

RESEARCH STATEMENT

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1. INTRODUCTION

My research interests lie in the interplay of dynamical systems and geometry of spaces with fractal nature. My work is on the dynamics of rational maps and more general branched covering maps on the topological 2-sphere, focusing on the measure of maximal entropy, equilibrium states, weak expansion properties, the equidistribution and large deviation principles for periodic and preimage points, dynamical zeta functions, and prime orbit theorems.

A dynamical system describes the law of evolution of points in a space over time. Despite the possible chaotic nature, one can often extract useful statistical information of the long-term behavior of the evolution of a system through dynamical investigations. Examples appear frequently in mathematics and the natural sciences.

Natural invariant measures that arise from dynamical systems, such as the measure of maximal entropy and equilibrium states, play an important role in connecting the dynamical and geometrical behavior of the system, the latter of which are often fractal in nature. The periodic orbits for a dynamical system serve the role of prime numbers in number theory, and are often related to natural invariant measures through their equidistribution. The analytic study of the dynamical zeta functions encoding periodic orbits yields precise quantitative understanding of the dynamical system.

In what follows, I will describe my previous results and provide an outline of the plan for my current and future research. Most of my thesis work has been summarized in a monograph *Ergodic theory of expanding Thurston maps* [Li17]. My more recently work centers around the dynamical zeta function and Prime Orbit Theorems [LZ18].

2. BACKGROUND AND RESULTS

Self-similar fractals have fascinated laymen and mathematicians alike due to their intrinsic beauty as well as mathematical sophistication. They appear naturally in mathematics and play important roles in the investigation of corresponding areas of research. One particularly abundant source of self-similar fractals is the study of holomorphic dynamics, where they arise as Julia sets of rational functions and limit sets of Kleinian groups.

The theory of complex dynamics dates back to the work of G. Kœnigs, E. Schröder, and others in the 19th century. This subject, concentrating on the study of iterated rational maps (i.e., quotients of polynomials) on the Riemann sphere, was developed into an active and fascinating area of research, with far-reaching connections to geometric function theory, number theory, geometry, probability, etc.

In the early 1980s, D. P. Sullivan introduced a “dictionary”, known as *Sullivan’s dictionary* nowadays, linking the theory of complex dynamics with another classical area of conformal dynamical systems, namely, geometric group theory, mainly concerning the study of Kleinian groups acting on the Riemann sphere. Many dynamical objects in both areas can be similarly defined and results similarly proved, yet essential and important differences remain.

A good starting point for the proposed research is the question: “*What is special about conformal dynamical systems in a wider class of dynamical systems characterized by a suitable metric-topological conditions?*” One can interpret this question from two perspectives:

- (1) Can one characterize conformal dynamical systems from their dynamical properties?
- (2) Can one characterize conformal dynamical systems from the metric properties of the associated fractal spaces?

W. P. Thurston gave an answer to the question from the first perspective via his celebrated combinatorial characterization theorem of postcritically-finite rational maps on the Riemann sphere among a class of more general topological maps, known as Thurston maps nowadays [DH93]. A *Thurston map* is a (non-homeomorphic) branched covering map on the topological 2-sphere S^2 whose finitely many critical points are all preperiodic. Thurston's theorem asserts that a Thurston map is essentially a rational map if and only if there exists no so-called *Thurston obstruction*, i.e., a collection of simple closed curves on S^2 subject to certain conditions [DH93]. Due to the important and fruitful applications of Thurston's theorem, many authors have worked on extending it beyond postcritically-finite rational maps using similar combinatorial obstructions. See, for example, [HSS09, CT11, ZJ09, Zh08, Wan14, CT15].

Via Sullivan's dictionary, the counterpart of Thurston's theorem in the geometric group theory is Cannon's Conjecture [Can94]. This conjecture predicts that a Gromov hyperbolic group G whose boundary at infinity $\partial_\infty G$ is a topological 2-sphere is essentially a Kleinian group, i.e., a group of Möbius transformations on the Riemann sphere. In the spirit of our question, Gromov hyperbolic groups can be considered as metric-topological systems generalizing the conformal systems in the context, namely, certain Kleinian groups. Recall that there are natural metrics d_{vis} on $\partial_\infty G$ called *visual metrics* that are quasimetrically equivalent to each other. From the second perspective to our question, Cannon's Conjecture can be reformulated in the following equivalent way: Let G be a Gromov hyperbolic group, then $\partial_\infty G$ is homeomorphic to the 2-sphere if and only if the metric space $(\partial_\infty, d_{\text{vis}})$ is *quasisymmetrically equivalent* to the Riemann sphere $\widehat{\mathbb{C}}$. Note that two metric spaces (X, d_X) and (Y, d_Y) are quasimetrically equivalent if there exists a quasimetric homeomorphism between them. Recall that a homeomorphism $f: X \rightarrow Y$ is called *quasisymmetric* if there exists a homeomorphism $\eta: [0, +\infty) \rightarrow [0, +\infty)$ such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta \left(\frac{d_X(x, y)}{d_X(x, z)} \right)$$

for all $x, y, z \in X$ with $x \neq z$. Roughly speaking, it requires that balls to be mapped to "round" sets with quantitative control for their "eccentricity".

Inspired by Sullivan's dictionary and their interest in Cannon's Conjecture, M. Bonk and D. Meyer, along with others, studied a subclass of Thurston maps by imposing some additional condition of expansion. A new characterization theorem of rational maps from a metric space point of view is established in this context by M. Bonk and D. Meyer [BM10], and P. Haïssinsky and K. Pilgrim [HP09]. Roughly speaking, we say that a Thurston map is expanding if for any two points $x, y \in S^2$, their preimages under iterations of the map get closer and closer. See [BM10, Proposition 8.2] for a list of equivalent definitions. For each expanding Thurston map, we can equip the 2-sphere S^2 with a natural class of metrics d , called *visual metrics*, that are quasimetrically equivalent to each other. As the name suggests, these metrics are constructed in a similar fashion as the visual metrics on the boundary $\partial_\infty G$ of a Gromov hyperbolic group. The following theorem was obtained in [BM10, BM17] and [HP09].

Theorem 2.1 (Bonk & Meyer, Haïssinsky & Pilgrim). *An expanding Thurston map is conjugate to a rational map if and only if the sphere (S^2, d) equipped with a visual metric d is quasimetrically equivalent to the Riemann sphere $\widehat{\mathbb{C}}$ equipped with the spherical metric.*

This theorem gives an answer to our question from a metric space point of view on the complex dynamics side of Sullivan's dictionary. It brought an alternative point of view to the program of characterizing rational maps using Thurston obstruction type criteria mentioned before.

The dynamics of iterations of an expanding Thurston map can be considered as a topological non-uniformly expanding version of the classical distance expanding continuous dynamical systems and forward-expansive dynamical systems, due to the topological obstruction from the presence of critical points of (non-homeomorphic) branched coverings on S^2 . I proved in [Li15] that actually expanding Thurston maps expand in a very subtle sense.

Theorem 2.2 (Li). *An expanding Thurston map is asymptotically h -expansive if and only if it has no periodic critical points. No expanding Thurston map is h -expansive.*

The concepts of h -expansiveness introduced by R. Bowen [Bow72b] and asymptotic h -expansiveness by M. Misiurewicz [Mis73] can be considered as weak forms of expansion properties in dynamical systems. The former implies the latter. A (forward-)expansive map is h -expansive [Bow72b], and C^∞ maps on a Riemannian manifold are asymptotically h -expansive [Bu97].

As a quantitative measure of the complexity of a dynamical system, the measure-theoretic entropy and its maximizing invariant Borel probability measures known as the *measures of maximal entropy* have been studied for various dynamical systems. For a rational map on the Riemann sphere of degree at least 2, M. Yu. Lyubich proved that there exists a unique measure of maximal entropy, which is the weak* limit of the distributions of the preimages and of the periodic points, counted with or without multiplicity. Similar results have been obtained for many dynamical systems (see, for example, [Pa64, Si72, Bow71, Bow72a, Ru89, PU10]), although for general dynamical systems, such results are still largely unknown.

Due to the important role played by the periodic points in determining the dynamical behavior, and their connection to the measure of maximal entropy, my study of expanding Thurston maps started with counting the number of periodic points [Li16].

Theorem 2.3 (Li). *Every expanding Thurston map $f: S^2 \rightarrow S^2$ has $1 + \deg f$ fixed points, counted with weight given by the local degree $\deg_f(x)$ of the map at each fixed point x .*

Here $\deg f$ is the topological degree of f . In the case of rational maps, this result is fairly easy to prove (see, for example, [Mil06, Lemma 12.1]). The proof for the general case is more difficult. It uses the correspondence between the fixed points of f and the 2-dimensional cells in the cell decomposition of S^2 induced by f and some special f -invariant Jordan curve, for the case when such curves exist. Careful analysis on the behavior near the invariant curve has to be carried out. In general, for each n large enough, there exists such a special f^n -invariant curve ([BM17, Theorem 15.1], [Li16, Lemma 3.11]). Finally, the general case follows from a number-theoretic argument.

Let f be an expanding Thurston map and μ_f its unique measure of maximal entropy. By applying Theorem 2.3, I showed in [Li16] the *equidistribution of preimages, periodic, and preperiodic points* with respect to μ_f as analogues of the corresponding results of M. Yu. Lyubich [Ly83].

Ergodic theory has been an important tool in the study of dynamical systems in general. The investigation of the existence and uniqueness of invariant measures and their properties has been a central part of ergodic theory. However, a dynamical system may possess a large class of invariant measures, some of which may be more interesting than others. It's therefore crucial to examine relevant invariant measures.

The *thermodynamical formalism* is one such mechanism to produce invariant measures with some nice properties under assumptions on the regularity of their *Jacobian functions*. More precisely, for a continuous transformation on a compact metric space, we can consider the *topological pressure* as a weighted version of the *topological entropy*, with the weights induced by a real-valued continuous function, called a *potential*. The Variational Principle identifies the topological pressure with the supremum of its measure-theoretic counterpart, the *measure-theoretic pressure*, over all invariant Borel probability measures [Bow75, Wal76]. Under additional regularity assumptions on the transformation and the potential, one gets existence and uniqueness of an invariant Borel probability measure maximizing the measure-theoretic pressure, called the *equilibrium state* for the given transformation and the potential. Often the Jacobian function for the transformation with respect to the equilibrium state is prescribed by a function induced by the potential. The existence and uniqueness of the equilibrium states and their various properties such as ergodic properties, equidistribution, fractal dimensions, etc., has been the main motivation of much research in the study of dynamical systems.

This theory, as a successful approach to choosing relevant invariant measures, was inspired by statistical mechanics, and created by D. Ruelle, Ya. Sinai, and others in the early seventies [Dob68, Si72, Bow75, Wal82]. Since then, the thermodynamical formalism has been applied in many classical contexts (see, for example, [Bow75, Ru89, Pr90, KH95, Zi96, MauU03, BS03, OI03, Yu03, PU10, MayU10]). However, beyond several classical dynamical systems, even the existence of equilibrium states is largely unknown, and for those dynamical systems that do possess equilibrium states, often the uniqueness is

unknown or at least requires additional conditions. The investigation of different dynamical systems from this perspective has been an active area of current research.

In my study of the thermodynamic formalism for our setting, I proved the existence and uniqueness of the equilibrium state with a Hölder continuous potential [Li18] by a careful study of the Ruelle (transfer) operator in our context.

Let f be an expanding Thurston map, d a visual metric, and ϕ, ψ real-valued Hölder continuous functions on (S^2, d) . In particular, f can be a postcritically-finite rational map on the Riemann sphere and d be the spherical metric (which cannot be a visual metric).

Theorem 2.4 (Li). *There exists a unique equilibrium state μ_ϕ for f and ϕ . Conversely, $\mu_\phi = \mu_\psi$ if and only if there exists a constant $K \in \mathbb{R}$ such that $\phi - \psi$ and K are co-homologous, i.e., $\phi - \psi - K = u \circ f - u$ for some real-valued continuous function u on S^2 .*

Actually μ_ϕ is a Gibbs measure (in an appropriate sense), absolutely continuous with respect to the unique probability eigen-measure m_ϕ of the adjoint of the Ruelle operator [Li18]. Moreover, I showed the *equidistribution of the preimages* with respect to μ_ϕ and m_ϕ , with the weight naturally induced by ϕ [Li18]. The *equidistribution of the periodic points* is, however, harder to conclude.

Fortunately, by a general result of H. Comman and J. Rivera-Letelier [CRL11, Theorem C], the existence and uniqueness of the equilibrium state μ_ϕ and the upper semi-continuity of the measure-theoretic entropy $h_\mu(f)$ as a function of μ establish the *level-2 large deviations principles* for preimages and periodic points, which implies immediately the corresponding equidistribution results. Thanks to Theorem 2.4, we only need to verify the upper semi-continuity of $\mu \mapsto h_\mu(f)$, which is in general guaranteed by the asymptotic h -expansiveness of f (see [Mis73]) in the absence of periodic critical points (see Theorem 2.2). We finally get in [Li15] the equidistribution results in this context.

My works [Li16, Li15, Li18] on the natural invariant measures, equidistributions, the Ruelle operator, and thermodynamical formalism in our setting, summarized in the monograph [Li17] set the stage for my more recent collaborated work [LZ18] on dynamical zeta functions and Prime Orbit Theorems in complex dynamics.

The idea of studying zeta functions was first introduced by A. Selberg in 1956 from number theory into geometry, and into dynamics by M. Artin and B. Mazur [AM65] in 1965 for diffeomorphisms and by S. Smale [Sm67] in 1967 for Anosov flows, where (perimitive) closed geodesics and periodic orbits serve the role of prime numbers. A related formulation of zeta functions for flows was later proposed and studied by D. Ruelle [Ru76a, Ru76b, Ru76c].

H. Huber established the first (effective) *Prime Geodesic Theorem*, as an analogue of the Prime Number Theorem, for surfaces of constant negative curvature in 1961 [Hu61].

Theorem 2.5 (H. Huber). *Let M be a compact surface of constant curvature -1 , and $\pi(T)$ be the number of primitive closed geodesics γ of length $l(\gamma) \leq T$. Then there exists $\alpha \in (0, 1)$ such that*

$$\pi(T) = \text{Li}(e^T) + O(e^{\alpha T}), \quad \text{as } T \rightarrow +\infty,$$

where $\text{Li}(y)$ is the Eulerian logarithmic integral function $\text{Li}(y) := \int_2^y \frac{1}{\log u} du$, $y > 0$.

Extensive researches have been carried out in geometry and dynamics in establishing Prime Orbit Theorems for various flows and other dynamical systems. With the “length” $l(\cdot)$ appropriately interpreted, we call a theorem in the form of Theorem 2.5 without (resp. with) the exponential error term a POT (resp. an effective POT). Here POT stands for Prime Orbit Theorem.

G. A. Margulis established in his thesis in 1970 [Mar04] (see also [Mar69]) a POT for the geodesic flows over compact Riemannian manifolds with variable negative curvature. Similar results were obtained by P. Sarnak in his thesis in 1980 for non-compact surfaces of finite volume [Sa80]. For geodesic flows over convex-cocompact surfaces of constant negative curvature, a POT was obtained conditionally by L. Guillopé [Gu86] and later unconditionally by S. P. Lalley [La89].

The exponential error term in the POTs in many contexts above were out of reach until D. Dolgopyat’s seminal work on the exponential mixing of Anosov flows in his thesis [Dol98], where he developed an

ingenuous approach to get new upper bounds on the norms of the Ruelle operators on some appropriate function spaces. M. Pollicott and R. Sharp [PoSh98] combined these bounds with techniques from number theory to get an effective POT for the geodesic flows over compact surfaces of variable negative curvature. These ideas of M. Pollicott, R. Sharp, and D. Dolgopyat have been adapted by many authors in various contexts, see for example, F. Naud [Na05], L. N. Stoyanov [St11], P. Giulietti, C. Liverani, and M. Pollicott [GLP13], H. Oh and D. Winter [OW16, OW17], D. Winter [Wi16], etc. Not surprisingly, in view of the connection between [Dol98] and [PoSh98], dynamical zeta functions are also closely related to the decay of correlations and resonances. For related researches on decay of correlations, see for example, D. Dolgopyat [Dol98], C. Liverani [Liv04], A. Avila, S. Gouëzel, and J. C. Yoccoz [AGY06], L. N. Stoyanov [St01, St11], V. Baladi and C. Liverani [BalLiv12], V. Baladi, M. Demers, and C. Liverani [BDL18].

F. Naud established effective POTs for both classical Fuchsian Schottky groups and some hyperbolic quadratic polynomials in [Na05], which can be considered a new correspondence in the Sullivan's dictionary. See related earlier works [Wad97] and [BJR02]. Despite active researches on dynamical zeta functions and POTs in many areas of dynamical systems, especially the works of L. N. Stoyanov [St11], G. A. Margulis, A. Mohammadi, and H. Oh [MMO14], and D. Winter [Wi16] on the group side of Sullivan's dictionary, and related works with different emphases such as J. Bourgain, A. Gamburd, and P. Sarnak [BGS11], F. Naud [Na14], H. Oh and D. Winter [OW16], S. Dyatlov and J. Zahl [DZ16], J. Bourgain and S. Dyatlov [BD17], the authors are not aware of similar entries in complex dynamics since F. Naud [Na05], until the recent work of H. Oh and D. Winter [OW17] on hyperbolic rational maps. A rational map is forward-expansive on some neighborhood of its Julia set if and only if it is hyperbolic.

In [LZ18], we establish an effective POT for expanding Thurston maps by a quantitative investigation on the holomorphic extension properties of the related dynamical zeta functions. As a special case, these results hold for postcritically-finite rational maps whose Julia set is the whole Riemann sphere. A stronger result is also obtained for Lattès maps due to deeper understanding of the relations between the visual, spherical, and canonical orbifold metrics in this context (see [LZ18, Theorem 1.12]).

Let $f: S^2 \rightarrow S^2$ be an expanding Thurston map, and d be a visual metric on S^2 . Let ϕ be a real-valued Hölder continuous function some condition that is shown to be generic [LZ18, Theorem 1.9]. Denote by s_0 the unique positive number with vanishing topological pressure $P(f, -s_0\phi) = 0$. Let $N_f \in \mathbb{N}$ be large enough depending only on f . Then for each integer $n \geq N_f$, for $F := f^n$ and $\Phi := \sum_{i=0}^{n-1} \phi \circ f^i$:

Theorem 2.6 (Li & Zheng). *There exists a constant $\delta \in (0, s_0)$ such that*

$$(2.1) \quad \pi(T) = \text{Li}(e^{s_0 T}) + O(e^{(s_0 - \delta)T}) \quad \text{as } T \rightarrow +\infty,$$

where $\pi(T)$ denotes the number of primitive periodic orbits τ of F with “length” $\sum_{x \in \tau} \Phi(x) \leq T$, and $\text{Li}(\cdot)$ is the Eulerian logarithmic integral function defined in Theorem 2.5.

As in number theory, such a precise asymptotic growth rate comes from quantitative study on the holomorphic extension of the dynamical zeta function

$$\zeta_{F, -\Phi}(s) := \exp\left(\sum_{n=1}^{+\infty} \frac{1}{n} \sum_{x=f^n(x)} e^{-s \sum_{i=0}^{n-1} \Phi(F^i(x))}\right), \quad s \in \mathbb{C}.$$

Theorem 2.7 (Li & Zheng). *There exists a constant $\epsilon_0 \in (0, s_0)$ such that $\zeta_{F, -\Phi}(s)$ converges on the half-plane $\{s \in \mathbb{C} \mid \Re(s) > s_0\}$ and extend to non-vanishing holomorphic functions on the half-plane $\{s \in \mathbb{C} \mid \Re(s) \geq s_0 - \epsilon_0\}$ except for the simple pole at $s = s_0$. Moreover, for each $\epsilon > 0$, there exist constants $C_\epsilon > 0$, $a_\epsilon \in (0, \epsilon_0]$, and $b_\epsilon \geq 2s_0 + 1$ such that*

$$\exp(-C_\epsilon |\Im(s)|^{2+\epsilon}) \leq |\zeta_{F, -\Phi}(s)| \leq \exp(C_\epsilon |\Im(s)|^{2+\epsilon})$$

for all $s \in \mathbb{C}$ with $|\Re(s) - s_0| < a_\epsilon$ and $|\Im(s)| \geq b_\epsilon$.

Special cases of Theorem 2.6 seems to be the first (effective) POT in complex dynamics outside of hyperbolic rational maps. We also want to emphasize that our setting is completely topological, without

any holomorphicity or smoothness assumptions on the dynamical systems or the potentials, with metric and geometric structures arising naturally from the dynamics of our maps, while most if not all of the previous results of POTs are established for smooth dynamical systems. Much of the difficulty in the study of the ergodic theory of complex dynamics comes from the singularities caused by critical points in the Julia set. In this sense, postcritically-finite maps are naturally the first class of rational maps to be considered after hyperbolic rational maps.

When we equip $\widehat{\mathbb{C}}$ with the spherical metric, even for a rational map f and a Hölder continuous potential ϕ , the existence of a critical point in the Julia set—so f is not hyperbolic—gives rise to the obstacles that the set of α -Hölder continuous functions on $\widehat{\mathbb{C}}$ is not invariant under the Ruelle operator (see [DPU96, Remark 3.1]) and that the *temporal distance function* induced by ϕ has worse regularity than ϕ . Similar obstacle of the lack of regularity appears in the seminal work of D. Dolgopyat [Dol98] in the study of exponential mixing of generic Anosov flows, in which jointly non-integrable C^1 strong stable and strong unstable foliations were assumed. More recently, C. Liverani [Liv04] established exponential mixing for contact Anosov flows, for which stronger regularity of the temporal distance function is known, by constructing spaces of distributions on which the Ruelle operator acts directly. Overcoming such obstacles lies in the core of current research, see [Fi11, Section 1] for an insightful account of historical developments. Our approach to avoid the above obstacles in our setting is to carry out the analysis of the Ruelle operator using the dynamically more natural visual metrics first before translating the final results back to spherical metrics, based on prior works of [BM17, HP09, Li17, Li18].

We believe that the techniques and approaches developed in [LZ18] can be used in the investigations of dynamical zeta functions and POTs for more general rational maps and other (non-smooth) branched covering maps on topological spaces, and may shed some light on related studies in other dynamical settings.

3. RESEARCH PLAN

3.1. Thermodynamical formalism and weak expansion properties of topological coarse expanding conformal systems. P. Haïssinsky and K. M. Pilgrim investigated in [HP09] a class of dynamical systems called *topological coarse expanding conformal maps*. Briefly speaking, such maps are branched coverings between Hausdorff, locally compact, locally connected topological spaces that satisfy three axioms, called Expansion, Irreducibility, and Degree. Expanding Thurston maps without periodic critical points form a subclass of such maps, and if we drop the Degree axiom, then they include all expanding Thurston maps. Topological coarse expanding conformal maps serve as a broad framework for the investigation of dynamics of general topological branched coverings with some natural expansion behavior. When restricted to the rational case, the class of topological coarse expanding conformal rational maps is much bigger than the class of rational expanding Thurston maps.

With my previous work on thermodynamical formalism discussed above, it would be natural to generalize them to topological coarse expanding conformal systems. Another promising direction is to show that every topological coarse expanding conformal map is asymptotically h -expansive.

One major difficulty is from the lack of the kind of combinatorial information we get for expanding Thurston maps from the existence of invariant Jordan curves that M. Bonk and D. Meyer obtained in [BM17]. However, since the proof of the asymptotic h -expansiveness of expanding Thurston maps without periodic critical points relies mainly on the non-recurrence of critical points and metric distortion estimates of the dynamics with respect to the visual metrics, both of which are shared by topological coarse expanding conformal maps, we expect a positive answer to the second problem. If it is indeed the case, we can then conclude that in this context, the measure-theoretic entropy considered as a function of the measure on the space of invariant Borel probability measures is upper semi-continuous, and therefore for each continuous potential, there exists at least one equilibrium state.

3.2. Dynamical zeta functions and Prime Orbit Theorems for Collet–Eckmann maps. Most of the literature on dynamical zeta functions and Prime Orbit Theorems on the complex dynamical side of Sullivan’s dictionary focuses on hyperbolic rational maps, with the exception of [BJR02]. Considering

the active researches on convex-cocompact and geometrically finite groups on the group side of Sullivan’s dictionary (see, for example, [Na05, St11, MMO14, Wi16]), it is interesting to see how far we can push our techniques in [LZ18] to other classes of rational maps such as semi-hyperbolic maps, sub-hyperbolic maps, Collet–Eckmann maps, and more generally, (non-smooth) branched covering maps such as topological coarse expanding conformal systems considered in [HP09].

Along this line of research, I am currently working with J. Rivera-Letelier on establishing Prime Orbit Theorems for both real and complex Collet–Eckmann maps. Rational Collet–Eckmann maps form a large class of non-uniformly hyperbolic rational maps. Considerable amount of effort has been made to understand their dynamics, see, for examples, works of F. Przytycki, J. Rivera-Letelier, S. Rohde, Weixiao Shen, and S. Smirnov [PR98, PRLS03, PRLS04, PRL07, PRL11, RLS14].

The results in [OW17] for hyperbolic rational maps f and the *geometric potential* $\log|f'|$ rely on the fact that $\log|f'|$ is Hölder continuous with respect to the spherical metric, which is not the case if there is at least one critical point in the Julia set. Beyond hyperbolic rational maps, one can aim to establish similar results for either the geometric potential or the class of Hölder continuous potentials. One key task is to establish a reasonable theory of thermodynamical formalism. While our success in [LZ18] for Hölder continuous potentials relies on the observation that visual metrics can serve as a rescue to the problem that the Ruelle operator does not preserve the class of α -Hölder continuous functions on $\widehat{\mathbb{C}}$ with the spherical metric, similar approach is yet to be proved fruitful for the geometric potential. Even though the geometric potential is less regular than Hölder continuous potentials, it is still trackable thanks to its advantage of being closely related to the geometry of the fractal Julia set due to the conformality of the dynamics (via Koebe’s distortion lemma). In order to obtain Prime Orbit Theorems for the geometric potential in the context of rational expanding Thurston maps, or more generally, Collet–Eckmann maps, we decide to take a different approach using the *inducing scheme* techniques adopted in [PRL07, PRL11, RLS14].

One challenge to our approach is that we have to fit the symbolic dynamics on a countably infinite alphabet with the delicate machinery of D. Dolgopyat. To our knowledge, all the literature on Prime Orbit Theorems mentioned above use symbolic dynamics on finite alphabets. While many obstacles are yet to be overcome, we have the confidence that our approach would lead to new versions of Prime Orbit Theorems for Collet–Eckmann maps, which complement the results from [LZ18], and moreover, serve as new entries in Sullivan’s dictionary with corresponding entries on the group side widely open.

3.3. Holonomies for expanding Thurston maps. The question of equidistribution of holonomy has been investigated in geometry and dynamics, and is closely related to Prime Orbit Theorems. See for example, P. Sarnak and M. Wakayama [SW99], G. A. Margulis, A. Mohammadi, and H. Oh [MMO14], H. Oh and D. Winter [OW16, OW17], D. Winter [Wi16] and references therein. For hyperbolic rational maps and the geometric potential, H. Oh and D. Winter defined the holonomy $\lambda_\theta(\tau) := \frac{\lambda(\tau)}{|\lambda(\tau)|} \in S^1$ for each primitive periodic orbit τ , where $\lambda(\tau) := (f^n)'(x)$ with $x \in \tau$ and $\text{card } \tau = n$. They showed in [OW17] that except for special maps, for each hyperbolic map, there exist $s_0 > 0$ and $\delta \in (0, s_0)$ such that for each $\psi \in C^4(S^1)$,

$$\sum_{|\lambda(\tau)| < T} \psi(\lambda_\theta(\tau)) = \text{Li}(T^{s_0}) \int_0^1 \psi(e^{2\pi i\theta}) d\theta + \mathcal{O}(T^{s_0-\delta}) \quad \text{as } T \rightarrow +\infty.$$

Given our work in [LZ18], we expect similar results to hold for expanding Thurston maps and Hölder continuous potentials.

Subproject 4: Effective Prime Orbit Theorem for hyperbolic rational maps and preperiodic points.

In his pioneering work [Wad97], S. Waddington considered a variation of the Ruelle zeta function encoding strictly preperiodic points instead of periodic points for hyperbolic rational maps, and obtained a form of Prime Orbit Theorem for strictly preperiodic points. As an application, he deduced an asymptotic counting formula in diophantine number theory.

With results from [Li16] and techniques from [OW17] and [LZ18], an exponential error term for the Prime Orbit Theorem in [Wad97] should be within reach, leading to a finer result in Diophantine number theory as an application.

3.4. Generalize the metric space point of view to bigger classes of Thurston maps. M. Bonk and D. Meyer’s philosophy in [BM10, BM17] is to use the combinatorial information of certain Markov partitions of the 2-sphere induced by an expanding Thurston map to study its properties. There are two main ingredients to this approach: one is the existence of certain forward invariant Jordan curves on S^2 that induces the Markov partitions; and the other is the existence of visual metrics on S^2 with respect to which the Markov partitions are very regular. The result on the existence of invariant Jordan curves asserts that for each expanding Thurston map f , there exists $N \in \mathbb{N}$ sufficiently large such that for each $n > N$, there exists a Jordan curve \mathcal{C} containing $\text{post } f$ such that $f^n(\mathcal{C}) \subseteq \mathcal{C}$.

Since a rational Thurston map is expanding if and only if the Julia set is the whole sphere, it is natural to ask whether it is possible to extend the metric and combinatorial theory of expanding Thurston maps to a larger subclass of Thurston maps whose “Julia set” is not the whole sphere. In other words, find appropriate subclass of Thurston maps M such that for each $f \in M$, there exist forward invariant Jordan curves or, more generally, forward invariant graphs for each high enough iterate of f . Active research on this problem is already being carried out by experts in the field, see for example, [GHMZ16]. The next step would be to establish the existence of nice metrics with quantitative bounds on the sets in the Markov partitions induced by the invariant graph, similar to the visual metrics for expanding Thurston maps. One of the final goals is to establish a characterization theorem of rational maps in this subclass of Thurston maps from a metric space point of view. Classes of maps considered in [HP09, ZJ09, CT11, CT15] serve as important reference and guidance in this direction. Ultimately, a more complete understanding of the visual metrics for these classes of maps can be used in the investigations of thermodynamical formalism and prime orbit theorems in a similar fashion as in [LZ18].

3.5. Metric gauges and the equilibrium states. Let f be an expanding Thurston map with no periodic critical points, and d a visual metric on S^2 . The *conformal gauge* $\mathcal{G}(f)$ of f is defined as the set of Ahlfors-regular metrics on S^2 that is quasisymmetrically equivalent to d . Recall that a metric space (X, d) is *Ahlfors regular* of dimension Q provided there is a Radon measure μ and a constant $C > 1$ such that $\frac{1}{C}r^Q \leq \mu(B_d(x, r)) \leq Cr^Q$ for $x \in X$ and $r \in (0, \text{diam}_d(X)]$. In this case, we can always choose μ to be the d -dimensional Hausdorff measure. The Ahlfors regular conformal dimension $\text{cdim}(f)$ of f is defined to be the infimum of the Hausdorff dimension of all metrics $d' \in \mathcal{G}(f)$ in the conformal gauge of f .

P. Haïssinsky and K. M. Pilgrim established in [HP14] that if there is a metric $d' \in \mathcal{G}(f)$ in the conformal gauge of f whose Hausdorff dimension is equal to the Ahlfors regular conformal dimension $\text{cdim}(f)$ of f , then f is conjugate to a rational map except for special cases of so-called obstructed Lattès examples. This provides a sufficient condition from the dimension theory of the dynamics for an expanding Thurston map to be (essentially) rational.

However, in general it is a difficult problem to determine the Ahlfors regular conformal dimension of a conformal gauge, and whether such dimension can be realized by a metric in the gauge. On the other hand, each equilibrium state μ_ϕ for a Hölder continuous potential ϕ , as a doubling measure, induces a family of metrics in the conformal gauge of d . More precisely, there exists $\epsilon_0 = \epsilon_0(\mu_\phi) > 0$ for which each $\epsilon \in (0, \epsilon_0]$ gives rise to a metric d_ϵ such that the identity map between (S^2, d_ϵ) and (S^2, d) is quasisymmetric, and d_ϵ is bi-Lipschitz equivalent to the quasi-metric q_ϵ on S^2 defined by $q_\epsilon(x, y) = (\mu_\phi(B(x, d(x, y)) \cup B(y, d(x, y))))^\epsilon$. Moreover, for each $\epsilon \leq \epsilon_0$, the metric measure space $(S^2, d_\epsilon, \mu_\phi)$ is Ahlfors $\frac{1}{\epsilon}$ -regular ([He01, Proposition 14.14]).

With the freedom to choose ϕ , one can first try to maximize $\epsilon_0(\mu_\phi)$ among Hölder continuous ϕ . If a condition on f can be obtained under which there exists ϕ with $\epsilon_0(\mu_\phi) = \frac{1}{2}$, then we can expect another sufficient condition for a characterization of rational maps among expanding Thurston maps.

In the context of smooth dynamical systems, R. Bowen established the first connection between topological pressure and Hausdorff dimension [Bow79]. He showed that for certain compact set $J \subseteq \mathbb{C}$

which arise as invariant sets of fractional linear transformation f of the Riemann sphere \mathbb{C} , the Hausdorff dimension $t = \dim_H J$ is the unique root of the equation

$$(3.1) \quad P(f|_J, -t \log |f'|),$$

where P denotes the topological pressure. Later, D. Ruelle established Bowen's formula (3.1) to $C^{1+\epsilon}$ conformal maps on a Riemannian manifold [Ru82], and D. Gatzouras and Y. Peres extended it to the C^1 case [GP97].

Recall that each rational Thurston map R has an invariant measure μ such that μ is absolutely continuous with respect to the Lebesgue measure and (S^2, d, μ) is Ahlfors 2-regular [GPS90]. Moreover, μ is an equilibrium state for the geometric potential which has singularities at critical points. Thus it makes sense to assume that the metric in $\mathcal{G}(f)$ that minimizes the Hausdorff dimension may correspond to potentials with (logarithmic) singularities at critical and postcritical points.

The thermodynamical formalism that we developed in [Li18] does not contain such potentials. In general, thermodynamical formalism for potentials with singularities is much more delicate and difficult to establish than the theory for Hölder continuous potentials (see, for example, [MayU10]). It leads to the following natural problem:

For suitably defined class of potentials that are Hölder continuous away from post f but with singularities at the points in post f , prove the existence and uniqueness of equilibrium states, and investigate their ergodic properties and fractal dimensions.

If this problem can be solved, we can then ask: Under what condition of f is there such a potential that corresponds to a metric $d' \in \mathcal{G}(f)$ whose Hausdorff dimension is equal to $\text{cdim}(f)$?

3.6. Random walks on the combinatorial structure. In probability theory, after the seminal work of I. Benjamini and O. Schramm [BS01], a lot of research has been done on the random walks on the Gromov–Hausdorff limit of random graphs and random surfaces (see, for example, [AS03, GR13, GGN13, NPS14]). In the spirit of using the combinatorial information to investigate the dynamical and geometrical properties of expanding Thurston maps, it is interesting to study the random walks on the Gromov–Hausdorff limit of the graphs induced from the dynamics rather than probability.

More precisely, by the existence of f^n -invariant Jordan curve $\mathcal{C}_n \subseteq S^2$ for each expanding Thurston map and each $n \in \mathbb{N}$ sufficiently large [BM10, Theorem 1.2], we get a sequence of graphs by taking the inverse images of \mathcal{C}_n under iterations of f^n . The recurrence and other finer properties of the random walk on the Gromov–Hausdorff limit of this sequence of graphs equipped with the graph metric bring us new perspectives to the study of the dynamics and geometry of expanding Thurston maps and the associated fractal sphere.

On the other hand, considering the large amount of recent research on Schramm–Loewner evolution, yet another approach along a similar line is to consider the scaling limits of the random walks on the graphs mentioned above. This is related to the Brownian motion on the Sierpinski gasket and the Sierpinski carpet (see, for example, [BP88, BB99]).

3.7. Iterated monodromy groups. The *iterated monodromy group* $\text{IMG}(f)$ of an expanding Thurston map f is the quotient of the fundamental group of $S^2 \setminus \text{post } f$ over the kernel of a natural action of this group. More precisely, let $\pi_1 = \pi_1(S^2 \setminus \text{post } f, p)$ be the fundamental group of $S^2 \setminus \text{post } f$ with a base point $p \in S^2 \setminus \text{post } f$. Then π_1 acts on the disjoint union $T = \bigsqcup_{n \geq 0} f^{-n}(p)$ in such a way that each $g \in \pi_1$ sends each $x \in f^{-n}(p)$ to the end point $y \in f^{-n}(p)$ of the lift by f^n starting at x of any loop γ in the homotopy class given by g . Let K be the kernel of this action. Then the iterated monodromy group is defined as $\text{IMG}(f) = \pi_1/K$.

If one considers the cell decompositions induced by an f -invariant Jordan curve \mathcal{C} containing post f , then there is a very concrete description of $\text{IMG}(f)$ in terms of a natural action of the 1-edges (i.e., the connected components of $\mathcal{C} \setminus \text{post } f$) on the 2-dimensional cells. One can for example enumerate relations in the free group generated by the 1-edges that corresponds to the trivial action on T .

There is much to be explored about the iterated monodromy groups of expanding Thurston maps. Basic questions such as the growth rate of $\text{IMG}(f)$, whether $\text{IMG}(f)$ is finitely presented, whether $\text{IMG}(f)$ is amenable, etc., are still unknown.

The techniques introduced in the recent breakthrough of a collaborator of mine, Tianyi Zheng [EZ18], gives a new perspective for the investigation of the growth rate of $\text{IMG}(f)$ that is worth pursuing.

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