Super-Brownian Motions with Absolutely Continuous Measure States[1]

Yan-Xia Ren[2]

Abstract

Suppose $X = \{X_t, P_\mu\}$ is a d-dimensional super-Brownian motion with branching rate function A and general branching mechanism ψ. We discuss conditions on A to guarantee that X_t has absolutely continuous states. For the particular case of $\psi(s, x, z) = z^2$, the analogous problem has been discussed by Dawson and Fleischmann (1995). We generalize and simplify the conditions of Dawson and Fleischmann based on an improvement on their argument.

Keywords Super-Brownian motion, fundamental solution, absolutely continuous state, branching rate functional

2000 Mathematics Subject Classification 60J80, 35K15, 60J65,

§1. Introduction

For every Borel measurable space $(E, \mathcal{B}(\mathcal{E}))$, we denote by $M(E)$ the set of all finite measures on $\mathcal{B}(\mathcal{E})$ endowed with the topology of weak convergence. The expression $< f, \mu >$ stands for the integral of f with respect to μ. We write $f \in \mathcal{B}(E)$ if f is a $\mathcal{B}(E)$-measurable function. Writing $f \in p\mathcal{B}(E)(b\mathcal{B}(E))$ means that, in addition, f is positive(bounded). We put $bp\mathcal{B}(E) =$

[1]Research of the author was supported in part by the National Natural Science Foundation of China (Grant No. 10001020) and Foundation for Authors awarded Excellent Ph. D. Dissertations.

[2]Department of Probability and Statistics, School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China, E-mail: yxren@math.pku.edu.cn

$(b\mathcal{B}(E) \cap p\mathcal{B}(E))$. If $E = \mathbf{R^d}$, we simply write $\mathcal{B}$ instead of $\mathcal{B}(\mathbf{R^d})$ and M instead of $M(\mathbf{R^d})$. We will use the symbol $\xrightarrow{bp}$ to denote bounded pointwise convergence. (Recall that functions converge boundedly pointwise if they are uniformly bounded and converge pointwise.)

Let $W := \{W, \Pi_{r,x}, r \geq 0, x \in \mathbf{R^d}\}$ denote the canonical Brownian motion in $\mathbf{R^d}$ with birth time α. $\Pi_{r,x}(\alpha = r, W_\alpha = x) = 1$. Set $\mathcal{F}^0_{\leq r} = \sigma(W_s, s \leq r); \mathcal{F}^0_{>r} = \sigma(W_s, s > r)$ and $\mathcal{F}^0_\infty = \bigvee\{\mathcal{F}^0_{\leq r}, r \geq 0\}$.

Set $S = [0, \infty) \times \mathbf{R^d}$. To every set $Q \subset S$ there corresponds the first exit time $\tau = \inf\{t : t \geq \alpha, (t, W_t) \notin Q\}$. Put $(r, x) \in Q^0$ if $\Pi_{r,x}\{\tau > r\} = 1$. A set $Q \in \mathcal{B}(S)$ is called finely open if $Q^0 = Q$. We denote by $\mathcal{T}$ the set of all exit times from finely open sets $Q \in \mathcal{B}(S)$. For $\tau \in \mathcal{T}$, Put $C \in \mathcal{F}^0_{\geq \tau}$ if $C \in \mathcal{F}^0_\infty$ and if, for each r, $\{C, \tau > r\} \in \mathcal{F}^0_{>r}$.

For $s, z \geq 0$,$x \in \mathbf{R^d}$, let

$$\psi(s, x, z) = a(s, x)z + b(s, x)z^2 + \int_0^\infty (e^{-uz} - 1 + uz)n(s, x, du), \tag{1.1}$$

where a, b are positive measurable functions, n is a kernel from $\mathbf{R^d}$ to $(0, \infty)$ such that for every finite interval Δ, $a(s, x), b(s, x)$ and $\int_0^\infty u \wedge u^2 n(s, x, du)$ are positive bounded Borel functions on $\Delta \times \mathbf{R^d}$.

Suppose A is a continuous additive functional of W. A is called a *branching rate functional* if there exists a time-inhomogeneous measure-valued Markov process $X = \{X_t, P_{r,\mu}, t \geq r \geq 0, \mu \in M\}$ with the Laplace functional

$$P_{r,\mu} \exp < -f, X_t >= \exp < -u(r, \cdot), \mu >, \quad 0 \leq r \leq t, \mu \in M, f \in bp\mathcal{B} \tag{1.2}$$

where u is the unique bounded solution of the integral equation

$$u(r,x) = \Pi_{r,x}(f(W_t)) - \Pi_{r,x}\int_r^t \psi(s, W_s, u(s, W_s))A(ds), \quad 0 \le r \le t, x \in \mathbf{R^d}. \tag{1.3}$$

We call $X = \{X_t, P_{r,\mu}, t \ge r \ge 0, \mu \in M\}$ a super-Brownian motion with parameters (A, ψ).

Particularly, when $\psi(s,x,z) = z^2$, Dawson and Fleischmann[2] investigated conditions on the additive functional A, which guarantee that X_t is absolutely continuous with respect to the Lebesgue measure in $\mathbf{R^d}$. But, if the branching mechanism ψ is given by the general form (1.1) and the branching rate function A is a general continuous additive functional, how to guarantee the state X_t is absolutely continuous? In this paper we are devoted to impose some conditions on A to guarantee that a super-Brownian motion X with general branching mechanism ψ has absolutely continuous states. Our approach in the present paper is an improvement over that of Dawson and Fleischmann[2] and the conditions imposed on A are simpler than that given by Dawson and Fleischmann.

§2. Main results

Assume that I is a *halfopen* interval $[L, T), 0 \le L < T$. We consider the absolute continuity of X_T. For $\nu \in M$ set

$$(S^I\nu)(r,y) := \int \nu(dz)p(T-r, y-z), \quad r \in I, y \in \mathbf{R^d}, \tag{2.1}$$

$$\nu_\epsilon(y) := \int \nu(dz)p(\epsilon, y-z) =: \nu \star p(\epsilon)(y), \quad y \in \mathbf{R^d}, \tag{2.2}$$

where $p(t,y) = (2\pi t)^{-d/2}\exp(-\frac{y^2}{2t}), t > 0, y \in \mathbf{R^d}$, is the Brownian transition density function. With an abuse of notation we use $\nu_\epsilon(dy)$ to denote the measure with density ν_ϵ given above. Note that for each $r \in I$,

$$S^I\nu_\epsilon(r,\cdot) \xrightarrow{bp} S^I\nu(r,\cdot) \text{ as } \epsilon \downarrow 0. \tag{2.3}$$

Consider the fundamental solutions of the integral equation (1.3). To be more precise, for $\nu \in M$ consider the integral equation in the form

$$u(r,x) = S^I\nu(r,x) - \Pi_{r,x}\int_r^T \psi(s, W_s, u(s,W_s))A(ds), \quad r \in I, x \in \mathbf{R^d}. \tag{2.4}$$

Definition 2.1 Fix a halfopen interval $I = [L,T), 0 \le L < T$. Let ν belong to M and A be a branching rate functional. A is called ν-regular for ψ, if there is a zero sequence $\{\epsilon(n), n \ge 1\}$ (depending on I, ν, A and ψ), called a ν-admissible sequence, such that for each fixed $r \in I$ and $x \in \mathbf{R^d}$,

$$\limsup_{n\to\infty} \Pi_{r,x}\int_t^T \psi(s, W_s, S^I\nu_{\epsilon(n)}(s,W_s))A(ds) \to 0 (t \uparrow T). \tag{2.5}$$

Note that there exists a constant C such that

$$\psi(s,x,z) \le C(z+z^2), \quad s \in I, z \ge 0, x \in \mathbf{R^d}. \tag{2.6}$$

If A is ν-regular for z^2 then it is also ν-regular for every ψ given by (1.1).

Definition 2.2 Fix a halfopen interval $I = [L,T), 0 \le L < T$. A branching rate functional A is said to be a.e.-regular for ψ if there exists a Borel subset N of Lebesgue measure zero such that A is ν-regular for ψ for all point measures ν on $\mathbf{R^d}$ with finite support contained in $\mathbf{R^d} \setminus \mathbf{N}$.

When $\psi = z^2$, Dawson and Fleischmann[2] also defined ν-regularity. For this particular case, our regularity condition is weaker than Dawson and Fleischmann's.

For fixed $I = [L,T]$, let $\mathcal{A}_0^I$ denote the set of all those continuous additive functionals A of the d-dimensional Brownian motion W satisfying

$$\Pi_{r,x}A(r,T) < \infty, \quad \text{for all } r \in I, x \in \mathbf{R^d}. \tag{2.7}$$

We write $A \in \mathcal{A}_0$, if $A \in \mathcal{A}_0^I$ for all finite interval I. We use notions $\mathcal{A}_0^I$ and $\mathcal{A}_0$ to differentiate notions $\mathcal{A}^I$ and $\mathcal{A}$ in Dawson and Fleischmann[2]. It is obvious that $\mathcal{A}_0^I \subset \mathcal{A}^I$ and $\mathcal{A}_0 \subset \mathcal{A}$.

Theorem 2.1(**Fundamental Solutions**) Suppose ψ is given by (1.1). Let ν belong to M and $A \in \mathcal{A}_0^I$ be ν-regular for ψ with respect to the interval $I = [L, T)$.

(1) (**Existence and Uniqueness**) There is exactly one measurable non-negative function $U^I[A, \nu]$ defined on $I \times \mathbf{R^d}$ which solves equation (2.4).

(2) (**Continuity of Regularization**) The solution $U^I[A, \nu]$ is continuous with respect to the operation of regulation of ν in the following sense: If $\{\epsilon(n), n \geq 1\}$ is a ν-admissible sequence then

$$U^I[A, \nu_{\epsilon(n)}](r, \cdot) \xrightarrow{bp} U^I(A, \nu)(r, \cdot), \text{ as } n \to \infty, \text{ for every } r \in I. \tag{2.8}$$

(3) (**First Derivative with Respect to Small Parameter**) If $a(s, x) \equiv 0$, then

$$\lambda^{-1} U(A, \lambda\nu)(r, \cdot) \xrightarrow{bp} S^I \nu(r, \cdot) \text{ as } \lambda \to 0, \text{ for every } r \in I. \tag{2.9}$$

Theorem 2.2 Suppose ψ is given by (1.1) with $a(s, x) \equiv 0$. Let $X = (X_t, P_{r,\mu})$ be a super-Brownian motion with parameters (A, ψ). Assume that A belong to $\mathcal{A}_\prime^\mathcal{I}$ and is a.e.-regular for ψ with respect to the interval $I = [L, T)$.

(1) For fixed time points $0 \leq r \leq L < T$ and $\mu \in M$, there exists a random measurable function x_T on $\mathbf{R^d}$ such that

$$P_{r,\mu}\{X_T(dz) = x_T(z)dz\} = 1.$$

(2) For each finite collection $z(1), \ldots, z(m)$ of points in $\mathbf{R^d} \setminus \mathbf{N}$, the Laplace function of the random vector $[x_T(z(1)), \ldots, x_T(z(m))]$ with respect to $P_{r,\mu}$ is given by

$$P_{r,\mu} \exp \left[\sum_{i=1}^{m} \lambda_i x_T(z(i)) \right] = \exp < -u(r, \cdot), \mu >, \quad \lambda_1, \ldots, \lambda_m \geq 0,$$

where u is the continuation of the fundamental solution $U^I[A,\nu]$ of (2.4) to the interval $[r,T]$, and $\nu = \sum_{i=1}^m \lambda_i \delta_{z(i)}$.

§3. Proof of the Main Results

Let us first state some lemmas. The following lemma 3.1 is taken from [4] with a slight modification and for completeness, we will give its proof below.

For $c \in p\mathcal{B}$, put

$$H^c(r_1, r_2) = \exp\left(\int_{r_2}^{r_1} c(s, W_s)A(ds)\right), \quad 0 \le r_1 \le r_2. \tag{3.1}$$

Lemma 3.1 Suppose $A(dt)$ is a non-negative continuous additive functional of the Brownian motion W in $\mathbf{R^d}$. Let $\tau \in \mathcal{T}$, and $c, g \in bp\mathcal{B}$. Assume that $\omega \in \mathcal{B}$ and $F \in \mathcal{F}'_{\ge\tau}$ satisfy

$$\Pi_{r,x}\int_r^\tau |\omega(s,W_s)|A(ds) < \infty; \quad \Pi_{r,x}|F| < \infty, \quad r \ge 0, x \in \mathbf{R^d}.$$

Then

$$g(r,x) = \Pi_{r,x}\left[H^c(r,\tau)F + \int_r^\tau H^c(r,s)\omega(s,W_s)A(ds)\right] \tag{3.2}$$

iff

$$g(r,x) + \Pi_{r,x}\int_r^\tau (cg)(s,W_s)A(ds) = \Pi_{r,x}\left[F + \int_r^\tau \omega(s,W_s)A(ds)\right]. \tag{3.3}$$

Proof Using the Markov property of the Brownian motion W, it is easy to check that

$$\begin{aligned}
&\Pi_{r,x}\int_r^\tau \omega(s,W_s)A(ds) \\
= \;& \Pi_{r,x}\int_r^\tau A(ds)H^c(r,s)c(s,W_s)\Pi_{s,W_s}\int_s^\tau \omega(s_1,W_{s_1})A(ds_1) \\
&+\Pi_{r,x}\int_r^\tau H^c(r,s)\omega(s,W_s)A(ds) \\
= \;& \Pi_{r,x}\int_r^\tau A(ds)c(s,W_s)\Pi_{s,W_s}\int_s^\tau H^c(s,s_1)\omega(s_1,W_{s_1})A(ds_1) \\
&+\Pi_{r,x}\int_r^\tau H^c(r,s)\omega(s,W_s)A(ds);
\end{aligned} \tag{3.4}$$

$$\begin{aligned}\Pi_{r,x}F = & \quad \Pi_{r,x}(H^c(r,\tau)F) + \Pi_{r,x}\int_r^\tau H^c(r,s)c(s,W_s)\Pi_{s,W_s}F \\ = & \quad \Pi_{r,x}(H^c(r,\tau)F) + \Pi_{r,x}\int_r^\tau c(s,W_s)\Pi_{s,W_s}(H^c(s,\tau)F).\end{aligned} \tag{3.5}$$

Using (3.4) and (3.5) we can get the result of this lemma. We omit the details here.

Using an analytic method, we can check that ψ has the following properties:

Lemma 3.2 Suppose ψ is given by the form (1.1).

(1) For fixed $s \geq 0$, and $x \in \mathbf{R^d}$, $\psi(s,x,z)$ is increasing and convex as a function of z, and for $z(1), \dots, z(l) \in \mathbf{R^d}$,

$$\psi(s,x,\sum_{j=1}^{l} z(j)) \leq 2^{l-1}\sum_{j=1}^{l}\psi(s,x,z(j)). \tag{3.6}$$

(2) For fixed $s \geq 0$, and $x \in \mathbf{R^d}$, $z^{-1}\psi(s,x,z)$ is increasing as a function of z.

(3) For $0 < \lambda \leq 1, s, z \leq 0, x \in \mathbf{R^d}$,

$$\lambda^{-1}\psi(s,x,\lambda z) \leq \psi(s,x,z); \tag{3.7}$$

(4) If $a(s,x) \equiv 0$, then

$$\lim_{\lambda \downarrow 0} \lambda^{-1}\psi(s,x,\lambda z) = 0. \tag{3.8}$$

Put

$$\lambda(s,x) = 2b(s,x) + \int_0^\infty u \wedge u^2 n(s,x,du), \quad s \geq 0, x \in \mathbf{R^d} \tag{3.9}$$

For $c \in p\mathcal{B}(\mathcal{S})$, define

$$R_{c,\psi}(s,x,z) = c(s,x)z - \psi(s,x,z), \quad s,z \geq 0, x \in \mathbf{R^d}. \tag{3.10}$$

Lemma 3.3 For all $M \geq 1$, $s \geq 0$, $0 \leq z_1, z_2 \leq M$ and $x \in \mathbf{R^d}$,

$$|R_{a+\lambda M,\psi}(s,x,z_1) - R_{a+\lambda M,\psi}(s,x,z_2)| \leq \lambda(s,x)M|z_1 - z_2|. \tag{3.11}$$

Proof For $x \in \mathbf{R^d}$, $0 \le z \le M$, $M \ge 1$,

$$\begin{aligned}\lambda(s,x)M \ge \; & [R_{a+\lambda M,\psi}(s,x,z)]'_z \\ = \; & \lambda(s,x)M - 2b(s,x)z - \int_0^\infty u(1-e^{-uz})n(s,x,du) \ge 0\end{aligned}$$

Thus the result of Lemma holds.

Proof of Theorem 2.1 Let $\{\epsilon(n), n \ge 1\}$ be a related ν-admissible zero sequence. Assume that u_n is a non-negative solution of the integral equation (2.4) with ν replaced by ν_n. We want to show that $u_n(r,\cdot), n \ge 1$ are uniformly bounded for each fixed $r \in I$ and $\{u_n(r,x); n \ge 1\}$ is a Cauchy sequence for fixed $r \in I, x \in \mathbf{R^d}$. First of all, for $n \ge 1$ and $r \in I$, we have the following domination:

$$0 \le u_n(r,\cdot) \le S^I \nu_n(r,\cdot) \le \parallel \nu \parallel p(\epsilon(n)+T-r,0), \quad \epsilon(n) \ge 0, \tag{3.12}$$

which means that $u_n(r,\cdot)$ are uniformly bounded for each fixed $r \in I$. Moreover, since $p(\epsilon+s,0) \le p(s,0) = p(1,0)s^{-d/2}, s \ge 0$, we have, for fixed $r \in I$,

$$0 \le u_n(r,\cdot) \le S^I \nu_n(r,\cdot) \le \parallel \nu \parallel p(1,0)(T-t)^{-d/2}, \quad \text{for } L \le r \le t. \tag{3.13}$$

For each fixed $t \in I$, let

$$M_t = (\parallel \nu \parallel p(1,0)(T-t)^{-d/2}) \vee 1; \quad c_t(s,x) = (a+\lambda M_t)(s,x).$$

Then by Lemma , for $s \ge 0$, $0 \le z_1,\, z_2 \le M_t$, $x \in \mathbf{R^d}$,

$$|R_{c_t,\psi}(s,x,z_1) - R_{c_t,\psi}(s,x,z_2)| \le \lambda(s,x)M_t|z_1-z_2| \le c_t(s,x)|z_1-z_2|, \tag{3.14}$$

where $R_{c_t,\psi}$ is defined by (3.10). Using Lemma with $c = c_t$, $F = \nu_n(W_T) - \int_t^T \psi(s,W_s,u_n(s,W_s))A(ds) \in \mathcal{F}^0_{\ge t}$, $g(r,x) = u_n(r,x), r \le t < T$ and $\omega(s,x) = c_t(s,x)u_n(s,x) - \psi(s,x,u_n(s,x)) = R_{c_t,\psi}(s,x,u_n(s,x))$, we get

$$
\begin{aligned}
u_n(r,x) = \quad & \Pi_{r,x}\left\{H^{c_t}(r,t)\left[\nu_n(W_T) - \int_t^T \psi(s,W_s,u_n(s,W_s))A(ds)\right]\right\} \\
& + \Pi_{r,x}\int_r^t A(ds)H^{c_t}(r,s)R_{c_t,\psi}(s,W_s,u_n(s,W_s)), \quad n \geq 1, r \leq t \leq T,
\end{aligned}
$$

For $r \in I, x \in \mathbf{R^d}$, put

$$
\begin{aligned}
h_{m,n}(r,x) = \quad & \Pi_{r,x}[H^{c_t}(r,t)(\nu_m - \nu_n)(W_T)]; \\
g_{m,n}(r,x) = \quad & \psi(r,x,S^I\nu_m(r,x)) + \psi(r,x,S^I\nu_n(r,x)).
\end{aligned}
\tag{3.15}
$$

By (3.14), for $m,n \geq 1, r \in I$ and $x \in \mathbf{R^d}$, we have

$$
\begin{aligned}
|u_m - u_n|(r,x) \leq \quad & |h_{m,n}(r,x)| + \Pi_{r,x}\left[H^{c_t}(r,t)\int_t^T g_{m,n}(s,W_s)A(ds)\right] \\
& + \Pi_{r,x}\int_r^t H^{c_t}(r,s)(c_t|u_m - u_n|)(s,W_s)A(ds).
\end{aligned}
\tag{3.16}
$$

Iterating this inequality $k \geq 1$ times and using the Markov property of W yields

$$
|u_m - u_n|(r,x) \leq |h_{m,n}(r,x)| + E + F + G, \tag{3.17}
$$

where we set

$$
\begin{aligned}
E := \quad & \Pi_{r,x}\textstyle\sum_{i=1}^k \displaystyle\int_r^t A(ds_1)\int_{s_1}^t A(ds_2)\ldots\int_{s_{i-1}}^t A(ds_i)\prod_{j=1}^{i-1} c_t(s_j,W_{s_j}) \\
& \cdot H^{c_t}(r,s_i)(c_t|h_{m,n}|)(s_i,W_{s_i}); \\
F := \quad & \Pi_{r,x}\sum_{i=1}^k \int_r^t A(ds_1)\int_{s_1}^t A(ds_2)\ldots\int_{s_{i-1}}^t A(ds_i)\prod_{j=1}^{i} c_t(s_j,W_{s_j}) \\
& \cdot H^{c_t}(r,t)\int_t^T g_{m,n}(s,W_s)A(ds) + \Pi_{r,x}\left[H^{c_t}(r,t)\int_t^T g_{m,n}(s,W_s)A(ds)\right]; \\
G := \quad & \Pi_{r,x}\int_r^t A(ds_1)\int_{s_1}^t A(ds_2)\ldots\int_{s_k}^t A(ds_{k+1})\prod_{j=1}^{k} c_t(s_j,W_{s_j}) \\
& \cdot H^{c_t}(r,s_{k+1})(c_t|u_m - u_n|)(s_{k+1},W_{s_{k+1}}),
\end{aligned}
$$

(with the interpretation: $\prod_{j=1}^0 c_t(s_j,W_{s_j}) = 1, s_0 = r$). Reversing the order of integration in all integrals in E, F and G, we can get

$$\begin{aligned}E = &\ \Pi_{r,x}\int_r^t H^{c_t}(r,s_1)(c_t|h_{m,n}|)(s_1,W_{s_1})\sum_{i=0}^{k-1}\frac{\left(\int_r^{s_1} c_t(s,W_s)A(ds)\right)^i}{i!}A(ds_1) \\ \leq &\ \Pi_{r,x}\int_r^t (c_t|h_{m,n}|)(s,W_s)A(ds);\end{aligned} \tag{3.18}$$

$$\begin{aligned}F = &\ \Pi_{r,x}\left[H^{c_t}(r,t)\int_t^T g_{m,n}(s,W_s)A(ds)\sum_{i=0}^{k}\frac{\left(\int_r^t c_t(s,W_s)A(ds)\right)^i}{i!}\right] \\ \leq &\ \Pi_{r,x}\int_t^T g_{m,n}(s,W_s)A(ds);\end{aligned} \tag{3.19}$$

$$G = \Pi_{r,x}\int_r^t H^{c_t}(r,s_1)(c_t|u_m-u_n|)(s_1,W_{s_1})\frac{\left(\int_r^{s_1} c_t(s,W_s)A(ds)\right)^k}{k!}A(ds_1).$$

By (3.13) and the definition of c_t,

$$G \leq C(\nu,t,T)\Pi_{r,x}\int_r^t H^{c_t}(r,s_1)\frac{\left(\int_r^{s_1} c_t(s,W_s)A(ds)\right)^k}{k!}A(ds_1) \tag{3.20}$$

where $C(\nu,t,T)$ is a constant depending only on ν, t and T. Letting $k\to\infty$ in (3.20), by (2.7) and the dominated convergence theorem, we get

$$G \to 0 \text{ as } k\to\infty. \tag{3.21}$$

Therefore by (3.17), (3.18), (3.19) and (3.21) we have

$$\begin{aligned}&|u_m-u_n|(r,x) \\ \leq &\ |h_{m,n}(r,x)| + \Pi_{r,x}\int_r^t (c_t|h_{m,n}|)(s,W_s)A(ds) + \Pi_{r,x}\int_t^T q_{m,n}(s,W_s)A(ds).\end{aligned} \tag{3.22}$$

For each $r\in I$,

$$\begin{aligned}q_{m,n}(r): &= \|(S^I\nu_m - S^I\nu_n)(r,\cdot)\|_\infty \\ &\leq \|\nu\|\|p(\epsilon(m)+T-r,\cdot)-p(\epsilon(n)+T-r,\cdot)\|_\infty.\end{aligned} \tag{3.23}$$

Therefore

$$\lim_{m,n\to\infty}\sup_{L\leq r\leq t} q_{m,n}(r) = 0, \quad t\in I. \tag{3.24}$$

By the Markov property of W,

$$h_{m,n}(r,x) = \Pi_{r,x}[H^{c_t}(r,t)(\nu_m - \nu_n)(W_T)] = \Pi_{r,x}[H^{c_t}(r,t)\Pi_{t,W_t}(\nu_m - \nu_n)(W_T)],$$

and therefore, by (3.24),

$$\begin{aligned} \sup_{L\le r\le t} \|h_{m,n}(r,\cdot)\|_\infty \le & \quad \|(S^I\nu_m - S^I\nu_n)(t,W_t)\|_\infty \\ = & \quad q_{m,n}(t) \to 0, \qquad \text{as } m,n\to\infty. \end{aligned} \tag{3.25}$$

Since A is ν-regular for ψ,

$$\limsup_{m,n\to\infty} \Pi_{r,x} \int_t^T q_{m,n}(s,W_s)A(ds) \to 0 (t\uparrow T), \quad \text{for all } r\in I, x\in \mathbf{R}^d. \tag{3.26}$$

Combining (3.22),(3.25), (3.26) and (2.7), we have $\{u_n(r,x), n\ge 1\}$ is a Cauchy sequence for fixed $r\in I, x\in \mathbf{R^d}$.

Summarizing, we established the existence of a non-negative measurable function u on $I\times \mathbf{R^d}$ such that

$$u_n(r,\cdot) \xrightarrow{bp} u(r,\cdot)(n\to\infty), \quad r\in I$$

Note that for any $r\in I, t\in (r,T)$ and $x\in \mathbf{R^d}$,

$$\begin{aligned} u_n(r,x) = & \quad S^I\nu_n(r,x) + \Pi_{r,x}\int_r^t \psi(s,W_s,u_n(s,W_s))A(ds) \\ & +\Pi_{r,x}\int_t^T \psi(s,W_s,u_n(s,W_s))A(ds). \end{aligned}$$

Letting $n\to\infty$ and then $t\uparrow T$ in the above equality, by noticing (2.5), we conclude that u solves equation(2.4).

Suppose u_1, u_2 are two solutions of (2.4). Repeating the procedure from the beginning with u_1 and u_2 instead of u_m and u_n, respectively, we get that

$$\begin{aligned} |u_1 - u_2| \le & \quad \Pi_{r,x}\left[H^{c_t}(r,t)\int_t^T 2\psi(s,W_s,S^I\nu(s,W_s))A(ds)\right] \\ & +\Pi_{r,x}\int_r^t H^{c_t}(r,s)(c_t|u_1-u_2|)(s,W_s)A(ds). \end{aligned}$$

Iterating the above inequality similarly as above we get

$$|u_1 - u_2| \le \Pi_{r,x} \int_t^T 2\psi(s, W_s, S^I\nu(s, W_s))A(ds).$$

By Fatou's lemma, (2.5) also holds for $\epsilon(n) = 0$ and therefore

$$\Pi_{r,x} \int_t^T 2\psi(s, W_s, S^I\nu(s, W_s))A(ds) \to 0(t \uparrow T).$$

So $u_1 = u_2$.

It remains to verify the asymptotic property (2.9). Since $\psi(s, x, \lambda z) \le \psi(s, x, z)$ for all $s, z \ge 0, x \in \mathbf{R^d}, \mathbf{0} < \lambda \le \mathbf{1}$, we know that the branching functional A is $\lambda\nu$-regular for all $0 < \lambda \le 1$. Fix $r \in I$, by equation(2.4) (with ν replaced by $\lambda\nu$),

$$|\lambda^{-1}U^I[A, \lambda\nu] - S^I\nu|(r, x) \le \Pi_{r,x} \int_r^T \lambda^{-1}\psi(s, W_s, \lambda S^I\nu(s, W_s))A(ds). \tag{3.27}$$

Letting $\lambda \downarrow 0$ in (3.27), by Fatou's lemma, and by noticing (3.7),(3.8) and (2.5), we get that $\limsup_{\lambda\downarrow 0} |\lambda^{-1}U^I[A, \lambda\nu] - S^I\nu|(r, x) = 0$. Thus, $\lambda^{-1}U^I[A, \lambda\nu](r, \cdot)@ >> \lambda \downarrow 0 > S^I\nu(r, \cdot)$ pointwisely. But $\lambda^{-1}U^I[A, \lambda\nu](r, \cdot)$ are all dominated by the same bounded function $S^I\nu(r, \cdot)$. The statement (3) follows.

Remark 3.1 Theorem 2.1 is a generalization of Theorem 2.5.2 in Dawson and Fleischmann[2] from a special branching mechanism $\psi(z) = z^2$ to a general ψ. The conditions imposed in this paper are simpler than that of Dawson and Fleischmann [2] because of an improvement over the proof of Dawson and Fleischmann. If $\psi(z) = z^2$ our regularity condition (2.5) is weaker than the regularity conditions in [2]. But we imposed another condition (2.7), which is easy to check.

Proof of Theorem 2.2 The proof of Theorem 2.2 is similar to that of Theorem 2.6.2 in Dawson and Fleischmann [2]. We omit the details here.

§4. Some particular cases

In this section we consider some particular cases such that the conditions of Theorem 2.2 hold. Recall that $I = [L, T)$. Throughout this section the branching rate functional A has the formal structure:

$$A = A_\xi(ds) := ds \int \xi(s, dy)\delta_y(W_s). \tag{4.1}$$

ξ is called the branching rate kernel. We use C to denote a positive constant which may change values from line to line.

Theorem 4.1 Suppose $\xi(s, dy) = \xi(s, y)dy$ with $\xi(\cdot, \cdot) \in bp\mathcal{B}(I \times \mathbf{R^d})$. And suppose, for fixed I, there exist branching mechanism $\psi_0(z) = b_0 z^2 + \int_0^\infty (e^{-uz} - 1 + uz)n_0(du)$ such that $\psi(s, x, z) \le \psi_0(z), s \in I, x \in \mathbf{R}^d$, where b_0 is a non-negative constant, $n_0(du)$ is a measure on $\mathbf{R}^d$ such that $\int_0^\infty u \wedge u^2 n_0(du) < \infty$. If

$$\int_t^T ds(T-s)^{d/2}\psi_0((T-s)^{-d/2}) \to 0(t \uparrow T), \tag{4.2}$$

then the results of Theorem 2.2 hold. In particular X_T is absolutely continuous.

Proof Suppose $\xi(s, y) \le C$. It is obvious that $A_\xi \in \mathcal{A}_0^I$. For $\nu = \sum_{j=1}^l \delta_{z(j)}$ with $z(1), \ldots, z(l) \in \mathbf{R^d}$, by property (1) of Lemma , we have

$$\begin{aligned} &\Pi_{r,x} \int_t^T \psi(s, W_s, S^I \nu_n(s, W_s))A_\xi(ds) \\ \le\ & C \int_t^T ds \int dy p(s-r, y-x) \sum_{j=1}^l \psi_0(p(\epsilon(n) + T - s, y - z(j))) \end{aligned} \tag{4.3}$$

By property (2) of Lemma , we have

$$\frac{\psi_0(p(\epsilon(n) + T - s, y - z(j)))}{p(\epsilon(n) + T - s, y - z(j)))} \le [2\pi(T-s)]^{d/2}\psi_0([2\pi(T-s)]^{-d/2}). \tag{4.4}$$

Note that

$$\begin{aligned}&\int p(s-r,y-x)p(\epsilon(n)+T-s,x-z(j)))dy \\ =\quad & p(\epsilon(n)+T-r,y-z(j))\leq (T-r)^{-d/2}\end{aligned} \tag{4.5}$$

Thus by (4.3), (4.4) and (4.5),

$$\begin{aligned}&\Pi_{r,x}\int_t^T \psi(s,W_s,S^I\nu_n(s,W_s))A_\xi(ds)\leq \\ &C(T-r)^{-d/2}\int_t^T (T-s)^{d/2}\psi_0([2\pi(T-s)]^{-d/2})ds\to 0(t\uparrow T).\end{aligned} \tag{4.6}$$

Therefore the results of Theorem 2.2 hold.

From Theorem 4.1 we easily have the following well-known results:

Corollary 4.2 Suppose ξ is bounded regular, i.e., $\xi(s,dy)=\xi(s,y)dy$ with $\xi(s,y)\in bp\mathcal{B}(\mathcal{I}\times\mathbf{R^d})$. Then under one of the following conditions, the corresponding super-Brownian motion X with parameters (ψ,A_ξ) has absolutely continuous states.

(1) $d<\frac{2}{\alpha-1}$ and ψ is given by

$$\psi(s,x,z)=\gamma(s,x)z^\alpha, 1<\alpha\leq 2, \gamma\in p\mathcal{B}, \tag{4.7}$$

where γ is bounded on $\Delta\times\mathbf{R^d}$ for each finite interval Δ;

(2) $d=1$ and ψ is given by (1.1) with $a\equiv 0$.

Theorem 4.3 Suppose ψ is given by (4.7) and $A_\xi\in\mathcal{A}_0^I$. If there exists a Borel subset N of Lebesgue measure 0 such that, for every $z\in\mathbf{R}^d\setminus N$, one of the following conditions (1) and (2) holds, then A_ξ is a.e.-regular, and the results of Theorem 2.2 hold. Particularly, X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^d$, $P_{r,\mu}$-a.e..

(1) $d<\frac{2}{\alpha}$, and there exists $\sigma_z>0$ such that

$$\sup_{s\in I}\xi(s,B(z,\sigma_z))<\infty. \tag{4.8}$$

(2) there exists $0 < \theta_z < \frac{\alpha d}{2} \wedge 1$ and $\sigma_z > 0$ such that

$$\sup_{s \in I} \int_{B(z,\sigma_z)} (y-z)^{(-\alpha d + 2\theta_z)} \xi(s,dy) < \infty. \tag{4.9}$$

Proof For $\nu = \sum_{j=1}^{l} \delta_{z(j)}, z(1), \cdots, z(l) \in \mathbf{R}^d \setminus N$, let

$$R_\xi(r,x,t,T) = \Pi_{r,x} \int_t^T (S^I \nu_n)^\alpha (s, W_s) A_\xi(ds), \tag{4.10}$$

Then

$$\begin{aligned} R_\xi &= \int_t^T ds \int \xi(s,dy) \int d\overline{y} p(s-r, \overline{y}-x) \delta_y(\overline{y}) \left[\sum_{j=1}^{l} p(\epsilon(n) + T - s, \overline{y} - z(j)) \right]^\alpha \\ &= \int_t^T ds \int \xi(s,dy) p(s-r,y-x) \left[\sum_{j=1}^{l} p(\epsilon(n) + T - s, y - z(j)) \right]^\alpha \\ &\le 2^\alpha \int_t^T ds \int \xi(s,dy) p(s-r,y-x) \sum_{j=1}^{l} p(\epsilon(n)+T-s, y-z(j))^\alpha \\ &\le 2^\alpha \sum_{j=1}^{l} [I_1(z(j)) + I_2(z(j))], \end{aligned} \tag{4.11}$$

where

$$I_1(z(j)) = \int_t^T ds \int_{B(z(j),\sigma_{z(j)})} \xi(s,dy) p(s-r,y-x) p(\epsilon(n)+T-s, y-z(j))^\alpha, \tag{4.12}$$

$$I_2(z(j)) = \int_t^T ds \int_{B(z(j),\sigma_{z(j)})^c} \xi(s,dy) p(s-r,y-x) p(\epsilon(n)+T-s, y-z(j))^\alpha. \tag{4.13}$$

Note that,

$$\begin{aligned} I_2(z(j)) &\le C_{\alpha,d} \int_t^T ds \int_{B(z(j),\sigma_{z(j)})^c} \xi(s,dy) p(s-r,y-x) \\ &\le C_{\alpha,d} A_\xi(t,T) \to 0 (t \uparrow T); \end{aligned} \tag{4.14}$$

$$\begin{aligned} I_1(z(j)) &\le \int_t^T ds \int_{B(z(j),\sigma_{z(j)})} \xi(s,dy) (2\pi(s-r))^{-d/2} p(\epsilon(n)+T-s, y-z(j))^\alpha \\ &\le C_d (t-r)^{-d/2} \int_t^T (T-s)^{-\theta} ds \cdot \\ &\quad \int_{B(z(j),\sigma_{z(j)})} \sup_{0<t<\infty} t^{-\frac{\alpha d}{2}+\theta} \exp\left(- \frac{\alpha (y-z(j))^2}{2t} \right) \xi(s,dy). \end{aligned} \tag{4.15}$$

where $C_{\alpha,d}$ denotes a constant depends only on α and d, and C_d denotes a constant depends only on d.

If for $z = z(j)$ condition (1) holds, then

$$
\begin{aligned}
I_1(z(j)) \leq\ & C_{\alpha,d}(t-r)^{-d/2} \sup_{s\in I} \xi(s, B(z(j), \sigma_{z(j)})) \int_t^T (T-s)^{-\alpha d/2} ds \\
\to\ & 0(t \uparrow T).
\end{aligned}
\tag{4.16}
$$

If for $z = z(j)$ condition (2) holds, choose $0 < \theta_z < \frac{\alpha d}{2} \wedge 1$. For fixed $\beta > 0, z \in \mathbf{R}^d$ and $\alpha > 0$, the function $g(t) := t^{-\beta} \exp\left(-\frac{\alpha z^2}{2t}\right)$ gets its maximum value $\left(\frac{\alpha z^2}{2\beta}\right)^{-\beta} \exp(-\beta)$ at $t_0 = \frac{\alpha}{2\beta} z^2$. Then we have

$$
\begin{aligned}
I_1(z(j)) \leq\ & C_{\alpha,d}(t-r)^{-d/2} \int_t^T (T-s)^{-\theta} ds \cdot \\
& \sup_{s\in I} \int_{B(z(j),\sigma_{z(j)})} (y - z(j))^{-\alpha d + 2\theta_z} \xi(s, dy) \to 0(t \uparrow T).
\end{aligned}
\tag{4.17}
$$

Thus, by (4.11), (4.14), (4.16) and (4.17),

$$
R_\xi(r, x, t, T) \to 0(t \uparrow T). \tag{4.18}
$$

Therefore, A is a.e.-regular. The results of Theorem 2.2 hold.

Now we give an example of application of Theorem 4.3. We consider *factored branching rate kernels* ξ, i.e.,

$$
\xi(s, dy) = \xi_{d-1}(s, y_{d-1}) dy_{d-1} \xi_1(s, dy_1), \quad s \in I, y = [y_{d-1}, y_1] \in \mathbf{R^{d-1}} \times \mathbf{R}, \tag{4.19}
$$

where ξ_1 is a one-dimensional kernel, whereas ξ_{d-1} is a bounded measurable function on $I \times \mathbf{R^{d-1}}$. Factorize the d-dimensional Brownian motion and transition density function as follows: for $t > 0$,$y = [y_{d-1}, y_1] \in \mathbf{R^{d-1}} \times \mathbf{R}$,

$$
W = [W^{d-1}, W^1]; \quad p_d(t, y) = p_{d-1}(t, y_{d-1}) p_1(t, y_1).
$$

(In the extreme case $d = 1$, we read W_s^1, ξ_1 and dy_{d-1} as W_s, ξ and δ_0, respectively.)

Suppose ξ_1 is given by

$$
\xi_1(s, dy_1) = \Gamma(0) = \sum_{i=1}^{\infty} \alpha_i \delta_{x(i)}, s \in \mathbf{R}, \tag{4.20}
$$

where $\Gamma(0)$ a stable random measure on $\mathbf{R}$ with index $\gamma \in (0,1)$, characterized by its Laplace functional

$$E\exp(\Gamma(0), -f) = \exp\left(\int -f(x)^\gamma dx\right), \quad f \geq 0. \tag{4.21}$$

Corollary 4.4 Suppose ψ is given by (4.7) and ξ is given by (4.19) and (4.20). If $1/\gamma > (\alpha-1)d-1$, then X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^d$, $P_{r,\mu}$-a.e..

Proof It is easy to check that $A_\xi \in \mathcal{A}_0^I$. Let

$$g(z) = \sup_{s\in I}\int_{B(z,\delta_z)}(y-z)^{(-\alpha d+2\theta_z)}\xi(s,dy). \tag{4.22}$$

Then

$$\begin{aligned}
E(g(z)) \leq\ & E\sup_{s\in I}\int_{z_1-1}^{z_1+1}\xi_1(s,dy_1)\int_{B(z_{d-1},1)}(y-z)^{-\alpha d+2\theta_z}dy_{d-1}\\
=\ & E\sup_{s\in I}\int_{-1}^{1}\xi_1(s,dy_1)\int_0^1(y_1^2+r^2)^{-\frac{\alpha d}{2}+\theta_z}r^{d-2}dr\\
=\ & E\sup_{s\in I}\int_{-1}^{1}\xi_1(s,dy_1)y_1^{(-\alpha d+2\theta_z+d-1)}\int_0^{1/y_1}(1+r^2)^{-\frac{\alpha d}{2}+\theta_z}r^{d-2}dr.
\end{aligned} \tag{4.23}$$

Choose $\theta_z < (\alpha-1)d/2+1/2$, then $\int_0^\infty(1+r^2)^{-\frac{\alpha d}{2}+\theta_z}r^{d-2}dr < \infty$. By (4.23),

$$E(g(z)) \leq \int_{-1}^{1} y_1^{(-\alpha d+2\theta_z+d-1)\gamma}dy_1. \tag{4.24}$$

If we choose $\theta_z > (\alpha-1)d/2+1/2-1/(2\gamma)$, then $(-\alpha d+2\theta_z+d-1)\gamma > -1$, and therefore, $E(g(z)) < \infty$. By the above discussions, if we choose θ_z satisfying

$$\begin{cases} (\alpha-1)d/2+1/2 > \theta_z > (\alpha-1)d/2+1/2-1/(2\gamma), \\ 0 < \theta_z < 1. \end{cases}$$

then $E(g(z)) < \infty$. This θ_z exists iff $1/\gamma > (\alpha-1)d-1$. Consequently, A_ξ is a.e.-regular and we are done.

Remark 4.1 Dawson & Fleischmann showed that if $\alpha = 2$, $d > 1$, then for $\gamma \in (0, 1/(2d-1))$, X_T is absolutely continuous, $P_{r,\mu}$-a.e. (see Example 4.4.4 in [2]). By our result, If $\alpha = 2$, $d > 1$, then for $\gamma \in (0, 1/(d-1))$, X_T is absolutely continuous, $P_{r,\mu}$-a.e.. Therefore our result is an improvement upon Dawson & Fleischmann's.

Acknowledgment The author would like to thank Professor Dawson for his careful reading of the original version of the paper and for his valuable suggestions which resulted in an improvement of the manuscript.

References

[1] D.A. Dawson, K. Fleischmann, A super-Brownian motion with a single point catalyst, Stochastic. Process. Appl. 49(1994) 3-40.

[2] D.A. Dawson, K. Fleischmann, Super-Brownian motion in higher dimensions with absolutely continuous measure states. J. Theoretical Probab. 8(1995) 179-206.

[3] D.A. Dawson, K. Fleischmann and S. Roelly, Absolute continuity for the measure states in a branching model with catalysts, Stochastic Processes. Proc. Semin. Vancouver CA (USA) 1990, Prog. Probab. 24(1991) 117-160.

[4] E.B.Dynkin, Superprocesses and partial differential equations. Ann. Probab. 21(1993) 1185-1262.