU^qf), X_t\rangle - \gamma_t\langle\phi_0qU^qf, X_t\rangle|$$

$$\leqslant 2\|f\|_\infty(1-\mathrm{e}^{-q\delta})\sup_{t\in[n\delta,(n+1)\delta]}\gamma_t\langle\phi_0,X_t\rangle.$$

Letting $n\to\infty$ and then $\delta\to 0$, the claim (6.33) follows immediately. Note that

$$\langle qU^q f\phi_0,\psi_0\rangle_m=\langle f\phi_0,\psi_0\rangle_m.$$

Thus, applying (6.25) with f replaced by $qU^q f$, we get

$$\lim_{\delta\to 0}\lim_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta]}|\gamma_t\langle\phi_0 T^{\phi_0}_{(n+1)\delta-t}(qU^q f),X_t\rangle-\langle\phi_0 f,\psi_0\rangle_m W|=0. \tag{6.34}$$

Now combining (6.33) and (6.34), (6.32) follows immediately. □

Theorem 6.12. *Suppose X is a superdiffusion on a domain $E\subset\mathbb{R}^d$ of finite Lebesgue measure with spatial motion being a killed diffusion in E with generator L given in* (6.30) *satisfying the conditions* (a) *and* (b)*. Suppose that Assumption* 1.1–1.4 *and* 6.1 *hold. Then there exists $\Omega_0\subset\Omega$ of probability one* (*i.e., $\mathbb{P}_\mu(\Omega_0)=1$ for every $\mu\in\mathcal{M}_F(E)$*) *such that, for every $\omega\in\Omega_0$ and for every bounded Borel function f on E satisfying* (a) *$|f|\leqslant c\phi_0$ for some $c>0$ and* (b) *the set of discontinuous points of f has zero m-measure, we have $\lim_{t\to\infty}\gamma_t\langle f,X_t\rangle(\omega)=\langle f,\psi_0\rangle_m W(\omega)$.*

Proof. With the preparation above, the proof of this theorem is similar to that of [18, Theorem 1.1]. We omit the details here. □

7 Concluding remarks

Suppose that $X=\{X_t,t\geqslant 0;\mathbb{P}_\mu\}$ is a supercritical superprocess in a locally compact separable metric space E such that the generator of the mean semigroup of X has discrete spectrum. Let ϕ_0 be a positive eigenfunction corresponding to the first eigenvalue λ_0 of the generator of the mean semigroup of X. Then $M_t:=\mathrm{e}^{-\lambda_0 t}\langle\phi_0,X_t\rangle$ is a positive martingale. Let M_∞ be the limit of M_t. It is known (see [17]) that M_∞ is non-degenerate if and only if the $L\log L$ condition is satisfied. In this paper, we prove that, under some further conditions, there exist a positive function γ_t on $[0,\infty)$ and a non-degenerate random variable W such that for any finite nonzero Borel measure μ on E,

$$\lim_{t\to\infty}\gamma_t\langle\phi_0,X_t\rangle=W,\quad \text{a.s.-}\mathbb{P}_\mu.$$

We also give the almost sure limit of $\gamma_t\langle f,X_t\rangle$ for a class of general test functions f.

In [24], a sequel to the present paper, we studied properties of the limit random variable W, such as absolute continuity and tail probabilities.

It would be interesting to extend the results of this paper and [24] to supercritical superprocesses with immigration.

The assumptions of this paper, particularly Assumption 1.2(ii), are pretty strong. For example, supercrtical super Brownian motion in $\mathbb{R}^d$ does not satisfy Assumption 1.2(ii). It would be interesting to consider corresponding results of this paper and [24] for supercritical superprocesses under weaker conditions. It would be very interesting to get rid of the assumption that the generator of the mean semigroup of X has discrete spectrum.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11671017, 11731009 and 11601354), Key Laboratory of Mathematical Economics and Quantitative Finance (Peking University), Ministry of Education, the Simons Foundation (Grant No. 429343) and Youth Innovative Research Team of Capital Normal University. The authors thank the referees for very careful reading of the paper and for very helpful comments.

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