On equilibrium states of CXC system

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- **Coarse expanding conforamal system:** Developed by P. Haïssinsky and K. Pilgrim.
- **Results:** The existence of equilibrium states of CXC system.
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- **Further study:** The statistical properties of CXC system.

Degree and local degree map:
Suppose X, Y are locally compact Hausdorff spaces, and let $f : X \to Y$ be a finite-to-one continuous map.

- The **degree** of f is

$$\deg(f) = \sup\left\{\text{card}\left(f^{-1}(y)\right) : y \in Y\right\}. \tag{1}$$

- For $x \in X$, the **local degree** of f at x is

$$\deg(f;x) = \inf_U \sup\left\{\text{card}\left(f^{-1}(z) \cap U\right) : z \in f(U)\right\}, \tag{2}$$

where U ranges over all neighborhoods of x.

Finite branched covering (FBC) map:
Suppose that X, Y are locally compact Hausdorff spaces, and let $f : X \to Y$ be a finite-to-one continuous map. The map f is a **finite branched covering** (abbr: FBC) map if $\deg(f) < \infty$ and

(i)
$$\sum_{x \in f^{-1}(y)} \deg(f;x) = \deg(f)$$
holds for all $y \in Y$;

(ii) for every $x_0 \in X$, there are compact neighborhoods U and V of x_0 and $f(x_0)$, respectively, such that
$$\sum_{x \in U, f(x) = y} \deg(f;x) = \deg(f;x_0)$$
holds for all $y \in V$.

An introduction to CXC system: Definition

Let $\mathfrak{X}_0, \mathfrak{X}_1$ be topological spaces satisfying the following assumptions:

(i) $\mathfrak{X}_0, \mathfrak{X}_1$ are Hausdorff, locally compact, locally connected spaces, each with finite many connected components.

(ii) $\mathfrak{X}_1$ is a open subset of $\mathfrak{X}_0$ and $\overline{\mathfrak{X}_1}$ is compact in $\mathfrak{X}_0$.

Let $f : \mathfrak{X}_1 \to \mathfrak{X}_0$ be an FBC map of degree $\deg(f) = d \geq 2$. For each $n \geq 0$ we define

$$\mathfrak{X}_{n+1} := f^{-1}(\mathfrak{X}_n)$$

and the *repellor* is defined as

$$X := \{x \in \mathfrak{X}_1 \mid f^n(x) \in \mathfrak{X}_1, \forall n \geq 0\}$$

and we make a technical assumption that $f|_X : X \to X$ is an FBC map of degree d.

the following properties hold for $\mathfrak{X}_n$ and X.

(1) $f|_{\mathfrak{X}_{n+1}} : \mathfrak{X}_{n+1} \to \mathfrak{X}_n$ is again an FBC map of degree d.

(2) $\overline{\mathfrak{X}_{n+1}} \subset \mathfrak{X}_n$, and $\overline{\mathfrak{X}_{n+1}}$ is compact in $\mathfrak{X}_n$ since f is proper.

(3) X is totally invariant: $f^{-1}(X) = X = f(X)$.

(4) $X = \bigcap_{n \in \mathbb{N}} \overline{\mathfrak{X}_n} = \bigcap_{n \in \mathbb{N}} \overline{\mathfrak{X}_{kn}}$ holds for all $k > 0$, thus f and $f^k|_{\mathfrak{X}_k} : \mathfrak{X}_k \to \mathfrak{X}_0$ have the same repellor X.

(5) The definition of repellor X and the compactness of $\mathfrak{X}_n$ implies that given any open set $Y \supset X$, $\mathfrak{X}_n \subset Y$ for all n sufficiently large.

An introduction to CXC system: Definition

Let $\mathcal{U}_0$ be a **finite** cover of X by **open, connected** subsets of $\mathfrak{X}_1$ whose intersection with X is nonempty. A *preimage* under f of a connected set A is defined as a **connected component** of $f^{-1}(A)$. Inductively, we define for each $n \geq 0$

$$\mathcal{U}_{n+1} := \{\widetilde{U} : \widetilde{U} \textit{ is a preimage of } U \textit{ for some } U \in \mathcal{U}_n\}.$$

The elements of $\mathcal{U}_n$ are connected components of $f^{-n}(U)$, where U ranges over $\mathcal{U}_0$. We note that, for each $\widetilde{U} \in \mathcal{U}_{n+1}$ and $U \in \mathcal{U}_n$, if $\widetilde{U}$ is a preimage of U, then $f|_U : \widetilde{U} \to U$ is surjective, and that $f^k(U) \in \mathcal{U}_{n-k}$ for all $k \leq n$. We can see that $\mathcal{U}_n$ is a finite open cover of X by connected open sets in $\mathfrak{X}_{n+1}$.

An introduction to CXC system: Definition

We say $f : (\mathfrak{X}_1, X) \to (\mathfrak{X}_0, X)$ is **coarse expanding conformal** with repellor X if there exist a finite cover $\mathcal{U}_0$ as above such that the following **axioms** hold:

1. **Expansion Axiom** (abbr: [**Expans**]) : For any finite open cover $\mathcal{V}$ of X by open sets of $\mathfrak{X}_0$, there exist N such that for all $n \geq N$ and $U \in \mathcal{U}_n$, there exist $V \in \mathcal{V}$ with $U \subset V$.
2. **Irreducibility Axiom** (abbr: [**Irred**]) : For any $x \in X$ and neighborhood W of x in $\mathfrak{X}_0$, there exist some n with $f^n(W) \supset X$.
3. **Degree Axiom** (abbr: [**Deg**]) : The set of degrees of maps of the form $f^k|_{\widetilde{U}} : \widetilde{U} \to U$ where $U \in \mathcal{U}_n$, $\widetilde{U} \in \mathcal{U}_{n+k}$ and n, k are arbitrary, has a finite maximum, denoted by p.

Example

Let $\widehat{\mathbb{C}}$ denote the Riemann sphere, and let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree $d \geq 2$ for which the critical points either converge under iteration to attracting cycles, or land on a repelling periodic cycle (such a function is called *subhyperbolic*). For such maps, every point on the sphere belongs either to the *Fatou set* and converges to an attracting cycle, or belongs to the *Julia set* $J(f)$. One may find a small closed neighborhood V_0 of the attracting periodic cycles such that $f(V_0) \subseteq \operatorname{int}(V_0)$. Set $\mathfrak{X}_0 = \widehat{\mathbb{C}} - V_0$ and $\mathfrak{X}_1 = f^{-1}(\mathfrak{X}_0)$. Then $f : \mathfrak{X}_1 \to \mathfrak{X}_0$ is an FBC of degree d, the repellor $X = J(f)$, and $f|_X : X \to X$ is an FBC of degree d.
Let $\mathcal{U}_0$ be a finite cover of $J(f)$ by open spherical balls contained in $\mathfrak{X}_1$, chosen so small that each ball contains at most one forward iterated image of a critical point. Then it can be checked that the axioms **[Expans], [Irred], [Deg]** hold.

Suppose we have a topological CXC system $f : \mathfrak{X}_1 \to \mathfrak{X}_0$ with repellor X and level-0 good open cover $\mathcal{U}_0$, and $\mathfrak{X}_0$ is endowed with a metric compatible with its topology. The metric dynamic system is called **metric CXC system** if it satisfies the following two axioms:

4. **Roundness distortion Axiom** (abbr: [**Round**]) : There exist continuous increasing embeddings $\rho_+ : [1,\infty) \to [1,\infty)$ and $\rho_- : [1,\infty) \to [1,\infty)$ such that for all $n, k \geq 0$ and $U \in \mathcal{U}_n$, $\widetilde{U} \in \mathcal{U}_{n+k}$, $y \in U$, $\widetilde{y} \in \widetilde{U}$, if

$$f^k(\widetilde{U}) = U,\ f^k(\widetilde{y}) = y,$$

then

$$\mathrm{Round}(\widetilde{U},\widetilde{y}) < \rho_-(\mathrm{Round}(U,y)),$$

$$\mathrm{Round}(U,y) < \rho_+(\mathrm{Round}(\widetilde{U},\widetilde{y})).$$

5. **Diameter distortion Axiom** (abbr: [**Diam**]) : There exist increasing homeomorhisms $\delta_+ : [0,1] \to [0,1]$ and $\delta_- : [0,1] \to [0,1]$ such that for all $n_0, n_1, k \geq 0$ and $U \in \mathcal{U}_{n_0}$, $U' \in \mathcal{U}_{n_1}$, $\widetilde{U} \in \mathcal{U}_{n_0+k}$, $\widetilde{U}' \in \mathcal{U}_{n_1+k}$, $\widetilde{U}' \subset \widetilde{U}$, $U' \subset U$, if

$$f^k(\widetilde{U}) = U, \ f^k(\widetilde{U}') = U',$$

then

$$\frac{\operatorname{diam} \widetilde{U}'}{\operatorname{diam} \widetilde{U}} < \delta_- \left(\frac{\operatorname{diam} U'}{\operatorname{diam} U} \right),$$

$$\frac{\operatorname{diam} U'}{\operatorname{diam} U} < \delta_+ \left(\frac{\operatorname{diam} \widetilde{U}'}{\operatorname{diam} \widetilde{U}} \right).$$

Equilibrium state

- Z = compact metric space
- $g: Z \to Z$ continuous map
- $\phi: Z \to \mathbb{R}$ continuous function ("potential")
- $\mathcal{M}_g = \{g\text{-invariant Borel probability measures on } Z\}$

Topological entropy: $h_{top}(g)$
Topological pressure: $P(g,\phi)$
Measure-theoretic entropy: $h_\nu(g)$
Measure-theoretic pressure: $P_\nu(g,\phi) = h_\nu(g) + \int \phi \, \mathrm{d}\nu$
Variational principle:

$$P(g,\phi) = \sup\{P_\nu(g,\phi) : \nu \in \mathcal{M}_g\}.$$

Equilibrium state: μ_ϕ: $P_{\mu_\phi}(g,\phi) = \sup\{P_\nu(g,\phi) : \nu \in \mathcal{M}_g\}$.

Equilibrium state of CXC system

We prove the existence of equilibrium states for metric CXC system by showing that metric CXC system is **asymptotically** *h*-**expansive**.

Asymptotic h-expansiveness

Topological conditional entropy: Let Z be a compact metric space and $g : Z \to Z$ be a continuous map. For each pair of open covers ξ and η of Z, we denote

$$H(\xi|\eta) = \log\left(\max_{A\in\eta}\left\{\min\left\{\operatorname{card}\xi_A : \xi_A \subseteq \xi,\ A \subseteq \bigcup \xi_A\right\}\right\}\right)$$

For a given open cover λ, the *topological conditional entropy* $h(g|\lambda)$ of g is defined as

$$h(g|\lambda) = \lim_{l\to+\infty}\lim_{n\to+\infty}\frac{1}{n}H\left(\bigvee_{i=0}^{n-1} g^{-i}(\xi_l)\,\middle|\,\bigvee_{j=0}^{n-1} g^{-j}(\lambda)\right),$$

where $\{\xi_l\}_{l\in\mathbb{N}_0}$ is an arbitrary refining sequence of open covers.

Topological tail entropy:
The *topological tail entropy* $h^*(g)$ of g is defined by

$$h^*(g) = \lim_{m\to+\infty} \lim_{l\to+\infty} \lim_{n\to+\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} g^{-i}(\xi_l) \,\middle|\, \bigvee_{j=0}^{n-1} g^{-j}(\eta_m) \right),$$

where $\{\xi_l\}_{l\in\mathbb{N}_0}$ and $\{\eta_m\}_{m\in\mathbb{N}_0}$ are two arbitrary refining sequences of open covers.

Asymptotically h-expansive:
We say a continuous map $g : Z \to Z$ on compact metric space Z is *asymptotically h-expansive* if $h^*(g) = 0$.

Asymptotic h-expansiveness

A consequence of asymptotic h-expansiveness is the existence of equilibrium state:

Proposition

Let $g : Z \to Z$ be a continuous map on a compact metric space Z. If g is asymptotically h-expansive, then the map

$$h_{\cdot}(g) : \mathcal{M}_g \to \mathbb{R}, \ \nu \mapsto h_\nu(g)$$

is upper semi-continuous.

Since $\mathcal{M}_g$ is weak* compact, there exist $\mu \in \mathcal{M}_g$ that attains the supremum

$$\sup\{P_\nu(g, \phi) : \nu \in \mathcal{M}_g\}.$$

and such μ is an equilibrium state.

Main result

We prove that a metric CXC system is asymptotically h-expansive under assumptions below

The Assumptions:

(1) $\mathfrak{X}_0, \mathfrak{X}_1$ are Hausdorff, locally compact, locally connected spaces, each with finite many connected components.

(2) $\mathfrak{X}_0$ is endowed with a metric compatible with its topology, and the minimum diameter of connected components of $\mathfrak{X}_0$ is positive.

(3) $\mathfrak{X}_1$ is an open subset of $\mathfrak{X}_0$ and $\overline{\mathfrak{X}_1}$ is compact in $\mathfrak{X}_0$.

(4) $f : \mathfrak{X}_1 \to \mathfrak{X}_0$ is an FBC map of degree $\deg(f) = d \geq 2$ with repellor X, and $f|_X : X \to X$ is an FBC map of degree d.

(5) There exists a cover $\mathcal{U}_0$ of X by open, connected subsets of $\mathfrak{X}_1$ whose intersection with X is nonempty such that axioms **[Expans]**, **[Deg]**, **[Round]**, **[Diam]** hold.

Main result

Theorem (Asymptotic h-expansiveness)

Let $f : (\mathfrak{X}_1, X) \to (\mathfrak{X}_0, X)$ be a metric coarse expanding conformal system with repellor X. If The Assumptions are satisfied, then $f|_X : X \to X$ is asymptotically h-expansive.

Theorem (Existence of equilibrium states)

Let $f : (\mathfrak{X}_1, X) \to (\mathfrak{X}_0, X)$ and $\mathcal{U}_0$ satisfy The Assumptions, then for each real-valued continuous function $\psi \in C(X)$, there exists at least one equilibrium state for the map $f|_X$ and potential ψ.

Main idea of the proof

We aim to prove

$$h^*(g) = \lim_{m\to+\infty} \lim_{l\to+\infty} \lim_{n\to+\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} g^{-i}(\xi_l) \,\middle|\, \bigvee_{j=0}^{n-1} g^{-j}(\eta_m) \right) = 0,$$

by giving an upper bound of

$$H(\xi|\eta) = \log\left(\max_{A\in\eta} \left\{ \min\left\{ \operatorname{card}\xi_A : \xi_A \subseteq \xi,\ A \subseteq \bigcup \xi_A \right\}\right\} \right)$$

for some refining sequence ξ_l, η_m of finite open covers of X.

Main idea of the proof

We define a sequence of finite open cover

$$\mathcal{W}_n = \{U \cap X : U \in \mathcal{U}_n\}$$

Note that $h*$ behave well under iteration, i.e. $h^*(g^n) = (h^*(g))^n$, we can choose $N \in \mathbb{N}$ such that $\{\mathcal{W}_{Nn}\}_{n \in \mathbb{N}_0}$ is a refining sequence. And we can check that $f^N : (\mathfrak{X}_N, X) \to (\mathfrak{X}_0, X)$ also satisfy The Assumptions we mentioned before.

WLOG, let $\mathcal{W}_n$ be a refining cover, it suffice to prove

$$h^*(g) = \lim_{m\to+\infty} \lim_{l\to+\infty} \lim_{n\to+\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} g^{-i}(\mathcal{W}_l) \,\middle|\, \bigvee_{j=0}^{n-1} g^{-j}(\mathcal{W}_m) \right) = 0$$

Main idea of the proof

We will give an upper bound for

$$H\left(\bigvee_{i=0}^{n-1} g^{-i}(\mathcal{W}_l) \,\middle|\, \bigvee_{j=0}^{n-1} g^{-j}(\mathcal{W}_m)\right)$$

$$=\log\left(\max_{A\in\bigvee_{j=0}^{n-1} g^{-j}(\mathcal{W}_m)}\left\{\min\left\{\operatorname{card}\xi_A : \xi_A\subseteq\bigvee_{i=0}^{n-1} g^{-i}(\mathcal{W}_l),\ A\subseteq\bigcup\xi_A\right\}\right\}\right)$$

For each $A\in\mathcal{W}_m$, set

$$A=\bigcap_{i=0}^{n} f^{-i}(W_i^m)=\left\{x\in W_0^m : f^i(x)\in W_i^m,\ i\in\{1,2,\dots,n\}\right\}$$

where $W_i^m\in\mathcal{W}_m$ for each $i\in\{0,1,\dots n\}$.

Main idea of the proof

First, we denote for all $m, n \in \mathbb{N}_0$, and $U_i^m \in \mathcal{U}_m$, $i \in \{0, 1, \ldots, n\}$

$$\begin{aligned} &E_m(U_0^m, \ldots, U_{n-1}^m; U_n^m) \\ &= \{U^{m+n} \in \mathcal{U}_{m+n} : f^n(U^{m+n}) = U_n^m, \text{ and} \\ &\quad f^i(U^{m+n}) \cap U_i^m \cap X \neq \varnothing, i \in \{0, \ldots, n-1\}\}, \end{aligned}$$

and we can prove that

Lemma

For all $m, n \in \mathbb{N}_0$, and $W_i^m = U_i^m \cap X$, $U_i^m \in \mathcal{U}_m$, $i \in \{0, 1, \ldots, n\}$,

$$\bigcap_{i=0}^{n} f^{-i}(W_i^m) \subseteq \bigcup_{U^{m+n} \in E_m(U_0^m, \ldots, U_{n-1}^m; U_n^m)} U^{m+n} \cap X. \tag{3}$$

Main idea of the proof

Next, we give an upper bound for

$$\text{card}\left(E_m(U_0^m, \ldots, U_{n-1}^m; U_n^m)\right)$$

We construct a "tree":

$$\mathcal{T} = \bigcup_{0 \le k \le n} \left\{ f^k(U) : U \in E_m(U_0^m, \ldots, U_{n-1}^m; U_n^m) \right\},$$

and define the *k-th layer* $\mathcal{L}_k$ of the $\mathcal{T}$ by

$$\mathcal{L}_k = \left\{ f^k(U) : U \in E_m(U_0^m, \ldots, U_{n-1}^m; U_n^m) \right\}.$$

Our result is: $\text{card}(\mathcal{L}_0) \le p^{\frac{n}{M_m}+1} \cdot \text{card}(\mathcal{L}_n)$, which is equivalent to

$$\text{card}\left(E_m(U_0^m, \ldots, U_{n-1}^m; U_n^m)\right) \le p^{\frac{n}{M_m}+1},$$

where $M_m \to \infty$ as $m \to \infty$.

Main idea of the proof

We can also prove that

Lemma

There exists $T_0 \geq 1$ *such that for each* $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, $W^n \in \mathcal{W}_n$, *there exist* $\mathcal{I} \subseteq \mathcal{W}_{n+k}$ *such that* $\operatorname{card}(\mathcal{I}) \leq (pT_0)^k$, *where* p *is the constant in axiom* **[Deg]**, *and*

$$W^n \subseteq \bigcup \mathcal{I},$$

Combined with our previous result, we have

$$\min \left\{ \operatorname{card} \xi_A : \xi_A \subseteq \bigvee_{j=0}^{n} f^{-j}(\mathcal{W}_l),\ A \subseteq \bigcup \xi_A \right\} \leq p^{\frac{n}{M_m}+1} (pT_0)^{l-m},$$

Main idea of the proof

Finally, we have

$$\begin{aligned} h^*(f) &= \lim_{m\to+\infty} \lim_{l\to+\infty} \lim_{n\to+\infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} f^{-j}(\mathcal{W}_l) \,\middle|\, \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{W}_m) \right) \\ &\leq \lim_{m\to+\infty} \lim_{l\to+\infty} \lim_{n\to+\infty} \frac{1}{n} \log\left(p^{\frac{n-1}{M_m}+1} (pT_0)^{l-m} \right) \\ &= \lim_{m\to+\infty} \frac{\log p}{M_m} \\ &= 0 \end{aligned}$$

Further research

To be filled...

End

Thank you!