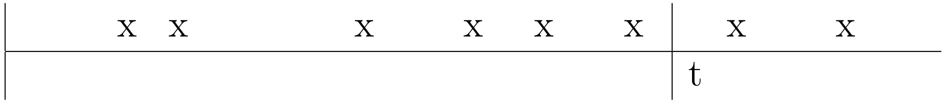
Stochastic equations for counting processes

Poisson processes

A Poisson process is a model for a series of random observations occurring in time.



Let $Y(t)$ denote the number of observations by time t. In the figure above, $Y(t) = 6$. Note that for $t < s$, $Y(s) - Y(t)$ is the number of observations in the time interval $(t, s]$. We make the following assumptions about the model.

1) Observations occur one at a time.
2) Numbers of observations in disjoint time intervals are independent random variables, i.e., if $t_0 < t_1 < \cdots < t_m$, then $Y(t_k) - Y(t_{k-1})$, $k = 1, \ldots, m$ are independent random variables.
3) The distribution of $Y(t + a) - Y(t)$ does not depend on t.

Characterization of a Poisson process

Theorem 1 *Under assumptions 1), 2), and 3), there is a constant* $\lambda > 0$ *such that, for* $t < s$, $Y(s) - Y(t)$ *is Poisson distributed with parameter* $\lambda(s-t)$, *that is,*

$$P\{Y(s) - Y(t) = k\} = \frac{(\lambda(s-t))^k}{k!} e^{-\lambda(s-t)}.$$

If $\lambda = 1$, then Y is a *unit* (or rate one) Poisson process. If Y is a unit Poisson process and $Y_\lambda(t) \equiv Y(\lambda t)$, then Y_λ is a Poisson process with parameter λ.

Formulating Markov models

Suppose $Y_\lambda(t) = Y(\lambda t)$ and $\mathcal{F}_t$ represents the information obtained by observing $Y_\lambda(s)$, for $s \leq t$.

$$P\{Y_\lambda(t+\Delta t) - Y_\lambda(t) = 1 | \mathcal{F}_t\} = P\{Y_\lambda(t+\Delta t) - Y_\lambda(t) = 1\} = 1 - e^{-\lambda \Delta t} \approx \lambda \Delta t$$

A continuous time Markov chain X taking values in $\mathbb{Z}^d$ is specified by giving its transition intensities that determine

$$P\{X(t+\Delta t) - X(t) = l | \mathcal{F}_t^X\} \approx \beta_l(X(t))\Delta t, \quad l \in \mathbb{Z}^d.$$

Counting process representation

If we write

$$X(t) = X(0) + \sum_l l N_l(t)$$

where $N_l(t)$ is the number of jumps of l at or before time t, then

$$P\{N_l(t+\Delta t) - N_l(t) = 1 | \mathcal{F}_t^X\} \approx \beta_l(X(t))\Delta t, \quad l \in \mathbb{Z}^d.$$

N_l is a *counting process* with intensity (*propensity* in the chemical literature) $\beta_l(X(t))$

Conditional intensities for counting processes

N is a *counting process* if $N(0) = 0$ and N is constant except for jumps of $+1$.

Assume N is adapted to $\{\mathcal{F}_t\}$.

$\lambda \geq 0$ is the $\{\mathcal{F}_t\}$-conditional intensity if (intuitively)

$$P\{N(t+\Delta t) > N(t)|\mathcal{F}_t\} \approx \lambda(t)\Delta t$$

or (precisely)

$$M(t) \equiv N(t) - \int_0^t \lambda(s)ds$$

is an $\{\mathcal{F}_t\}$-local martingale, that is, if τ_k is the kth jump time of N,

$$E[M((t+s)\wedge\tau_k)|\mathcal{F}_t] = M(t\wedge\tau_k)$$

for all $s, t \geq 0$ and all k.

Lemma 2 *If N has $\{\mathcal{F}_t\}$-intensity λ, then there exists a unit Poisson process (may need to enlarge the sample space) such that*

$$N(t) = Y(\int_0^t \lambda(s)ds)$$

Modeling with counting processes

Specify $\lambda(t) = \gamma(t, N)$, where γ is nonanticipating in the sense that $\gamma(t, N) = \gamma(t, N(\cdot \wedge t))$.

Martingale problem. Require

$$N(t) - \int_0^t \gamma(s, N) ds$$

to be a local martingale.

Time change equation. Require

$$N(t) = Y(\int_0^t \gamma(s, N) ds).$$

These formulations are equivalent in the sense that the solutions have the same distribution.

Systems of counting processes

Lemma 3 *(Meyer [9], Kurtz [8]) Assume* $N = (N_1, \ldots, N_m)$ *is a vector of counting processes with no common jumps and* λ_k *is the* $\{\mathcal{F}_t\}$*-intensity for* N_k*. Then there exist independent unit Poisson processes* $Y_1, \ldots, Y_m$ *(may need to enlarge the sample space) such that*

$$N_k(t) = Y_k(\int_0^t \lambda_k(s)ds)$$

Specifying nonanticipating intensities $\lambda_k(t) = \gamma_k(t, N)$:

$$N_k(t) = Y_k(\int_0^t \gamma_k(s, N)ds)$$

Representing continuous time Markov chains

If

$$P\{X(t+\Delta t) - X(t) = l | \mathcal{F}_t^X\} \approx \beta_l(X(t))\Delta t, \quad l \in \mathbb{Z}^d.$$

then we can write

$$N_l(t) = Y_l(\int_0^t \beta_l(X(s))ds),$$

where the Y_l are independent, unit Poisson processes. Consequently,

$$\begin{aligned} X(t) &= X(0) + \sum_l l N_l(t) \\ &= X(0) + \sum_l l Y_l(\int_0^t \beta_l(X(s))ds). \end{aligned}$$

Random jump equation

Alternatively, setting $\bar{\beta}(k) = \sum_l \beta_l(k)$,

$$N(t) = Y(\int_0^t \bar{\beta}(X(s))ds)$$

and

$$X(t) = X(0) + \int_0^t F(X(s-), \xi_{N(s-)})dN(s)$$

where Y is a unit Poisson process, $\{\xi_i\}$ are iid uniform $[0,1]$, and

$$P\{F(k,\xi) = l\} = \frac{\beta_l(k)}{\bar{\beta}(k)}.$$

Connections to simulation schemes

Simulating the random-jump equation gives Gillespie's [4, 5] *direct method* (the *stochastic simulation algorithm* SSA).

Simulating the time-change equation gives the *next reaction* (next jump) method as defined by Gibson and Bruck [3].

For $0 = \tau_0 < \tau_1 < \cdots$,

$$\hat{X}(\tau_n) = X(0) + \sum_l l Y_l \left(\sum_{k=0}^{n-1} \beta_l(\hat{X}(\tau_k))(\tau_{k+1} - \tau_k) \right)$$

gives Gillespie's [6] τ-leap method

Application of the LLN and CLT to Poisson processes

Theorem 4 *If Y is a unit Poisson process, then for each $u_0 > 0$,*

$$\lim_{K\to\infty} \sup_{u\le u_0} \left|\frac{Y(Ku)}{K} - u\right| = 0 \quad a.s.$$

The central limit theorem suggests that for large K

$$\frac{Y(Ku) - Ku}{\sqrt{K}} \approx W(u), \qquad \frac{Y(Ku)}{K} \approx u + \frac{1}{\sqrt{K}} W(u)$$

where W is *standard Brownian motion.* More precisely, W can be constructed so that

$$\left|\frac{Y(Ku)}{K} - \left(u + \frac{1}{K} W(Ku)\right)\right| \le \Gamma \frac{\log(Ku+2)}{K}$$

for a random variable Γ independent of u and K.

Example: SIR epidemic model

$S \to I$ at rate $\frac{\lambda}{N} si = N\lambda \frac{s}{N}\frac{i}{N}$

$I \to R$ at rate $\mu i = N\frac{i}{N}$

where s is the number of susceptible individuals, i the number of infectives, and N some measure of the size of the region in which the population lives. Then

$$\begin{aligned} S(t) &= S(0) - Y_1(N\lambda \int_0^t \frac{S(s)}{N}\frac{I(s)}{N} ds) \\ I(t) &= I(0) + Y_1(N\lambda \int_0^t \frac{S(s)}{N}\frac{I(s)}{N} ds) - Y_2(N\mu \int_0^t \frac{I(s)}{N} ds) \end{aligned}$$

The normalized population sizes become

$$\begin{aligned} C_S^N(t) &= C_S^N(0) - N^{-1} Y_1(N\lambda \int_0^t C_S^N(s) C_I^N(s) ds) \\ C_I^N(t) &= C_I^N(0) + N^{-1} Y_1(N\lambda \int_0^t C_S^N(s) C_I^N(s) ds) - N^{-1} Y_2(N\mu \int_0^t C_I^N(s) ds) \end{aligned}$$

First scaling limit

Assume $\beta_l^N(X^N(t)) \approx N\lambda_l(N^{-1}X^N(t))$

Setting $C^N(t) = N^{-1}X^N(t)$

$$\begin{aligned} C^N(t) &= C^N(0) + \sum_l lN^{-1}Y_k(\int_0^t \beta_l^N(X^N(s))ds) \\ &\approx C^N(0) + \sum_l lN^{-1}Y_k(N\int_0^t \lambda_l(C^N(s))ds) \end{aligned}$$

The law of large numbers for the Poisson process implies $N^{-1}Y(Nu) \approx u$,

$$C^N(t) \approx C^N(0) + \sum_l \int_0^t l\lambda_l(C^N(s))ds,$$

which gives $C^N \to C$ satisfying

$$\dot{C}(t) = \sum_l l\lambda_l(C(s))ds \equiv F(C(t)).$$

Central limit theorem/Van Kampen Approximation

$$
\begin{aligned}
V^N(t) &\equiv \sqrt{N}(C^N(t) - C(t)) \\
&\approx V^N(0) + \sqrt{N}(\sum_k l N^{-1} Y_l(N \int_0^t \lambda_l(C^N(s))ds) - \int_0^t F(C(s))ds) \\
&= V^N(0) + \sum_l l \frac{1}{\sqrt{N}} \tilde{Y}_l(N \int_0^t \tilde{\lambda}_l(C^N(s))ds) \\
&\qquad + \int_0^t \sqrt{N}(F^N(C^N(s)) - F(C(s)))ds \\
&\approx V^N(0) + \sum_l W_k(\int_0^t \lambda_l(C(s))ds) + \int_0^t \nabla F(C(s))V^N(s)ds
\end{aligned}
$$

Gaussian limit

V^N converges to the solution of

$$V(t) = V(0) + \sum_k lW_l(\int_0^t \lambda_l(C(s))ds) + \int_0^t \nabla F(C(s)))V(s)ds$$

$$C^N(t) \approx C(t) + \frac{1}{\sqrt{N}} V(t)$$

Diffusion approximation

Since $\frac{Y(Ku)-Ku}{\sqrt{K}} \approx W(u)$

$$\begin{aligned} C^N(t) &= C^N(0) + \sum_k lN^{-1}Y_l(N\int_0^t \lambda_l(X^N(s))ds) \\ &\approx C^N(0) + \sum_k lN^{-1/2}W_l(\int_0^t \lambda_l(C^N(s))ds) \\ &\qquad + \int_0^t F(C^N(s))ds, \end{aligned}$$

where

$$F(c) = \sum_k l\lambda_l(c).$$

The diffusion approximation is given by the equation

$$\tilde{C}^N(t) = \tilde{C}^N(0) + \sum_k lN^{-1/2}W_l(\int_0^t \lambda_l(\tilde{C}^N(s))ds) + \int_0^t F(\tilde{C}^N(s))ds.$$

Itô formulation

The time-change formulation is equivalent to the Itô equation

$$\begin{aligned}\tilde{C}^N(t) &= \tilde{C}^N(0) + \sum_k lN^{-1/2}\int_0^t \sqrt{\lambda_l(\tilde{C}^N(s))}d\tilde{W}_l(s) \\ &\quad + \int_0^t F(\tilde{C}^N(s))ds \\ &= \tilde{C}^N(0) + N^{-1/2}\int_0^t \sigma(\tilde{C}^N(s))d\tilde{W}(s) + \int_0^t F(\tilde{C}^N(s))ds,\end{aligned}$$

where $\sigma(c)$ is the matrix with columns $\sqrt{\tilde{\lambda}_l(c)}l$.

See Kurtz [7], Ethier and Kurtz [1], Chapter 10, Gardiner [2], Chapter 7, and Van Kampen, [10].

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Abstract

Stochastic equations for counting processes

A counting process is usually specified by its conditional intensity. The conditional intensity then determines the martingale properties of the process and uniquely determines its distribution. The connection between the specified intensity and the process can also be established by formulating an appropriate stochastic equation. Two natural forms for the stochastic equation will be described. The relationship between the stochastic equations and simulation methods will be given, and ways of exploiting the stochastic equations to obtain limit theorems will be described.