QUASISYMMETRIC UNIFORMIZATION OF CELLULAR BRANCHED COVERS

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ABSTRACT. In this paper, we introduce a class of branched covering maps on topological n-spheres $\mathcal{S}^n$ called expanding cellular Markov branched covers which might be viewed as analogs of expanding Thurston maps. Similarly as expanding Thurston maps, expanding cellular Markov branched covers admit visual metrics. In this paper, we study the quasisymmetric uniformization problem corresponding to visual metrics of expanding cellular Markov branched covers. More precisely, for a cellular Markov branched cover $f\colon \mathbb{S}^n \to \mathbb{S}^n$ and a visual metric ϱ of f, we prove that ϱ is quasisymmetrically equivalent to the chordal metric if and only if f is uniformly quasiregular, and the local multiplicity of the family $\{f^m\}_{m\in\mathbb{N}}$ is bounded uniformly.

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1. Introduction

Backgrounds and contents. In [BM17], Bonk and Meyer develop a theory of visual metrics for expanding Thurston maps. In particular, they show that for an expanding Thurston map $f\colon \mathcal{S}^2 \to \mathcal{S}^2$ on a topological sphere $\mathcal{S}^2$ and a visual metric ϱ for f, the visual-sphere $(\mathcal{S}^2, \varrho)$ is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$ if and only if f is topologically conjugate to a rational map ([BM17, Theorem 18.1]). Recall that the visual metrics for expanding Thurston maps form a class of metrics on $\mathcal{S}^2$ having a close relation to the notion of visual metrics on the boundary of a Gromov hyperbolic space. Recall also that a homeomorphism $h\colon (X, d_1) \to (Y, d_2)$ between metric spaces is a quasisymmetry if there is a homeomorphism $\eta\colon [0, +\infty) \to [0, +\infty)$ satisfying

$$\frac{d_2(h(x), h(y))}{d_2(h(x), h(z))} \leqslant \eta\!\left(\frac{d_1(x, y)}{d_1(x, z)}\right)$$

for all distinct x, y, $z \in X$. Two metrics d_1, d_2 on a topological space X are quasisymmetrically equivalent if the identity map $\mathrm{id}\colon (X, d_1) \to (X, d_2)$ is a quasisymmetry.

The Bonk–Meyer theorem is an instance of a more general problem:

THE QUASISYMMETRIC UNIFORMIZATION PROBLEM. *Suppose that X is a metric space homeomorphic to some "standard" metric space Y. Under what condition is X quasisymmetrically equivalent to Y?*

In this paper, we extend the discussion in [BM17] to higher dimensional spheres. More precisely, we introduce a topological dynamics on topological n-spheres analogous to Thurston maps for which we obtain both metric and conformal versions of the aforementioned Bonk–Meyer theorem.

First, we introduce expanding cellular Markov branched covers as higher dimensional analogs of expanding Thurston map. We give a conceptual definition and postpone the precise definition to Section 3.

Definition. A sense-preserving, discrete and open continuous map $f\colon \mathcal{S}^n \to \mathcal{S}^n$ on a topological n-sphere $\mathcal{S}^n$ is called a *cellular Markov branched cover* if there exists a pair $(\mathcal{D}_1, \mathcal{D}_0)$ of cell decompositions of $\mathcal{S}^n$ such that, $\mathcal{D}_1$ is a refinement of $\mathcal{D}_0$ and the induced map $f_*\colon \mathcal{D}_1 \to \mathcal{D}_0$, $c \mapsto f(c)$, maps each cell in $\mathcal{D}_1$ homeomorphically to a cell in $\mathcal{D}_0$.

The pair $(\mathcal{D}_1, \mathcal{D}_0)$ is called a *Markov partition of f*. A cellular Markov branched cover admits a consecutive sequence of cellular Markov partitions. Precisely, we prove the following in Section 3.

Proposition. *If f is a cellular Markov branched cover, then there exists a sequence of cell decompositions $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ such that $(\mathcal{D}_{m+1}, \mathcal{D}_m)$ is a cellular Markov partition of f for each $m \in \mathbb{N}_0$.*

We call the sequence $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ an *essential sequence*. The map f is *expanding* if there exists an essential sequence $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ such that $\bigcap_{m\in\mathbb{N}} \bigcup\{Q \in \mathcal{D}_m : x \in Q\} = \{x\}$ holds for each $x \in \mathcal{S}^n$.

Similarly as expanding Thurston maps, expanding cellular Markov branched covers induce a family of visual metrics on $\mathcal{S}^n$; see Section 4 for a more detailed discussion.

The main theorem in this paper reads as follows; in the statement, the spherical metric σ on the standard n-sphere $\mathbb{S}^n$ is defined as usual.

Theorem 1.1. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be an expanding cellular Markov branched cover and let ϱ be a visual metric of f. Then the following conditions are equivalent:*

  (i) $\mathrm{id}\colon (\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \varrho)$ *is quasisymmetric.*
  (ii) $f\colon (\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \sigma)$ *is uniformly local UBQS.*
 (iii) f *is uniformly quasiregular, and*

$$\sup\{N_{\mathrm{loc}}(f^m, x) : m \in \mathbb{N},\ x \in \mathbb{S}^n\} < +\infty.$$

In Theorem 1.1, we actually discuss the quasisymmetric uniformization of visual metrics from two perspectives. Statement (ii) concerns

the metrical behavior of the map, and statement (iii) concerns the quasiregular property of the map. Therefore, later in this paper, we shall divide Theorem 1.1 into two parts, and have discussions in Sections 6 and 7, respectively.

On the metric side of the dynamics, we have the branched quasisymmetries introduced by Guo and Williams in [GW16] as a metric generalization of quasiregular mappings. A map $f\colon X \to Y$ between metric spaces is a *branched quasisymmetry (BQS)* if there is a homeomorphism $\eta\colon [0, +\infty) \to [0, +\infty)$ having the property that, for all intersecting continua E and E' in X,

$$\operatorname{diam}(f(E')) \leqslant \eta\left(\frac{\operatorname{diam} E'}{\operatorname{diam} E}\right) \operatorname{diam}(f(E)).$$

The name for these mappings stems from the observation that they also give a non-injective generalization of quasisymmetric maps. Indeed, for bounded turning spaces X and Y, a homeomorphism $X \to Y$ is quasisymmetric if and only if it is a branched quasisymmetry; see [GW16, Section 6.7] or [LP19, Section 3] for discussion and terminology. Uniformly local UBQS maps are BQS maps with a uniformity condition, which will be introduced later.

On the conformal side, we have quasiregular maps. A continuous map $f\colon M \to N$ between oriented Riemannian n-manifolds M and N is *quasiregular* if f is in the Sobolev space $W^{1,n}_{\mathrm{loc}}(M, N)$, and there exists $K \geqslant 1$ such that the distortion inequality

$$\|Df\|^n \leqslant K J_f \quad \text{a.e. in } M$$

holds. Here $\|Df\|$ and J_f are the operator norm and the Jacobian determinant of the differential Df, respectively. A quasiregular map $f\colon M \to M$ is *uniformly quasiregular* if f^k is quasiregular with a common constant $K \geqslant 1$ in the distortion inequality for all $k \in \mathbb{N}$.

Regarding the general theory of quasiregular maps in higher dimensional Euclidean spaces and Riemannian manifolds, we refer to monographs of Iwaniec and Martin [IM01], Reshetnyak [Re89], Rickman [Ri93], and Vuorinen [Vu88]. The dynamics of uniformly quasiregular mappings may be seen as a version of complex dynamics in higher dimensional Riemannian manifolds and UQR dynamics of $\mathbb{S}^n$ as a higher dimensional version of rational dynamics in $\mathbb{S}^2$. We refer to a survey of Martin [Ma13] for a detailed discussion on these connections.

Branched quasirymmetries and quasiregular maps have a close relation. In fact, a map $M \to N$ between oriented closed Riemannian manifolds is a branched quasisymmetry if and only if it is quasiregular; see [LP19, Appendix A].

As in the case of uniformly quasiregular maps, we consider branched quasisymmetries with uniformity. It is a natural thought to call a map $f\colon X \to X$ on a metric space X a *uniform branched quasisymmetry (UBQS)* if the iterates $f^k\colon X \to X$ are branched quasisymmetries with respect to the same homeomorphism $\eta\colon [0, +\infty) \to [0, +\infty)$. This condition is, however, too strong and we consider mappings satisfying a uniform local version of this condition. We say that a mapping $f\colon X \to X$ is *uniformly local UBQS map* if there exists a a homeomorphism $\eta\colon [0, +\infty) \to [0, +\infty)$, and a finite cover $\mathcal{U}$ of X by connected open sets for which $f^k|_{\widetilde{U}}\colon \widetilde{U} \to X$ is an η-BQS for all $k \in \mathbb{N}$ and each connected component $\widetilde{U}$ of $f^{-k}(U)$ for $U \in \mathcal{U}$.

Whereas the dynamics of uniformly quasiregular mappings may be seen as a version of complex dynamics in higher dimensional Riemannian manifolds, the dynamics of UBQS mappings may be viewed as the analog of uniformly quasiregular mappings in the quasiconformal geometry of metric spaces.

At the end of this paper, we discuss cubical maps on $\mathcal{S}^n$ similar to the "pillow" example [BM17]; see Section 8 for the definition of cubical maps. Also, this class of cubical maps is a special instance of the coarse expanding conformal (CXC) dynamics of Haïsinsky and Pilgrim [HP09]. The class of cubical maps gives intuitive examples of cellular Markov branched covers with bounded local multiplicity.

Corollary 1.2. *Let $f\colon (\mathbb{S}^n, \mathcal{D}_1, L_1) \to (\mathbb{S}^n, \mathcal{D}_0, L_0)$ be an expanding cubical map and let ϱ be a visual metric for f. Then* $\mathrm{id}\colon (\mathbb{S}^n, \varrho) \to (\mathbb{S}^n, \sigma)$ *is a quasisymmetry if and only if f is uniformly quasiregular.*

Organization of the article. The article is organized as follows.

Sections 2 and 3 are dedicated to preliminaries on cellular Markov maps and related notions. In Section 4, we introduce expanding cellular Markov maps and visual metrics for such maps. Section 5 is preliminary on branched quasisymmetries and quasiregular maps, where we also introduce branched quasisymmetries with uniformity. In Section 6 we consider the quasisymmetric uniformization of visual metrics in terms of branched quasisymmetries, and prove the metric part of Theorem 1.1. In Section 7, we furthermore study the quasisymmetric uniformization of visual metrics in terms of uniformly quasiregular maps, and prove the quasiregular part of Theorem 1.1. In the last section, Section 8, we discuss cubical structures and cubical maps, as an example.

Notation. Let $\mathbb{N} \coloneqq \{1, 2, 3, \dots\}$ be the set of positive integers, and $\mathbb{N}_0 \coloneqq \{0\} \cup \mathbb{N}$. As usual, we denote by $\log_a$ the logarithm to the base $a > 0$, and, in particular, by $\log$ the logarithm function to the base e.

Given a topological space X, the closure, interior, and boundary of a set $A \subseteq X$ (regarded as a subset of the topological space X) are denoted by $\overline{A}$, $\operatorname{int}(A)$, and ∂A, respectively. For a metric space (X, d), the distance between a and b is denoted by $|a - b|$ when the metric is clear from the context. The diameter of a subset A is $\operatorname{diam}_d(A) \coloneqq \sup\{d(a, b) : a, b \in A\}$. The distance between subsets A and B is $\operatorname{dist}_d(A, B) \coloneqq \inf\{d(a, b) : a \in A, \ b \in B\}$.

We denote by $\mathbb{B}^n \coloneqq \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < 1\}$ the (open) unit ball in $\mathbb{R}^n$, and by $\mathbb{B}^n(a, r)$ the open ball in $\mathbb{R}^n$ at $a \in \mathbb{R}^n$ with radius $r > 0$. Denote by

$$\mathbb{S}^n \coloneqq \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

the n-dimensional unit sphere in $\mathbb{R}^{n+1}$, and by $\mathcal{S}^n$ a topological n-sphere, i.e., a topological space homeomorphic to $\mathbb{S}^n$.

The spherical (chordal) metric σ on $\mathbb{S}^n$ is defined as usual. A metric $\widetilde{\sigma}$ on $\mathcal{S}^n$ is called a spherical metric on $\mathcal{S}^n$ if there exists a isometry $\rho \colon (\mathcal{S}^n, \widetilde{\sigma}) \to (\mathbb{S}^n, \sigma)$.

2. Preliminaries on cell decompositions

In this section, we introduce cell decompositions. Our definitions of cells and cell decompositions follow [BM17, Chapter 5], where the reader can find a more detailed discussion. In this section, the topological space $\mathfrak{X}$ will always be locally compact and Hausdorff.

2.1. **Topological cells.** Recall that, for each $n \in \mathbb{N}$, a subset c of $\mathfrak{X}$ homeomorphic to $[0, 1]^n$ (or $\overline{\mathbb{B}^n}$, equivalently) is called an *(compact topological) n-cell*, and $\dim(c) \coloneqq n$ is called the dimension of c. We denote $\partial_\circ c$ the set of points corresponding to $[0, 1]^n \smallsetminus (0, 1)^n$ under a homeomorphism between c and $[0, 1]^n$. The set $\partial_\circ c$ is independent of the choice of the homeomorphism, and thus is well-defined. We call $\partial_\circ c$ the *(cell) boundary* and $\operatorname{int}_\circ(c) \coloneqq c \smallsetminus \partial_\circ c$ the *(cell) interior* of c. Note that boundary and interior will generally not agree with the boundary and interior of c regarded as a subset of the topological space $\mathfrak{X}$. In particular, a 0-cell is a subset consisting of a single point in $\mathfrak{X}$. For a 0-cell c, we set $\partial_\circ c \coloneqq \emptyset$ and $\operatorname{int}_\circ(c) \coloneqq c$. Note that the definitions

of cell boundaries and interiors differ between 0-cells and n-cells where $n > 0$.

We recall some basic properties of cells. First, a simple observation is that a cell in $\mathfrak{X}$ is a closed subset.

Lemma 2.1. *Each cell $c \subseteq \mathfrak{X}$ is closed. Moreover, $c = \overline{\text{int}_\circ(c)}$.*

Proof. Let $c \subseteq \mathfrak{X}$ be a cell. Since c is a compact subset of the Hausdorff space $\mathfrak{X}$, it is a closed subset. When $\dim(c) = 0$, it is clear that $c = \overline{\text{int}_\circ(c)}$. Suppose now that $\dim(c) = d > 0$ and let $\varphi \colon c \to \overline{\mathbb{B}^d}$ be a homeomorphism. For each $x \in \partial_\circ c$, consider a sequence $\{y_k\}_{k \in \mathbb{N}}$ of points in $\mathbb{B}^d$ converging to $\varphi(x)$, then $\{\varphi^{-1}(y_k)\}_{k \in \mathbb{N}}$ is a sequence of points in $\text{int}_\circ(c)$ converging to x, and $x \in \overline{\text{int}_\circ(c)}$. Therefore, $c \subseteq \overline{\text{int}_\circ(c)}$, and thus $c = \overline{\text{int}_\circ(c)}$ since c is closed. $\square$

In this paper, we generally focus on cells in topological spheres. For n-cells in the n-spheres, the cell interior and cell boundary agree with their topological counterparts. More precisely, we have the following.

Lemma 2.2. *Let X be an n-cell in $\mathcal{S}^n$. Then $\text{int}_\circ(X)$ and $\partial_\circ X$ agree with the interior and boundary of X regarded as a subset of topological space $\mathcal{S}^n$, respectively.*

Proof. It is clear that $X \neq \mathcal{S}^n$, and that $\mathcal{S}^n \smallsetminus X$ is a non-empty open set. Let $\varphi \colon \mathcal{S}^n \to \mathbb{S}^n$ be a homeomorphism, and $\pi \colon \mathbb{S}^n \smallsetminus \{\varphi(p)\} \to \mathbb{R}^n$ be a stereographic projection, where p is an arbitrary point in $\mathcal{S}^n \smallsetminus X$.

Consider $B := \pi(\varphi(X))$. Then B is an n-cell in $\mathbb{R}^n$. Let $h \colon \overline{\mathbb{B}^n} \to B$ be a homeomorphism. By Brouwer's invariance of domain theorem, $\text{int}_\circ(B) = h(\mathbb{B}^n)$ is an open subset of $\mathbb{R}^n$, thus, $\text{int}_\circ(B) \subseteq \text{int}(B)$. Also, it follows from Brouwer's invariance of domain theorem that $h^{-1}(\text{int}(B))$ is an open subset of $\mathbb{R}^n$ that is contained in $\overline{\mathbb{B}^n}$, and thus $h^{-1}(\text{int}(B)) \subseteq \mathbb{B}^n$. Therefore, $\text{int}_\circ(B) = \text{int}(B)$, and $\partial_\circ B = \overline{B} \smallsetminus \text{int}(B) = B \smallsetminus \text{int}_\circ(B) = \partial B$.

Since $\pi \circ \varphi \colon \mathcal{S}^n \smallsetminus \{p\} \to \mathbb{R}^n$ is a homeomorphism, it is clear that $\text{int}_\circ(X)$ and $\partial_\circ X$ are the interior and boundary of X regarded as a subset of topological space $\mathcal{S}^n$, respectively. $\square$

Recall that, since an n-cell is a topological cone over its boundary, we have the following extension property.

Lemma 2.3. *Let X and Y be n-cells and let $\varphi \colon \partial_\circ X \to \partial_\circ Y$ be a homeomorphism. Then there exists a homeomorphism $\widetilde{\varphi} \colon X \to Y$ that extends φ, i.e., $\widetilde{\varphi}|_{\partial_\circ X} = \varphi$.*

2.2. **Cell decompositions.** Our definition for a cell decomposition of a locally compact Hausdorff space reads as follows.

Definition 2.4 (Cell decompositions)**.** A collection $\mathcal{D}$ of cells in a locally compact Hausdorff space $\mathfrak{X}$ is a *cell decomposition of $\mathfrak{X}$* if the following conditions are satisfied:

(i) The union of all cells in $\mathcal{D}$ is equal to $\mathfrak{X}$.

(ii) We have $\mathrm{int}_\circ(\sigma) \cap \mathrm{int}_\circ(\tau) = \emptyset$ for all σ, $\tau \in \mathcal{D}$ with $\sigma \neq \tau$.

(iii) For each $\tau \in \mathcal{D}$, the boundary $\partial_\circ \tau$ is a union of cells in $\mathcal{D}$.

(iv) Every point in $\mathfrak{X}$ has a neighborhood that meets only finitely many cells in $\mathcal{D}$.

For a cell decomposition $\mathcal{D}$ and $k \in \mathbb{N}_0$, we denote $\mathcal{D}^{[k]} := \{c \in \mathcal{D} : \dim(c) = k\}$ and $\mathcal{D}^{(k)} := \bigcup_{l \leqslant k} \mathcal{D}^{[l]}$. We call the subset $\bigcup \mathcal{D}^{(k)}$ of X the *k-skeleton of $\mathcal{D}$.*

We record some elementary properties of cell decompositions, and we refer the reader to [BM17, Section 5.1] for details.

Lemma 2.5 ([BM17, Lemma 5.2])**.** *Let $\mathcal{D}$ be a cell decomposition of $\mathfrak{X}$. Then the following statements are true:*

(i) *For each $k \in \mathbb{N}_0$, the k-skeleton of $\mathcal{D}$ is equal to $\bigcup\{\mathrm{int}_\circ(c) : c \in \mathcal{D}, \ \dim(c) \leqslant k\}$.*

(ii) $\mathfrak{X} = \bigcup\{\mathrm{int}_\circ(c) : c \in \mathcal{D}\}$.

(iii) *For each $\tau \in \mathcal{D}$, we have $\tau = \bigcup\{\mathrm{int}_\circ(c) : c \in \mathcal{D}, \ c \subseteq \tau\}$.*

Lemma 2.6 ([BM17, Lemma 5.3])**.** *Let $\mathcal{D}$ be a cell decomposition of $\mathfrak{X}$.*

(i) *If σ and τ are two distinct cells in $\mathcal{D}$ with $\sigma \cap \tau \neq \emptyset$, then one of the following three statements holds: $\sigma \subseteq \partial_\circ \tau$, $\tau \subseteq \partial_\circ \sigma$, or $\sigma \cap \tau = \partial_\circ \sigma \cap \partial_\circ \tau$ and the intersection consists of cells in $\mathcal{D}$ of dimension strictly less than $\min\{\dim(\sigma), \dim(\tau)\}$.*

(ii) *If σ, τ_1, ..., τ_k are cells in $\mathcal{D}$, and $\mathrm{int}_\circ(\sigma) \cap \bigcup_{j=1}^{k} \tau_j \neq \emptyset$, then $\sigma \subseteq \tau_i$ for some $i \in \{1, \ldots, k\}$.*

Let $\mathcal{D}$ be a cell decomposition of $\mathfrak{X}$. For each $c \in \mathcal{D}$, we denote

$$\mathcal{D}|_c := \{c' \in \mathcal{D} : c' \subseteq c\},$$

and call it the *restriction of $\mathcal{D}$ on c.* It is clear that $\mathcal{D}|_c$ is a cell decomposition of c. We have the following observation that $\mathcal{D}|_c$ contains cells of arbitrary dimensions not more than $\dim(c)$.

Lemma 2.7. *Let $\mathcal{D}$ be a cell decomposition of $\mathfrak{X}$. Then, for each $c \in \mathcal{D}$ and each $k \in \mathbb{N}_0$ with $k \leqslant \dim(c)$, there exists $\sigma \in \mathcal{D}|_c$ for which $\dim(\sigma) = k$.*

We postpone the proof of the lemma to Subsection 2.3.

2.3. **Cell decompositions on topological spheres.**
Recall that a *topological n-sphere* is a topological space homeomorphic to the unit n-sphere $\mathbb{S}^n := \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1 \right\}$.

Since $\mathcal{S}^n$ is an n-manifold, the dimension of each cell in a cell decomposition of $\mathcal{S}^n$ is not more than n. More precisely, we have the following properties.

Lemma 2.8. *Let $\mathcal{D}$ be a cell decomposition of $\mathcal{S}^n$. Then the following statements hold:*

(i) *$\mathcal{D}$ is a finite set.*

(ii) *$\mathcal{S}^n = \bigcup \{c \in \mathcal{D} : \dim(c) = n\}$.*

(iii) *Each cell in $\mathcal{D}$ is contained in an n-cell in $\mathcal{D}$.*

Proof. (i) For each x, by Definition 2.4 (iv), we may choose a neighborhood U_x in $\mathcal{S}^n$ that meets only finitely many cells in $\mathcal{D}$. Since $\mathcal{S}^n$ is compact, suppose that $\mathcal{S}^n = U_{x_1} \cup \cdots \cup U_{x_m}$ for some $x_1, \ldots, x_m \in \mathcal{S}^n$. For each $c \in \mathcal{D}$, there eixsts $i \in \{1, \ldots, m\}$ such that $U_{x_i} \cap c \neq \emptyset$. It follows that $\mathcal{D} = \bigcup_{i=1}^{m} \{c \in \mathcal{D} : c \cap U_i \neq \emptyset\}$ is a finite set.

(ii) Since $\mathcal{D}$ is a finite set by (i), and the interior $\mathrm{int}(c)$ of each $c \in \mathcal{D}$ with $\dim(c) < n$ (regarded as a subset of the topological space $\mathcal{S}^n$) is empty, we have that

$$\mathrm{int}(\bigcup \{c \in \mathcal{D} : \dim(c) < n\}) = \emptyset.$$

As a consequence, the union $\bigcup \{c \in \mathcal{D} : \dim(c) = n\}$ is dense in $\mathcal{S}^n$. Since each $c \in \mathcal{D}$ is a closed subset of $\mathcal{S}^n$, also $\bigcup \{c \in \mathcal{D} : \dim(c) = n\}$ is closed. Thus $\mathcal{S}^n = \bigcup \{c \in \mathcal{D} : \dim(c) = n\}$.

(iii) Let $c \in \mathcal{D}$ be arbitrary. By (ii), there exists $X \in \mathcal{D}$ with $\dim(X) = n$ and $\mathrm{int}_\circ(c) \cap X \neq \emptyset$. Then by Lemma 2.6 (ii), we have that $c \subseteq X$. $\qquad\square$

Now we are ready to prove Lemma 2.7.

Proof of Lemma 2.7. For each $c \in \mathcal{D}$ with $\dim(c) = 0$, the claim is clear. It suffices to show that for each $c \in \mathcal{D}$ with $\dim(c) > 0$, there exists $\sigma \in \mathcal{D}|_c$ with $\dim(\sigma) = \dim(c) - 1$, then the claim follows by induction.

Let $c \in \mathcal{D}$ with $\dim(c) = d > 0$ be arbitrary. It is clear that $\partial(\mathcal{D}|_c) := \{c' \in \mathcal{D} : c' \subseteq \partial_\circ c\}$ is a cell decomposition of $\partial_\circ c$ and that $\partial(\mathcal{D}|_c) \subseteq$

$\mathcal{D}|_c$. Since $\partial_\circ c$ is a topological $(d-1)$-sphere, by Lemma 2.8, $\partial_\circ c$ is the union of $(d-1)$-cells in $\partial(\mathcal{D}|_c)$. Thus, there exists $\sigma \in \mathcal{D}|_c$ with $\dim(\sigma) = d - 1$. □

Let $\mathcal{D}$ be a cell decomposition of $\mathcal{S}^n$. We have by Lemma 2.7 that $\mathcal{D}^{[k]}$ is non-empty for each $0 \leqslant k \leqslant n$. Since $\mathcal{D}^{[n]}$, $\mathcal{D}^{[n-1]}$, and $\mathcal{D}^{[0]}$ play important roles in this paper, we introduce the following terminologies. For a 0-cell $\{v\} \in \mathcal{D}$, the unique point $v \in \mathcal{S}^n$ is a *vertex*. We call the n-cells of $\mathcal{D}$ *rooms* and $(n-1)$-cells *faces*.

The n-cells of $\mathcal{D}$ form a cover of $\mathcal{S}^n$, but they are not open. Therefore, to construct an open cover of $\mathcal{S}^n$ from $\mathcal{D}$, we introduce the notion of a *flower*, which is an open neighborhood at a vertex.

Definition 2.9. The *flower* $\mathrm{Fl}(p) \subseteq \mathcal{S}^n$ of a vertex p of a cell decomposition $\mathcal{D}$ of $\mathcal{S}^n$ is

$$\mathrm{Fl}(p) := \bigcup \{\mathrm{int}_\circ(c) : c \in \mathcal{D}, \ p \in c\}.$$

We denote by $\mathcal{F}(\mathcal{D}) := \{\mathrm{Fl}(p) : p \text{ is a vertex of } \mathcal{D}\}$ the collection of all flowers of $\mathcal{D}$.

Remark. For flowers of a cell decomposition $\mathcal{D}$ of $\mathcal{S}^n$, we have the following basic properties:

(i) For each p, the flower $\mathrm{Fl}(p)$ is path-connected, since $\{p\} \cup \mathrm{int}_\circ(c)$ is path-connected for each $c \in \mathcal{D}$ with $p \in c$.

(ii) The collection $\mathcal{F}(\mathcal{D})$ forms a cover of $\mathcal{S}^n$. The reason is, for each $x \in \mathcal{S}^n$, it follows from Lemmas 2.5 and 2.7 that there exists $c \in \mathcal{D}$ with $x \in \mathrm{int}_\circ(c)$ and there exists a vertex p with $p \in c$.

The following lemma shows that each flower is open, and thus the collection $\mathcal{F}(\mathcal{D})$ is a cover of $\mathcal{S}^n$ by connected open subsets.

Lemma 2.10. *Let $\mathcal{D}$ be a cell decomposition of $\mathcal{S}^n$ and let $p \in \mathcal{D}$ be a vertex. Then the following statements hold:*

(i) $\mathrm{Fl}(p)$ *contains no other vertex, and we have*

$$\mathrm{Fl}(p) = \mathcal{S}^n \smallsetminus \bigcup \{c \in \mathcal{D} : p \notin c\}.$$

In particular, $\mathrm{Fl}(p)$ is an open subset of $\mathcal{S}^n$.

(ii) $\overline{\mathrm{Fl}(p)} = \bigcup \{X \in \mathcal{D}^{[n]} : p \in X\}$ *and $\partial\,\mathrm{Fl}(p)$ is the union of all $c \in \mathcal{D}$ with the following property: $p \notin c$, and $c \subseteq X$ for some $X \in \mathcal{D}^{[n]}$ with $p \in X$.*

Proof. (i) Let $x \in \mathrm{Fl}(p)$ be arbitrary. Then, there exists a unique $\sigma \in \mathcal{D}$ with $p \in \sigma$ and $x \in \mathrm{int}_\circ(\sigma)$. If $x \in \tau$ for some $\tau \in \mathcal{D}$ with $p \notin \tau$, then

$\text{int}_\circ(\sigma) \cap \tau \neq \emptyset$. By Lemma 2.6, we have $\sigma \subseteq \tau$, which contradicts $p \notin \tau$. Thus, we have $\text{Fl}(p) \subseteq \mathcal{S}^n \smallsetminus \bigcup \{c \in \mathcal{D} : p \notin c\}$.

For the other direction, let $x \in \mathcal{S}^n \smallsetminus \bigcup \{c \in \mathcal{D} : p \notin c\}$ be arbitrary, and suppose that $x \in \text{int}_\circ(\sigma)$ for some $\sigma \in \mathcal{D}$ (see Lemma 2.5). Then $p \in \sigma$, and thus $x \in \text{Fl}(p)$. Hence we have $\text{Fl}(p) \supseteq \mathcal{S}^n \smallsetminus \bigcup \{c \in \mathcal{D} : p \notin c\}$. The proof of (i) is complete.

(ii) By Lemma 2.8, each cell in $\mathcal{D}$ is contained in an n-cell. Then, since n-cells are closed, we have

$$\overline{\text{Fl}(p)} \subseteq \bigcup \{X \in \mathcal{D}^{[n]} : p \in X\}.$$

Since $X = \overline{\text{int}_\circ(X)} \subseteq \overline{\text{Fl}(p)}$ for each $X \in \mathcal{D}^{[n]}$ with $p \in X$, we have that $\bigcup \{X \in \mathcal{D}^{[n]} : p \in X\} \subseteq \overline{\text{Fl}(p)}$, and thus $\overline{\text{Fl}(p)} = \bigcup \{X \in \mathcal{D}^{[n]} : p \in X\}$.

Now we cosider $\partial \text{Fl}(p)$. Denote by F the union of all $c \in \mathcal{D}$ with $p \notin c$, and $c \subseteq X$ for some $X \in \mathcal{D}^{[n]}$ with $p \in X$. By (i) we know $\text{Fl}(p)$ is an open set, thus

$$\partial \text{Fl}(p) = \overline{\text{Fl}(p)} \smallsetminus \text{Fl}(p) = \bigcup \{X \in \mathcal{D}^{[n]} : p \in X\} \cap \bigcup \{c \in \mathcal{D} : p \notin c\}.$$

First consider arbitrary $X \in \mathcal{D}^{[n]}$ and $c \in \mathcal{D}$ satisfying $p \in X$, $p \notin c$ and $c \cap X \neq \emptyset$. By Lemma 2.6, $c \cap X$ is a union of cells in $\mathcal{D}$. For each cell $\sigma \subseteq c \cap X$, it is clear that $p \notin \sigma$ and $\sigma \subseteq X$. Hence $c \cap X \subseteq F$, and it follows that $\partial \text{Fl}(p) \subseteq F$. On the other hand, consider $c \in \mathcal{D}$ with $p \notin c$ and $c \subseteq X$ for some $X \in \mathcal{D}^{[n]}$ with $p \in X$, then we have $c \subseteq \partial \text{Fl}(p)$ since $c = c \cap X$. It follows that $F \subseteq \partial \text{Fl}(p)$. The proof of (ii) is complete. $\qquad\square$

As a corollary of Lemma 2.10, we observe that flowers in $\mathcal{F}(\mathcal{D})$ separate disjoint cells in $\mathcal{D}$ in the following sense.

Corollary 2.11. *Let $\mathcal{D}$ be a cell decomposition of $\mathcal{S}^n$, and let $\sigma, \tau \in \mathcal{D}$ be disjoint. Then, for each flower $U \in \mathcal{F}(\mathcal{D})$, either $U \cap \sigma = \emptyset$, or $U \cap \tau = \emptyset$.*

Proof. Suppose on the contrary that $\sigma \cap \text{Fl}(p) \neq \emptyset$ and $\tau \cap \text{Fl}(p) \neq \emptyset$ for a vertex p. If $p \notin \sigma$, then by Lemma 2.10 (i), we have that $\sigma \subseteq \mathcal{S}^n \smallsetminus \text{Fl}(p)$, which yields a contradiction. Hence $p \in \sigma$, and $p \in \tau$ by the same argument. This contradicts $\sigma \cap \tau = \emptyset$. $\qquad\square$

We finish this subsection by introducing the term *joining opposite sides* and recording a relevant property of flowers, which generalizes a joining condition of Bonk and Meyer for cell decompositions on $\mathcal{S}^2$; see [BM17, Section 5.7].

Definition 2.12 (Joining opposite sides)**.** Let $\mathcal{D}$ be a cell decomposition of $\mathcal{S}^n$. A subset $A \subseteq \mathcal{S}^n$ *joins opposite sides* of $\mathcal{D}$ if

$$\bigcap \{c \in \mathcal{D} : A \cap c \neq \emptyset\} = \emptyset.$$

Lemma 2.13. *Let $\mathcal{D}$ be a cell decomposition of $\mathcal{S}^n$. Then, a subset $A \subseteq \mathcal{S}^n$ joins opposite sides of $\mathcal{D}$ if and only if A is not contained in any flower in $\mathcal{F}(\mathcal{D})$.*

Proof. First, suppose that $A \subseteq \mathrm{Fl}(p)$ for some vertex p. Let $\sigma \in \mathcal{D}$ with $\sigma \cap A \neq \emptyset$ be arbitrary, then $\sigma \cap \mathrm{Fl}(p) \neq \emptyset$. If $p \notin \sigma$, then by Lemma 2.10 (i), we have that $\sigma \subseteq \mathcal{S}^n \smallsetminus \mathrm{Fl}(p)$, which yields a contradiction. Since σ is arbitrary, we have $p \in \bigcap \{\sigma \in \mathcal{D} : A \cap \sigma \neq \emptyset\} \neq \emptyset$.

For the other direction, suppose now that $\bigcap \{\sigma \in \mathcal{D} : A \cap \sigma \neq \emptyset\} \neq \emptyset$. By Lemma 2.6, $\bigcap \{\sigma \in \mathcal{D} : A \cap \sigma \neq \emptyset\}$ is a union of cells in $\mathcal{D}$. It follows from Lemma 2.7 that each cell in $\mathcal{D}$ contains at least one vertex. Let $p \in \bigcap \{\sigma \in \mathcal{D} : A \cap \sigma \neq \emptyset\}$ be a vertex. Consider arbitrary $x \in A$ and assume that $x \in \mathrm{int}_\circ(c)$ for some $c \in \mathcal{D}$. Since $c \cap A \neq \emptyset$, we have that $p \in c$, and thus $x \in \mathrm{Fl}(p)$. Since $x \in A$ is arbitrary, we have that $A \subseteq \mathrm{Fl}(p)$. $\qquad\square$

3. Cellular Markov maps

In this section, we introduce *cellular Markov maps* on topological n-sphere and study some general properties of these maps. The definition of such maps is inspired by the work of Bonk and Meyer in [BM17].

3.1. **Cellular maps between cellular spaces.**
We first introduce cellular maps. Note that these terms have various meanings in different contexts, and here we follow the definitions in [BM17, Chapter 5].

Definition 3.1 (Cellular maps)**.** Let $\mathcal{D}'$ and $\mathcal{D}$ be cell decompositions of $\mathfrak{X}'$ and $\mathfrak{X}$, respectively, and $f \colon \mathfrak{X}' \to \mathfrak{X}$ be a continuous map. We say that f is *cellular* for $(\mathcal{D}', \mathcal{D})$ if, for each $c \in \mathcal{D}'$, the restriction $f|_c \colon \sigma \to f(c)$ is a homeomorphism and $f(c)$ is a cell in $\mathcal{D}$.

For a cell decomposition $\mathcal{D}$ of $\mathfrak{X}$, a homeomorphism $\phi \colon \mathfrak{X}' \to \mathfrak{X}$ between locally compact Hausdorff spaces induces a cell decomposition

$$(3.1) \qquad\qquad \phi^*(\mathcal{D}) \coloneqq \left\{ \phi^{-1}(c) : c \in \mathcal{D} \right\}$$

of $\mathfrak{X}'$. We call $\phi^*(\mathcal{D})$ the *pull back of $\mathcal{D}$ under ϕ*. It is clear that $\mathcal{D} = (\phi^{-1})^*(\phi^*\mathcal{D})$.

Given a homeomorphism $\mathfrak{X}' \to \mathfrak{X}$ and a cell decomposition $\mathcal{D}$ of $\mathfrak{X}$, the cellularity of the homeomorphism is encoded into the pull back of $\mathcal{D}$. More precisely, we have the following lemma.

Lemma 3.2. *Let $\mathcal{D}'$ and $\mathcal{D}$ be cell decompositions on $\mathfrak{X}'$ and $\mathfrak{X}$, respectively. Then a homeomorphism $\phi\colon \mathfrak{X}' \to \mathfrak{X}$ is cellular for $(\mathcal{D}', \mathcal{D})$ if and only if $\mathcal{D}' = \phi^*(\mathcal{D})$.*

Proof. It is clear that if $\mathcal{D}' = \phi^*(\mathcal{D})$, then ϕ is cellular for $(\mathcal{D}', \mathcal{D})$.

Suppose now that ϕ is cellular for $(\mathcal{D}', \mathcal{D})$. Then, for each $\sigma' \in \mathcal{D}'$, $\sigma := \phi(\sigma')$ is a cell in $\mathcal{D}$, and $\sigma' = \phi^{-1}(\sigma) \in \phi^*(\mathcal{D})$. On the other hand, let $\sigma \in \mathcal{D}$ be arbitrary, and denote $\sigma' := \phi^{-1}(\sigma)$. It is clear that σ' is a cell in $\mathfrak{X}'$, and $\mathrm{int}_\circ(\sigma') = \phi^{-1}(\mathrm{int}_\circ(\sigma))$. By Lemma 2.5 (ii), we have that

$$\mathrm{int}_\circ(\sigma') \subseteq \bigcup\{\mathrm{int}_\circ(c') : c' \in \mathcal{D}', \ \mathrm{int}_\circ(\sigma') \cap \mathrm{int}_\circ(c') \neq \emptyset\}.$$

Consider arbitrary $c' \in \mathcal{D}'$ with $\mathrm{int}_\circ(\sigma') \cap \mathrm{int}_\circ(c') \neq \emptyset$. Denote $c = \phi(c')$, then c is a cell in $\mathcal{D}$ with $\mathrm{int}_\circ(c) \cap \mathrm{int}_\circ(\sigma) \neq \emptyset$. It follows that $c = \sigma$. Then $\sigma' = c'$ is a cell in $\mathcal{D}'$. Thus $\mathcal{D}' = \{\phi^{-1}(\sigma) : \sigma \in \mathcal{D}\} = \phi^*(\mathcal{D})$. $\square$

Since the restriction of a cellular map on a cell is a homeomorphism, we have the following local property as an immediate corollary of Lemma 3.2

Corollary 3.3. *Let $f\colon \mathfrak{X}' \to \mathfrak{X}$ be a cellular map for $(\mathcal{D}', \mathcal{D})$. Then $\mathcal{D}'|_{c'} = (f|_{c'})^*(\mathcal{D}|_{f(c')})$ for each $c' \in \mathcal{D}'$.*

3.2. **Markov property.** The topological dynamics is based on the following notion of refinement of a cell decomposition.

Definition 3.4 (Refinements). A cell decomposition $\mathcal{D}_1$ of $\mathfrak{X}$ is a *refinement* of a cell decomposition $\mathcal{D}_0$ of $\mathfrak{X}$ if:

(i) For each $\sigma \in \mathcal{D}_1$, there exists $\tau \in \mathcal{D}_0$ satisfying $\sigma \subseteq \tau$.
(ii) Each cell $\tau \in \mathcal{D}_0$ is the union of cells $\sigma \in \mathcal{D}_1$ satisfying $\sigma \subseteq \tau$.

In this case, we also say that $\mathcal{D}_1$ is a *refinement of $\mathcal{D}_0$*.

Let $\mathcal{D}_1$ be a refinement of $\mathcal{D}_0$. For each $\tau \in \mathcal{D}$, we denote

$$\mathcal{D}_1|_\tau := \{\sigma \in \mathcal{D}_1 : \sigma \subseteq \tau\}.$$

It is clear that $\mathcal{D}_1|_\tau$ forms a cell decomposition of τ, and that $\mathcal{D}_1|_\tau$ is a refinement of $\mathcal{D}_0|_\tau$.

We are now ready to define the Markov property of cellular maps.

Definition 3.5 (Cellular Markov partitions, cellular Markov maps). If $f\colon \mathfrak{X} \to \mathfrak{X}$ is cellular for $(\mathcal{D}', \mathcal{D})$ and $\mathcal{D}'$ is a refinement of $\mathcal{D}$, then $(\mathcal{D}', \mathcal{D})$ is a *cellular Markov partition* for f. We call $f\colon \mathfrak{X} \to \mathfrak{X}$ a *cellular Markov map* if there exists a cellular Markov partition for f.

Definition 3.6 (Essential cell decomposition). For a cellular Markov map $f\colon \mathfrak{X} \to \mathfrak{X}$, a cell decomposition $\mathcal{D}$ of $\mathfrak{X}$ is an *essential (cell)*

decomposition of f if there exists a cell decomposition $\mathcal{D}'$ of $\mathfrak{X}$ such that $(\mathcal{D}', \mathcal{D})$ is a cellular Markov partition

Although an essential cell decomposition in Definition 3.6 is not unique, the cell decomposition $\mathcal{D}'$ is uniquely determined by the given essential cell decomposition $\mathcal{D}$. For this observation, we record some basic properties of cellular Markov maps for further use.

Lemma 3.7. *Let $f \colon \mathfrak{X} \to \mathfrak{X}$ be a cellular Markov map and $(\mathcal{D}_1, \mathcal{D}_0)$ be a cellular Markov partition for f. Then the following statements hold:*

(i) *For each $\tau \in \mathcal{D}_0$, we have*

$$f^{-1}(\mathrm{int}_\circ(\tau)) = \bigcup\{\mathrm{int}_\circ(\sigma) : \sigma \in \mathcal{D}_1, \, f(\sigma) = \tau\}.$$

(ii) *For each $\tau \in \mathcal{D}_0$, and each $\sigma \in \mathcal{D}_1$ with $\tau = f(\sigma)$, we have that $\mathrm{int}_\circ(\sigma)$ is a connected component of $f^{-1}(\mathrm{int}_\circ(\tau))$.*

(iii) *$\mathcal{D}_1 = \bigcup_{\tau \in \mathcal{D}_0} \{\bar{c} : c \text{ is a connected component of } f^{-1}(\mathrm{int}_\circ(\tau))\}$.*

Proof. (i) Let $\tau \in \mathcal{D}_0$ be arbitrary. For each $x \in f^{-1}(\mathrm{int}_\circ(\tau))$, suppose that $x \in \mathrm{int}_\circ(\sigma)$ for some $\sigma \in \mathcal{D}_1$ (see Lemma 2.5), then $f(x) \in \mathrm{int}_\circ(\tau) \cap \mathrm{int}_\circ(f(\sigma))$, and thus $f(\sigma) = \tau$. Therefore $f^{-1}(\mathrm{int}_\circ(\tau)) \subseteq \bigcup\{\mathrm{int}_\circ(\sigma) : \sigma \in \mathcal{D}_1, \, f(\sigma) = \tau\}$. On the other hand, it is clear that $\bigcup\{\mathrm{int}_\circ(\sigma) : \sigma \in \mathcal{D}_1, \, f(\sigma) = \tau\} \subseteq f^{-1}(\mathrm{int}_\circ(\tau))$, thus

$$f^{-1}(\mathrm{int}_\circ(\tau)) = \bigcup\{\mathrm{int}_\circ(\sigma) : \sigma \in \mathcal{D}_1, \, f(\sigma) = \tau\}.$$

(ii) Let $\tau \in \mathcal{D}_0$ be arbitrary. We show that if $\sigma, \sigma' \in \mathcal{D}_1$ satisfy $\sigma \neq \sigma'$, $\tau = f(\sigma) = f(\sigma')$, then $\mathrm{int}_\circ(\sigma) \cap \sigma' = \emptyset$. Assume on the contrary that $\mathrm{int}_\circ(\sigma) \cap \sigma' \neq \emptyset$, then, by Lemma 2.6 (ii), we have that $\sigma \subseteq \sigma'$. If $\sigma \cap \mathrm{int}_\circ(\sigma') \neq \emptyset$, then $\sigma' \subseteq \sigma$, which contradicts $\sigma \neq \sigma'$. Hence $\sigma \subseteq \partial_\circ \sigma'$, which contradicts $\dim(\sigma) = \dim(\sigma')$. Therefore $\mathrm{int}_\circ(\sigma) \cap \sigma' = \emptyset$.

Fix arbitrary $\sigma \in \mathcal{D}_1$ with $\tau = f(\sigma)$. Since $\sigma \cap \mathrm{int}_\circ(\sigma') = \emptyset$ for each $\sigma' \in \mathcal{D}_1$ with $\tau = f(\sigma')$ and $\sigma' \neq \sigma$, it follows from (i) that $\mathrm{int}_\circ(\sigma) = \sigma \cap f^{-1}(\mathrm{int}_\circ(\tau))$. Since σ is a closed subset of $\mathfrak{X}$ (see Lemma 2.1), $\mathrm{int}_\circ(\sigma) = \sigma \cap f^{-1}(\mathrm{int}_\circ(\tau))$ is a closed subset of $f^{-1}(\mathrm{int}_\circ(\tau))$.

Then $\{\mathrm{int}_\circ(\sigma) : \sigma \in \mathcal{D}_1, \, f(\sigma) = \tau\}$ is a collection of closed subset of $f^{-1}(\mathrm{int}_\circ(\tau))$ that are pairwise disjoint. Hence for each $c \in \mathcal{D}_1$ with $f(c) = \tau$, $\mathrm{int}_\circ(c)$ is a connected component of $f^{-1}(\mathrm{int}_\circ(\tau))$.

(iii) For each $\sigma \in \mathcal{D}_1$, we have that $\sigma = \overline{\mathrm{int}_\circ(\sigma)}$ by Lemma 2.1, and that $\mathrm{int}_\circ(\sigma)$ is a connected component of $f^{-1}(\mathrm{int}_\circ(f(\sigma)))$ by (ii).

On the other hand, for each $\tau \in \mathcal{D}_0$ and each connected component c of $f^{-1}(\mathrm{int}_\circ(\tau))$, by (i) and (ii) we have that $c = \mathrm{int}_\circ(\sigma)$ for some $\sigma \in \mathcal{D}_1$. Thus $\bar{c} = \sigma \in \mathcal{D}_1$.   □

Corollary 3.8. *Let $f\colon \mathfrak{X} \to \mathfrak{X}$ be a cellular Markov map, and $\mathcal{D}$ be an essential cell decomposition of f. Then there exists a unique cell decomposition $\mathcal{D}'$ of $\mathfrak{X}$ such that $(\mathcal{D}', \mathcal{D})$ is a cellular Markov partition of f.*

We call the cell decomposition $\mathcal{D}'$ the *pull back* of $\mathcal{D}$ under f.

Definition 3.9. The *pull back $f^*(\mathcal{D})$ of an essential cell decomposition $\mathcal{D}$ of a cellular Markov map $f\colon \mathfrak{X} \to \mathfrak{X}$* is the unique cell decomposition of $\mathfrak{X}$ for which $(\mathcal{D}', \mathcal{D})$ is a cellular Markov partition of f.

Remark. Note that, if a cellular Markov map $f\colon \mathfrak{X} \to \mathfrak{X}$ is a homeomorphism, then the definitions in (3.1) and Definition 3.9 for a pull back $f^*(\mathcal{D})$ agree.

We are now ready to show that a cellular Markov map $\mathfrak{X} \to \mathfrak{X}$ satisfies the essential sequence condition in the definition in the Introduction. First, we give a formal definition of an essential sequence, which is equivalent to the definition in the Introduction. Then, we establish the existence.

Definition 3.10. A sequence $\{\mathcal{D}_m\}_{m\in\mathbb{N}}$ of cell decompositions of $\mathfrak{X}$ is *an essential sequence* for a cellular Markov map $f\colon \mathfrak{X} \to \mathfrak{X}$ if, for each $m \in \mathbb{N}$, $(\mathcal{D}_m, \mathcal{D}_{m-1})$ is a cellular Markov partition of f. We also say that $\{\mathcal{D}_m\}_{m\in\mathbb{N}}$ is an *essential sequence in $\mathfrak{X}$* if $\{\mathcal{D}_m\}_{m\in\mathbb{N}}$ is an essential sequence for a cellular Markov map $\mathfrak{X} \to \mathfrak{X}$.

Proposition 3.11. *Let $f\colon \mathfrak{X} \to \mathfrak{X}$ be a cellular Markov map and $\mathcal{D}$ an essential decomposition for f. Then there exists a unique essential sequence $\{\mathcal{D}_m\}_{m\in\mathbb{N}}$ for f satisfying $\mathcal{D}_0 = \mathcal{D}$.*

The key is to show that the pull back of an essential decomposition of a cellular Markov map f is also an essential decomposition with respect f.

Lemma 3.12. *Let $f\colon \mathfrak{X} \to \mathfrak{X}$ be a cellular Markov map, and $(\mathcal{D}_1, \mathcal{D}_0)$ be a cellular Markov partition for f, then*

$$\mathcal{D}_2 := \bigcup_{\sigma \in \mathcal{D}'} (f|_\sigma)^*(\mathcal{D}_1|_{f(\sigma)})$$

is a cell decomposition of $\mathfrak{X}$ and $(\mathcal{D}_2, \mathcal{D}_1)$ is a cellular Markov partition for f. Hence $\mathcal{D}_1$ is an essential decomposition of f, and $\mathcal{D}_2 = f^*(\mathcal{D}_1)$.

Proof of Proposition 3.11 assuming Lemma 3.12. We inductively define, for each $m \in \mathbb{N}$, a cell decomposition $\mathcal{D}_m = f^*(\mathcal{D}_{m-1})$. By Lemma 3.12, each $(\mathcal{D}_m, \mathcal{D}_{m-1})$ is a cellular Markov partition. $\square$

To prove Lemma 3.12, we recall a cell-wise minimality property for refinements from [BM17].

Lemma 3.13 ([BM17, Lemma 5.7]). *Let $\mathcal{D}$ be a cell decomposition of a locally compact Hausdorff space $\mathfrak{X}$ and let $\mathcal{D}'$ be a refinement of $\mathcal{D}$. Then, for each $\sigma \in \mathcal{D}'$, there exists a minimal cell $\tau \in \mathcal{D}$ with $\sigma \subseteq \tau$, i.e., if $\sigma \subseteq \widetilde{\tau}$ for some $\widetilde{\tau} \in \mathcal{D}$, then $\tau \subseteq \widetilde{\tau}$. Moreover, τ is the unique cell in $\mathcal{D}$ with $\mathrm{int}_\circ(\sigma) \subseteq \mathrm{int}_\circ(\tau)$.*

Proof of Lemma 3.12. It is clear that $\mathcal{D}_2$ is a collection of cells. We verify conditions (i) through (iv) in Definition 2.4 one by one.

(i) The union of cells in $\mathcal{D}_1$ is $\mathfrak{X}$. It follows from Definition 3.9 that each cell in $\mathcal{D}_1$ is a union of cells in $\mathcal{D}_2$. Therefore, $\mathfrak{X} = \bigcup \mathcal{D}_2$.

(ii) Let σ, $\tau \in \mathcal{D}_2$ with $\sigma \neq \tau$ be arbitrary. By Definition 3.9, $f(\sigma)$ and $f(\tau)$ are cells in $\mathcal{D}_1$. If $f(\sigma) \neq f(\tau)$, then $\mathrm{int}_\circ(f(\sigma)) \cap \mathrm{int}_\circ(f(\tau)) = \emptyset$, and thus $\mathrm{int}_\circ(\sigma) \cap \mathrm{int}_\circ(\tau) = \emptyset$.

In what follows, assume that $f(\sigma) = f(\tau)$, and denote $c_1 \coloneqq f(\sigma) = f(\tau)$. Suppose that c_0 is the unique cell in $\mathcal{D}_0$ with $\mathrm{int}_\circ(c_1) \subseteq \mathrm{int}_\circ(c_0)$ and $c_1 \subseteq c_0$ (see Lemma 3.13). Suppose that $\sigma \in (f|_{\sigma_1})^*(\mathcal{D}_1|_{f(\sigma_1)})$ and $\tau \in (f|_{\tau_1})^*(\mathcal{D}_1|_{f(\tau_1)})$ for σ_1, $\tau_1 \in \mathcal{D}_1$, then $\sigma \subseteq \sigma_1$ and $\tau \subseteq \tau_1$. Hence $\mathrm{int}_\circ(c_0) \cap f(\sigma_1) \cap f(\tau_1) \neq \emptyset$. By Lemma 2.6 (i), $f(\sigma_1) \cap f(\tau_1)$ is a union of cells in $\mathcal{D}_0$, then it follows from Lemma 2.6 (ii) that $c_0 \subseteq f(\sigma_1) \cap f(\tau_1)$. By Corollary 3.3, $\mathcal{D}_1|_{\sigma_1} = (f|_{\sigma_1})^*(\mathcal{D}_0|_{f(\sigma_1)})$. Thus, there is $\sigma_1' \in \mathcal{D}_1|_{\sigma_1}$ satisfying $f(\sigma_1') = c_0$. By the same argument, let $\tau_1' \in \mathcal{D}_1|_{\tau_1}$ satisfies $f(\sigma_1') = c_0$. Then we have that $\sigma \subseteq \sigma_1'$, $\tau \subseteq \tau_1'$, $\mathrm{int}_\circ(\sigma) \subseteq \mathrm{int}_\circ(\sigma_1')$, and $\mathrm{int}_\circ(\tau) \subseteq \mathrm{int}_\circ(\tau_1')$. If $\sigma_1' = \tau_1' = c_1'$ for some $c_1' \in \mathcal{D}_1$, then since $f|_{c_1'} \colon c_1' \to c_0$ is a homeomorphism, we have $\sigma = (f|_{c_1'})^{-1}(c_1) = \tau$, which contradicts $\tau \neq \sigma$. Therefore, we have that $\sigma_1' \neq \tau_1'$, $\mathrm{int}_\circ(\sigma_1') \cap \mathrm{int}_\circ(\tau_1') = \emptyset$, and thus $\mathrm{int}_\circ(\sigma) \cap \mathrm{int}_\circ(\tau) = \emptyset$.

(iii) Let $\sigma \in \mathcal{D}_2$ be arbitrary. Set $\sigma \in (f|_\tau)^*(\mathcal{D}_1|_{f(\tau)})$ for some $\tau \in \mathcal{D}_1$. Since $\partial_\circ f(\sigma)$ is a union of cells in $\mathcal{D}_1|_{f(\tau)}$, we have that $\partial_\circ \sigma = (f|_\tau)^{-1}(\partial_\circ f(\sigma)) = \bigcup \{(f|_\tau)^{-1}(\sigma') : \sigma' \in \mathcal{D}_1, \ \sigma' \subseteq \partial_\circ f(\sigma)\}$ is a union of cells in $\mathcal{D}_2$.

(iv) We first show that, for each $\tau \in \mathcal{D}_0$, the set $\mathcal{D}_1|_\tau$ is a finite collection of cells in $\mathcal{D}_1$. Fix an arbitrary $\tau \in \mathcal{D}_0$. For each $x \in \tau$, let U_x be a neighborhood of x that meets only finitely many cells in $\mathcal{D}_1$. Since τ is a compact subset of $\mathfrak{X}$, consider a subset $\{x_1, \ldots, x_k\} \subseteq \tau$ that satisfies $\tau \subseteq U_{x_1} \cup \cdots \cup U_{x_k}$. If $\mathcal{D}_1|_\tau$ is infinite, then there exists U_{x_i} that meets infinitely many cells in $\mathcal{D}_1$, which leads to a contradiction. Hence $\mathcal{D}_1|_\tau$ is finite.

As a consequence, $(f|_\sigma)^*(\mathcal{D}_1|_{f(\sigma)})$ is finite for each $\sigma \in \mathcal{D}_1$. Fix arbitrary $x \in \mathfrak{X}$, and consider a neighborhood U of x that meets only

finitely many cells in $\mathcal{D}_1$. For each $c \in \mathcal{D}_2$ with $c \cap U \neq \emptyset$, there exists $\sigma \in \mathcal{D}_1$ with $c \in (f|_\sigma)^*(\mathcal{D}_1|_{f(\sigma)})$. Since

$$\bigcup_{\sigma \in \mathcal{D}_1, \, \sigma \cap U \neq \emptyset} (f|_\sigma)^*(\mathcal{D}_1|_{f(\sigma)})$$

is a finite collection of cells in $\mathcal{D}_2$, we have that U meets only finitely many cells in $\mathcal{D}_2$.

Now, we have verified that $\mathcal{D}_2$ is a cell decomposition of $\mathfrak{X}$. It is clear from the construction of $\mathcal{D}_2$ that $\mathcal{D}_2$ is a refinement of $\mathcal{D}_1$ and that f is cellular for $(\mathcal{D}_2, \mathcal{D}_1)$. Hence $(\mathcal{D}_2, \mathcal{D}_1)$ is a cellular Markov partition for f. Now the proof is complete. $\square$

3.3. **Cellular Markov branched covers.** Recall that, in the quasiregular literature, a discrete, open, and sense-preserving map $f \colon \mathcal{S}^n \to \mathcal{S}^n$ is called a *(generalized) branched cover*; see e.g. Heinonen and Rickman [HR02] for a detailed discussion on these assumptions in the theory of branched covers.

Having this definition of branched covers, we may define that a *cellular Markov branched cover* $f \colon \mathfrak{X} \to \mathfrak{X}$ is a branched cover that is also a cellular Markov map. By Proposition 3.11, this definition is equivalent to the definition given in the introduction. Note that, a cellular Markov branched cover on $\mathcal{S}^n$ is a closed map, also it is surjective since $\mathcal{S}^n$ is connected.

Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov branched cover and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ an essential sequence of f. For $m \in \mathbb{N}_0$, we call a vertex, a face, and a room in $\mathcal{D}_m$ a *level-m vertex*, a *level-m face*, and a *level-m room*, respectively. We call an element in $\mathcal{F}(\mathcal{D}_m)$ a level-m flower, and denote the level-m flower of p by $\mathrm{Fl}_m(p)$. It is clear that for all $m, k \in \mathbb{N}_0$, each level-m vertex is a level-$(m+k)$ vertex.

The following proposition gives the basic forward and backward invariance properties of flowers under cellular Markov branched covers.

Proposition 3.14. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov branched cover and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ an essential sequence of f. Then the following statements are true for all $m, k \in \mathbb{N}_0$:*

(i) *If p is a level-$(m+k)$ vertex, then $f^k(\mathrm{Fl}_{m+k}(p)) = \mathrm{Fl}_m\big(f^k(p)\big)$.*

(ii) *If q is a level-m vertex, then the connected components of $f^{-k}(\mathrm{Fl}_m(q))$ are the level-$(m+k)$ flowers $\mathrm{Fl}_{m+k}(p)$, $p \in f^{-k}(q)$.*

We separate a technical lemma for the proof of Proposition 3.14.

Lemma 3.15. *Let $f\colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov branched cover and $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ an essential sequence of f. Let $m,\, k \in \mathbb{N}_0$ be arbitrary. Let also p be a level-$(m+k)$ vertex, and $Y \in \mathcal{D}_m^{[n]}$ satisfies $Y \cap f^k(\mathrm{Fl}_{m+k}(p)) \neq \emptyset$. Then the following are true:*

   (i) $\mathrm{int}_\circ(Y) \cap f^k(\mathrm{Fl}_{m+k}(p)) \neq \emptyset$.

   (ii) *There exists $\widetilde{Y} \in \mathcal{D}_{m+k}^{[n]}$ with $p \in \widetilde{Y}$ such that $Y = f^k\big(\widetilde{Y}\big)$.*

Proof. (i) By Lemma 2.10 (i), the subset $\mathrm{Fl}_{m+k}(p)$ is open. Since f^k is an open map, also $f^k(\mathrm{Fl}_{m+k}(p))$ is open. Then, since $Y = \overline{\mathrm{int}_\circ(Y)}$, the claim follows.

(ii) By (i), suppose that $\widetilde{Y} \in \mathcal{D}_{m+k}^{[n]}$ satisfies $p \in \widetilde{Y}$ and $f^k\big(\widetilde{Y}\big) \cap \mathrm{int}_\circ(Y) \neq \emptyset$. Since $\dim(Y) = \dim\big(f^k\big(\widetilde{Y}\big)\big) = n$, it follows from Lemma 2.6 that $Y = f^k\big(\widetilde{Y}\big)$. $\qquad\qquad\square$

We also recall an elementary topological fact; see e.g. [BM17, Lemma 5.4].

Lemma 3.16. *Let $A \subseteq \mathfrak{X}$ be a closed set of a locally compact Hausdorff space $\mathfrak{X}$, and $U \subseteq \mathfrak{X} \smallsetminus A$ be a non-empty connected open set with $\partial U \subseteq A$. Then U is a connected component of $\mathfrak{X} \smallsetminus A$.*

Proof of Proposition 3.14. (i) Let $m,k \in \mathbb{N}_0$ and let $p \in \mathcal{D}_m$ be a vertex. We denote $q \coloneqq f^k(p)$. Let $X \in \mathcal{D}_m$ be a cell containing q. Let also $Y \in \mathcal{D}_m^{[n]}$ be a level-m room containing X. Then $Y \cap f^k(\mathrm{Fl}_{m+k}(p)) \neq \emptyset$ since $q \in Y$. It follows from Lemma 3.15 (ii) that there exists $\widetilde{Y} \in \mathcal{D}_{m+k}^{[n]}$ satisfies $p \in \widetilde{Y}$ and $Y = f^k\big(\widetilde{Y}\big)$.

Since f^k is cellular for $(\mathcal{D}_{m+k}, \mathcal{D}_m)$, we have by Corollary 3.3 that $\mathcal{D}_{m+k}|_{\widetilde{Y}} = (f^k|_{\widetilde{Y}})^*(\mathcal{D}_m|_Y)$. Denote $\widetilde{X} \coloneqq (f^k|_{\widetilde{Y}})^{-1}(X) \in \mathcal{D}_{m+k}|_{\widetilde{Y}}$, then $\mathrm{int}_\circ(X) = f^k\big(\mathrm{int}_\circ\big(\widetilde{X}\big)\big) \subseteq f^k(\mathrm{Fl}_{m+k}(p))$. Since $X \in \mathcal{D}_m$ with $q \in X$ is arbitrary, we have $\mathrm{Fl}_m(q) \subseteq f^k(\mathrm{Fl}_{m+k}(p))$. On the other hand, it is clear that $f^k(\mathrm{Fl}_{m+k}(p)) \subseteq \mathrm{Fl}_m(q)$, then the proof of (i) is complete.

(ii) It is clear that $f^{-k}(q)$ is a collection of level-$(m+k)$ vertices. We first fix an arbitrary $p \in f^{-k}(q)$, and consider $\mathrm{Fl}_{m+k}(p)$. By Lemma 2.10 (i), the subset $\mathrm{Fl}_{m+k}(p)$ is open. Since $\mathrm{int}_\circ(c)\cup\{p\}$ is connected for each $c \in \mathcal{D}_m$ with $p \in c$, we get that $\mathrm{Fl}_{m+k}(p)$ is connected. By Lemma 3.16, It suffices to show that $\partial\,\mathrm{Fl}_{m+k}(p) \subseteq \mathcal{S}^n \smallsetminus f^{-k}(\mathrm{Fl}_m(q))$. Suppose on the contrary that $f^{-k}(\mathrm{Fl}_m(q)) \cap \partial\,\mathrm{Fl}_{m+k}(p) \neq \emptyset$. By Lemma 2.10 (ii), there exists $c \in \mathcal{D}_{m+k}$ with the property that $c \cap f^{-k}(\mathrm{Fl}_m(q)) \neq \emptyset$, $p \notin c$, and $c \subseteq X$ for some $X \in \mathcal{D}_m$ with $p \in X$. Then $f^k(c) \subseteq f^k(X)$, and $q \notin f^k(c)$. By Lemma 2.10 (i), we have

$f^k(c) \subseteq \mathcal{S}^n \smallsetminus \mathrm{Fl}_m(q)$, which contradicts $c \cap f^{-k}(\mathrm{Fl}_m(q)) \neq \emptyset$. Therefore, $\partial \mathrm{Fl}_{m+k}(p) \subseteq \mathcal{S}^n \smallsetminus f^{-k}(\mathrm{Fl}_m(q))$, and thus $\mathrm{Fl}_{m+k}(p)$ is a connected component of $f^{-k}(\mathrm{Fl}_m(q))$.

Conversely, we denote by $\mathcal{U}$ the collection of all connected components of $f^{-k}(\mathrm{Fl}_m(q))$. Then each element of $\mathcal{U}$ is an open subset of $\mathcal{S}^n$. Suppose that $U \in \mathcal{U}$, and denote

$$F := U \cup \left(\mathcal{S}^n \smallsetminus f^{-k}(\mathrm{Fl}_m(q)) \right) = \mathcal{S}^n \smallsetminus \bigcup_{V \in \mathcal{U} \smallsetminus \{U\}} V.$$

Clearly F is a closed subset of $\mathcal{S}^n$.

Since f^k is open, the subset $f^k(U)$ is an open subset of $\mathrm{Fl}_m(q)$. It is easy to veryfy that f^k is closed, and thus $f^k(U) = f^k(F) \cap \mathrm{Fl}_m(q)$ is a closed subset of $\mathrm{Fl}_m(q)$. Then by the connectivity of $\mathrm{Fl}_m(q)$ we have $f^k(U) = \mathrm{Fl}_m(q)$. Let $p \in U$ satisfies $f^k(p) = q$. Since $\mathrm{Fl}_{m+k}(p)$ is a connected component of $f^{-k}(\mathrm{Fl}_m(q))$, we have that $U = \mathrm{Fl}_{m+k}(p)$. $\square$

As an immediate corollary of Proposition 3.14, we have the following containment property for connected sets under the Markov branched covers.

Corollary 3.17. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov branched covers and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence of f. Then for all $m, k \in \mathbb{N}_0$, a connected set $A \subseteq \mathcal{S}^n$ is contained in a level-$(m+k)$ flower if and only if $f^k(A)$ is contained in a level-m flower.*

Proof. Let $A \subseteq \mathcal{S}^n$ be a connected subset.

If A is contained in $\mathrm{Fl}_{m+k}(p)$ for some level-$(m+k)$ vertex p, then, by Proposition 3.14 (i), we have that $f^k(A)$ is contained in $f^k(\mathrm{Fl}_{m+k}(p)) = \mathrm{Fl}_m(f^k(p))$.

Conversely, if $f^k(A)$ is contained in $\mathrm{Fl}_m(q)$ for some level-m vertex q, then A is contained in some connected component of $f^{-k}(\mathrm{Fl}_m(q))$. By Proposition 3.14 (ii), we have that A is contained in some level-$(m+k)$ flower. $\square$

As a second corollary, we obtain that flowers are normal domains for cellular Markov branched covers. Recall that a domain $U \subseteq \mathcal{S}^n$ is a normal domain of f if $f(\partial U) = \partial f(U)$.

Corollary 3.18. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov branched covers and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence of f. Let $m, k \in \mathbb{N}_0$, and let p be a level-$(m+k)$ vertex. Then we have $f^k(\partial \mathrm{Fl}_{m+k}(p)) = \partial \mathrm{Fl}_m\left(f^k(p)\right)$.*

Proof. First suppose that $c \in \mathcal{D}_{m+k}$ and $X \in \mathcal{D}^{[n]}_{m+k}$ satisfy $c \subseteq X$ and $p \in X \smallsetminus c$. Then, since the restriction $f^k|_X \colon X \to f^k(X)$ is a

homeomorphism, we have $f^k(c) \subseteq f^k(X)$, and $f^k(p) \in f^k(X) \smallsetminus f^k(c)$. Then, by Lemma 2.10 (ii), we have $f^k(\partial \operatorname{Fl}_{m+k}(p)) \subseteq \partial \operatorname{Fl}_m(f^k(p))$.

For the other direction, suppose now that $c \in \mathcal{D}_m$, $X \in \mathcal{D}_m^{[n]}$ satisfy $c \subseteq X$ and $f^k(p) \in X \smallsetminus c$. By Lemma 3.15 (ii), we may assume that $\widetilde{X} \in \mathcal{D}_{m+k}^{[n]}$ satisfies $p \in \widetilde{X}$ and $f^k(\widetilde{X}) = X$. Consider $\widetilde{c} := (f^k|_{\widetilde{X}})^{-1}(c)$, then $\widetilde{c} \in \mathcal{D}_{m+k}|_{\widetilde{X}}$ satisfies $p \notin \widetilde{c}$ and $f^k(\widetilde{c}) = c$. It follows from Lemma 2.10 (ii) that $\partial \operatorname{Fl}_m(f^k(p)) \subseteq f^k(\partial \operatorname{Fl}_{m+k}(p))$. $\qquad\square$

We also obtain an invariance property of nested flowers at a vertex.

Lemma 3.19. *Let* $f \colon \mathcal{S}^n \to \mathcal{S}^n$ *be a cellular Markov branched cover, and* $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ *be an essential sequence of* f*. Then, for all* m*,* m_0*,* $l \in \mathbb{N}_0$ *with* $m_0 \leqslant m$*, and each level-m vertex* p*, we have that* $\operatorname{Fl}_{m+l}(p) \subseteq \operatorname{Fl}_m(p)$*, and that*

$$\overline{\operatorname{Fl}_{m+l}(p)} = \overline{\operatorname{Fl}_m(p)} \cap f^{-(m-m_0)}\big(\overline{\operatorname{Fl}_{m_0+l}(f^{m-m_0}(p))}\big).$$

Proof. Fix arbitrary m, m_0, $l \in \mathbb{N}_0$ with $m_0 \leqslant m$, and a level-m vertex p. It is clear that p is also a level-$(m+l)$ vertex. Denote $k := m - m_0$ and $q := f^k(p)$ for simplicity.

For each $c \in \mathcal{D}_{m+l}$ with $p \in c$, by Lemma 3.13, there exists $\sigma \in \mathcal{D}_m$ for which $\operatorname{int}_{\circ}(c) \subseteq \operatorname{int}_{\circ}(\sigma)$, and thus $p \in \sigma$. Since $\operatorname{Fl}_m(p) = \bigcup\{\operatorname{int}_{\circ}(c) : c \in \mathcal{D}_m,\ p \in c\}$, we have $\operatorname{Fl}_{m+l}(p) \subseteq \operatorname{Fl}_m(p)$.

It is clear that $\overline{\operatorname{Fl}_{m+l}(p)} \subseteq \overline{\operatorname{Fl}_m(p)} \cap f^{-k}\big(\overline{\operatorname{Fl}_{m_0+l}(q)}\big)$. On the other hand, suppose that $x \in \overline{\operatorname{Fl}_m(p)}$ satisfies $f^k(x) \in \overline{\operatorname{Fl}_{m_0+l}(q)}$. Consider $Y \in \mathcal{D}_{m_0+l}^{[n]}$ with $q \in Y$ and $f^k(x) \in Y$. Consider also $Y' \in \mathcal{D}_{m_0}^{[n]}$ with $Y \subseteq Y'$. By Lemma 3.15 (ii), there is $X' \in \mathcal{D}_m^{[n]}$ with $p \in X'$ such that $f^k(X') = Y'$. Since $\mathcal{D}_{m+l}|_{X'} = (f^k|_{X'})^*(\mathcal{D}_{m_0+l}|_{Y'})$, we have that $X := (f^k|_{X'})^{-1}(Y) \in \mathcal{D}_{m+l}^{[n]}$ satisfies $p \in X$, and $x \in X$. Thus $x \in \overline{\operatorname{Fl}_{m+l}(p)}$. It follows that $\overline{\operatorname{Fl}_{m+l}(p)} \supseteq \overline{\operatorname{Fl}_m(p)} \cap f^{-k}\big(\overline{\operatorname{Fl}_{m_0+l}(q)}\big)$, since x is arbitrary. $\qquad\square$

We finish this section with a combinatorial distance property for different essential sequences $\{\mathcal{D}_m\}_{m \in \mathbb{N}}$ and $\{\mathcal{C}_m\}_{m \in \mathbb{N}}$ of the same cellular Markov branched cover on $\mathcal{S}^n$. This property yields the independence of the gauge of the visual metrics, discussed in the next section, on the choice of the essential sequence.

Proposition 3.20. *Let* $f \colon \mathcal{S}^n \to \mathcal{S}^n$ *be a cellular Markov branched cover, and let* $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ *and* $\{\mathcal{C}_m\}_{m \in \mathbb{N}_0}$ *be essential sequences of* f*. Then there exists* $M \in \mathbb{N}$ *having the following property: for each* $m \in \mathbb{N}_0$ *and each* $X \in \mathcal{C}_m$*, there exists a cover* $\mathcal{U}$ *of* X *by flowers in* $\mathcal{F}(\mathcal{D}_m)$ *for which* $\operatorname{card}(\mathcal{U}) \leqslant M$*.*

Proof. For each $X \in \mathcal{C}_0$, we fix a finite cover $\mathcal{A}_X$ of X by connected subsets of X such that each element of $\mathcal{A}_X$ is contained in some flower in $\mathcal{F}(\mathcal{D}_0)$. For example, one may consider

$$\mathcal{A}_X := \{\phi_X^{-1}(I) : I \in \{[0, 1/h], [1/h, 2/h], \ldots, [(h-1)/h, 1]\}^n\},$$

where $\phi_X \colon X \to [0,1]^d$, $d := \dim(X)$, is a homeomorphism and $h \in \mathbb{N}$ is sufficiently large. Denote $M := \max\{\mathrm{card}(\mathcal{A}_X) : X \in \mathcal{C}_0\}$.

Fix arbitrary $m \in \mathbb{N}_0$, and $X \in \mathcal{C}_m$. Denote $Y := f^m(X) \in \mathcal{C}_0$. Since $f^m|_X \colon X \to Y$ is a homeomorphism, $\mathcal{A}_X := \{(f^m|_X)^{-1}(A) : A \in \mathcal{A}_Y\}$ is a finite cover of X by connected subset of X with $\mathrm{card}(\mathcal{A}_X) = \mathrm{card}(\mathcal{A}_Y)$. Since each element of $\mathcal{A}_Y$ is contained in a flower in $\mathcal{F}(\mathcal{D}_0)$, it follows from Corollary 3.17 that each element in $\mathcal{A}_X$ is contained in a flower in $\mathcal{F}(\mathcal{D}_m)$. Hence X can be covered by not more than M flowers in $\mathcal{F}(\mathcal{D}_m)$. □

3.4. **Local multiplicity.** At the end of this section, we consider the (local) multiplicity of a cellular Markov map in terms of its combinatorial data. First, we fix some notations.

For a map $f \colon \mathfrak{X} \to \mathfrak{Y}$ and a subset $A \subseteq \mathfrak{X}$, we denote

$$(3.2) \qquad N(f, A) := \sup\{\mathrm{card}(f^{-1}(y) \cap A) : y \in f(A)\}.$$

The *local multiplicity* of f at a point $x \in \mathfrak{X}$ is defined as

$$(3.3) \qquad N_{\mathrm{loc}}(f, x) := \inf\{N(f, U) : U \subseteq \mathfrak{X} \text{ is a neighborhood of } x\}.$$

The notion is also named local index, local degree, etc., in different contexts.

Given a cell decomposition $\mathcal{D}$ of $\mathcal{S}^n$, we denote

$$(3.4) \qquad N(\mathcal{D}, x) := \mathrm{card}(\{X \in \mathcal{D}^{[n]} : x \in X\}).$$

For a cellular Markov map $f \colon \mathcal{S}^n \to \mathcal{S}^n$ and a cellular Markov partition $(\mathcal{D}_1, \mathcal{D}_0)$, we have a two-sided bound for local multiplicity.

Lemma 3.21. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov map, and $(\mathcal{D}_1, \mathcal{D}_0)$ be a cellular Markov partition. Then for each $x \in \mathcal{S}^n$,*

$$N_{\mathrm{loc}}(f, x) \leqslant N(\mathcal{D}_1, x) \leqslant N(\mathcal{D}_0, f(x)) \cdot N_{\mathrm{loc}}(f, x).$$

Proof. Fix arbitrary $x \in \mathcal{S}^n$. Denote $\mathcal{V}_1 := \{X \in \mathcal{D}_1^{[n]} : x \in X\}$ and $V := \bigcup \mathcal{V}_1$. Denote also $\mathcal{V}_0 := \{Y \in \mathcal{D}_0^{[n]} : f(x) \in Y\}$. It is clear that $\{f(X) : X \in \mathcal{V}_1\} \subseteq \mathcal{V}_0$.

First, we show that $x \in \mathrm{int}(V)$. Indeed, since $V' := \bigcup\{X \in \mathcal{D}_1^{[n]} : x \notin X\}$ is a closed subset of $\mathcal{S}^n$ with $x \notin V'$, the complement $\mathcal{S}^n \smallsetminus V'$ is an open neighborhood of x contained in V. Then, since $f|_X$ is injective for each $X \in \mathcal{V}_1$, we have that $N(f, V) \leqslant \mathrm{card}(\mathcal{V}_1) = N(\mathcal{D}_1, x)$, and

thus $N_{\mathrm{loc}}(f, x) \leqslant N(f, V) \leqslant N(\mathcal{D}_1, x)$. The first inequality is now verified.

Now we verify the second inequality. For each $Y \in \mathcal{V}_0$, set $\mathcal{U}(Y) := \{X \in \mathcal{V}_1 : f(X) = Y\}$. Then, it is clear that

$$\mathrm{card}(\mathcal{V}_1) = \sum_{Y \in \mathcal{V}_0} \mathrm{card}(\mathcal{U}(Y)) \leqslant \mathrm{card}(\mathcal{V}_0) \cdot \max_{Y \in \mathcal{V}_0}\{\mathrm{card}(\mathcal{U}(Y))\}.$$

Suppose that $Y_0 \in \mathcal{V}_0$ satisfies $\mathrm{card}(\mathcal{U}(Y_0)) \geqslant \mathrm{card}(\mathcal{V}_1)/\mathrm{card}(\mathcal{V}_0)$.

Let U be an arbitrary open neighborhood of x. Then $W := \bigcap\{f(U \cap \mathrm{int}_\circ(X)) : X \in \mathcal{U}(Y_0)\}$ is a non-empty open subset of Y_0, and thus $W \subseteq \mathrm{int}_\circ(Y_0)$. Set $y \in W$, then for each $X \in \mathcal{U}(Y_0)$, there exists a preimage of y in $\mathrm{int}_\circ(X)$. Thus $\mathrm{card}(f^{-1}(y) \cap U) \geqslant \mathrm{card}(\mathcal{U}(Y_0)) \geqslant \mathrm{card}(\mathcal{V}_1)/\mathrm{card}(\mathcal{V}_0)$. Then the second inequality follows. □

Remark. In particular, we have that $N(\mathcal{D}_1, x) \leqslant \mathrm{card}(\mathcal{D}_0) \cdot N_{\mathrm{loc}}(f, x)$.

In addition, we show that the upper bound $\sup_{x \in \mathcal{S}^n} N(\mathcal{D}, x)$ is attained only if x is a vertex.

Lemma 3.22. *Let $\mathcal{D}$ be a cell decomposition of $\mathcal{S}^n$. Then for each $x \in \mathcal{S}^n$, there exists a vertex p with $N(\mathcal{D}, p) \geqslant N(\mathcal{D}, x)$.*

Proof. Let $x \in \mathcal{S}^n$ be arbitrary. Suppose that $x \in \mathrm{int}_\circ(c)$ for some $c \in \mathcal{D}$ and p be a vertex with $p \in c$. For each $X \in \mathcal{D}^{[n]}$, if $x \in X$, then since $\mathrm{int}_\circ(c) \cap X \neq \emptyset$, it follows from Lemma 2.6 that $c \subseteq X$, and thus $p \in X$. Then the claim follows. □

Lemmas 3.21 and 3.22 yield the following corollary.

Corollary 3.23. *Let $f : \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov map and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence of f. Then the following are equivalent:*

(i) $\sup\{N_{\mathrm{loc}}(f^m, x) : m \in \mathbb{N}, \ x \in \mathcal{S}^n\} < +\infty$.

(ii) $\sup\{N(\mathcal{D}_m, x) : m \in \mathbb{N}, \ x \in \mathcal{S}^n\} < +\infty$.

(iii) $\sup\{N(\mathcal{D}_m, p) : m \in \mathbb{N}, \ p \text{ is a vertex of } \mathcal{D}_m\} < +\infty$.

4. Visual metric

4.1. **Expansion.** Now we introduce *expanding cellular Markov branched covers*. Here the term *expanding* does not agree with distance expanding in the usual sense. More precisely, it refers to the expansion property in the sense of essential sequences.

Definition 4.1 (Expansion). An essential sequence $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ of a cellular Markov map on $\mathcal{S}^n$ is *expanding* if for each open cover $\mathcal{U}$ of $\mathcal{S}^n$, there exists $m_0 \in \mathbb{N}_0$ with the property that for each $m > m_0$, and each

$X \in \mathcal{D}_m$, it holds $X \subseteq U$ for some $U \in \mathcal{U}$. Moreover, a cellular Markov map $f \colon \mathcal{S}^n \to \mathcal{S}^n$ is *expanding* if there exists an expanding essential sequence of f.

For a metric n-sphere $(\mathcal{S}^n, d)$, we denote

$$\mathrm{mesh}_d(\mathcal{D}) \coloneqq \max\{\mathrm{diam}_d(c) : c \in \mathcal{D}\}$$

for a cell decomposition $\mathcal{D}$ of $(\mathcal{S}^n, d)$. If d is clear from the context, we may write $\mathrm{mesh}(\mathcal{D}) \coloneqq \mathrm{mesh}_d(\mathcal{D})$. When $\mathcal{S}^n$ is equipped with a metric compatible with its topology, the expansion of essential sequences can be characterized in terms of metric.

Proposition 4.2 (Expansion in the metric sense). *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov map and $\{\mathcal{D}_m\}$ be an essential sequence. Let d be a metric on $\mathcal{S}^n$ compatible with the given topology of $\mathcal{S}^n$. Then $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ is expanding if and only if $\mathrm{mesh}_d(\mathcal{D}_m) \to 0$ as $m \to +\infty$.*

Proof. First assume that $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ is expanding. For each $\epsilon > 0$, consider the open cover $\mathcal{U}_\epsilon \coloneqq \{B_d(x, \epsilon) : x \in \mathcal{S}^n\}$, then $\mathrm{diam}_d(X) \leqslant 2\epsilon$ for each $X \in \mathcal{D}_m$ whenever $m \in \mathbb{N}$ is sufficiently large. Hence $\mathrm{mesh}_d(\mathcal{D}_m) \to 0$ as $m \to +\infty$.

For the other direction, if $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ satisfies $\mathrm{mesh}_d(\mathcal{D}_m) \to 0$ as $m \to +\infty$, then for each open cover $\mathcal{U}$ of $\mathcal{S}^n$, when m is sufficiently large, the diameter of each elements in $\mathcal{D}_m$ is less than the Lebesgue number of $\mathcal{U}$. It follows that $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ is expanding. $\square$

The following proposition shows that for a cellular Markov branched cover, the definition of expansion does not rely on the choice of the essential sequence.

Proposition 4.3. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov branched cover. Let $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ and $\{\mathcal{C}_m\}_{m \in \mathbb{N}_0}$ be essential sequences of f. Then $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ is expanding if and only if $\{\mathcal{C}_m\}_{m \in \mathbb{N}_0}$ is expanding.*

A characterization of expanding cellular Markov branched covers follows from Proposition 4.3.

Corollary 4.4. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov branched cover. Then the following statements are equivalent:*
  (i) *The map f is expanding.*
  (ii) *There exists an expanding essential sequence of f.*
  (iii) *Each essential sequence of f is expanding.*

Before proving Proposition 4.3, we recall an elementary property of connected spaces. More precisely, given a finite open (or closed) cover,

each pair of points can be joined by a chain consisting of elements in the cover.

Lemma 4.5. *Let $\mathfrak{X}$ be a connected space and $\mathcal{U}$ be a finite cover of $\mathfrak{X}$. If $\mathcal{U}$ is open (or closed), then for each pair x, $y \in \mathfrak{X}$, there exists a finite collection $U_1, \ldots, U_n \in \mathcal{U}$ with the following property: $x \in U_1$, $y \in U_n$, and $U_i \cap U_{i+1} \neq \emptyset$ for each $i \in \{1, \ldots, n-1\}$.*

Proof. Let x, $y \in \mathfrak{X}$ be arbitrary. Denote

$\Omega \coloneqq \{V \in \mathcal{U} :$ there exist $V_1, \ldots, V_l \in \mathcal{U}$ such that $x \in V_1$,

$\qquad\qquad V = V_l$, and $V_i \cap V_{i+1} \neq \emptyset$ for each $i \in \{1, \ldots, l-1\}\}$.

It suffices to show that $\Omega = \mathcal{U}$ when $\mathcal{U}$ is open (or closed).

Suppose that $\mathcal{U}$ is open (or closed), then $\bigcup \Omega$ and $\bigcup(\mathcal{U} \smallsetminus \Omega)$ are both open (or closed). It is clear that $\Omega \neq \emptyset$. It is also clear that if $V \in \mathcal{U} \smallsetminus \Omega$, then $V \cap \bigcup \Omega = \emptyset$. Thus, $\bigcup(\mathcal{U} \setminus \Omega) = \mathfrak{X} \smallsetminus \bigcup \Omega$. If $\Omega \neq \mathcal{U}$, then $\bigcup \Omega$ and $\mathfrak{X} \smallsetminus \bigcup \Omega = \bigcup(\mathcal{U} \smallsetminus \Omega)$ are both non-empty, open (or close) subsets, which contradicts the assumption that $\mathfrak{X}$ is connected. $\qquad\square$

Proof of Proposition 4.3. Assume that $\mathrm{mesh}(\mathcal{D}_m) \to 0$ as $m \to +\infty$.

Fix arbitrary $m \in \mathbb{N}_0$ and $X \in \mathcal{C}_m$. By Proposition 3.20, the connected subset X can be covered by M elements in $\mathcal{F}(\mathcal{D}_m)$. Then, Lemma 4.5 and Lemma 2.10 (ii) imply that $\mathrm{diam}(X) \leqslant 2M \cdot \mathrm{mesh}(\mathcal{D}_m)$. Since X is arbitrary, we have $\mathrm{mesh}(\mathcal{C}_m) \leqslant 2M \cdot \mathrm{mesh}(\mathcal{D}_m)$. Therefore, we have that $\mathrm{mesh}(\mathcal{C}_m) \to 0$ as $m \to +\infty$.

Similarly, if $\mathrm{mesh}(\mathcal{C}_m) \to 0$, then $\mathrm{mesh}(\mathcal{D}_m) \to 0$ as $m \to +\infty$. $\qquad\square$

Given cell decompositions $\mathcal{C}$ and $\mathcal{D}$, we denote by $J(\mathcal{C}, \mathcal{D})$ the minimal number of elements in $\mathcal{C}^{[n]}$ required to form a connected set joining opposite sides of $\mathcal{D}$. More precisely, we set

(4.1)
$$J(\mathcal{C}, \mathcal{D}) \coloneqq \min\{\mathrm{card}(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{C}^{[n]}, \text{ and } \bigcup \mathcal{A} \text{ is a connected}$$
$$\text{set that joins opposite sides of } \mathcal{D}\}.$$

Clearly $J(\mathcal{C}, \mathcal{D}) < +\infty$, since $\mathcal{C}^{[n]}$ forms a finite cover of $\mathcal{S}^n$.

Lemma 4.6. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov map, and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an expanding essential sequence. Then $J(\mathcal{D}_m, \mathcal{D}_0) \to +\infty$ as $m \to +\infty$.*

Proof. To be concise, we denote $J_m \coloneqq J(\mathcal{D}_m, \mathcal{D}_0)$ for each $m \in \mathbb{N}_0$.

Let d be a metric on $\mathcal{S}^n$ compatible with the topology. Denote by δ_0 the Lebesgue number of $\mathcal{F}(\mathcal{D}_0)$. Let $N \in \mathbb{N}$ be arbitrary. Since $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ is expanding, there exists $m_0 \in \mathbb{N}_0$ such that $\mathrm{mesh}_d(\mathcal{D}_m) < \delta_0/N$ for each $m > m_0$. Fix an arbitrary $m > m_0$, and suppose that

$J_m \leqslant N$. Let $X_1, \ldots, X_{J_m}$ be level-m rooms such that $A = \bigcup_{i=1}^{J_m} X_i$ is connected and joins opposite sides of $\mathcal{D}_0$. Then, by Lemma 4.5,

$$\operatorname{diam}_d(A) \leqslant J_m \operatorname{mesh}_d(\mathcal{D}_m) < N \cdot \delta_0/N = \delta_0$$

and A is contained in a level-0 flower. However, by Lemma 2.13, we know that A cannot be contained in a level-0 flower, which leads to a contradiction. Therefore $J_m > N$ for each $m > m_0$. Since N is arbitrary, we have that $J_m \to +\infty$ as $m \to +\infty$. □

4.2. **The joining level $m_{f,\mathcal{D}}$ and visual metrics.** In this section, we introduce a natural class of metrics that we call *visual metrics*. We refer the reader to [BM17, Chapter 8] for a similar discussion.

Fix a cellular Markov map $f \colon \mathcal{S}^n \to \mathcal{S}^n$, an expanding essential sequence $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$, and two points $x, y \in \mathcal{S}^n$. If X and Y are level-m rooms containing x and y, respectively, then X and Y must be disjoint whenever m is sufficiently large. We define $m_{f,\mathcal{D}_0}(x,y)$ to be the maximal level of two intersecting rooms that contain x and y, and this yields an approximation for the "combinatorial distance" between x and y with respect to $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$. Later, the number $m_{f,\mathcal{D}_0}(x,y)$ will be used for the definition of visual metrics. Note that $m_{f,\mathcal{D}_0}$ is similar to the Gromov products on Gromov hyperbolic spaces.

Definition 4.7. Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov map, and $\{\mathcal{D}_l\}_{l \in \mathbb{N}_0}$ be an essential sequence of f. We define $m_{f,\mathcal{D}_0} \colon \mathcal{S}^n \times \mathcal{S}^n \to \mathbb{N}_0 \cup \{+\infty\}$ by
(4.2)
$$m_{f,\mathcal{D}_0}(x,y) \coloneqq \sup\big(\{0\} \cup \big\{l \in \mathbb{N} : \text{there exist } X, Y \in \mathcal{D}_l^{[n]} \text{ with}$$
$$x \in X, \ y \in Y, \text{ and } X \cap Y \neq \emptyset\big\}\big),$$

When f and $\mathcal{D}_0$ are clear from the context, we may write $m(x,y) \coloneqq m_{f,\mathcal{D}_0}(x,y)$.

Remark. Since the sequence $\{\mathcal{D}_l\}_{l \in \mathbb{N}_0}$ is determined by $\mathcal{D}_0$ and f, it suffices to indicate $\mathcal{D}_0$ in the notation $m_{f,\mathcal{D}_0}$.

We give some elementary properties of the function $m_{f,\mathcal{D}_0}$. First, we have a characterization of expanding essential sequences.

Lemma 4.8. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov map and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence of f. Then the following are equivalent:*

  (i) *The sequence $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ is expanding.*
  (ii) *For all $x, y \in \mathcal{S}^n$ with $x \neq y$, we have $m_{f,\mathcal{D}_0}(x,y) < +\infty$.*

Proof. It is clear that (i) implies (ii). Consider now the converse implication. It suffices to show that if $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ is not expanding, there exist $x, y \in \mathcal{S}^n$ with $x \neq y$ and $m_{f,\mathcal{D}_0}(x, y) = +\infty$.

Equip $\mathcal{S}^n$ with a metric d compatible with the given topology. It is clear that the sequence $\{\text{mesh}(\mathcal{D}_m)\}_{m \in \mathbb{N}_0}$ of positive real numbers is decreasing. Suppose that $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ is not expanding. Then we have $\delta \coloneqq \inf\{\text{mesh}(\mathcal{D}_m) : m \in \mathbb{N}_0\} > 0$.

For each $m \in \mathbb{N}_0$, pick $X_m \in \mathcal{D}_m$ such that $\text{diam}(X_m) > \delta$. Suppose that $Y_0 \in \mathcal{D}_0$ contains infinitely many elements in $\text{card}(\{m \in \mathbb{N} : X_m \subseteq Y_0\}) = +\infty$. Enumerate $\{m \in \mathbb{N} : X_m \subseteq Y_0\} = \{i_1^0, i_2^0, \dots\}$, where $i_1^0 < i_2^0 < \cdots$ is a strictly increasing sequence of positive integers. Denote $X_m^0 \coloneqq X_{i_m^0}$ for each $m \in \mathbb{N}$.

We define inductively a sequence $\{Y_j \in \mathcal{D}_j\}_{j \in \mathbb{N}_0}$ and a subsequence $\{X_m^{j-1}\}_{m \in \mathbb{N}}$ of $\{X_m\}_{m \in \mathbb{N}_0}$, for which $\{X_m^{j-1} : m \in \mathbb{N}\} \subseteq Y_{j-1}$, in the following way. Let $j \in \mathbb{N}$ be arbitrary. With $Y_{j-1} \in \mathcal{D}_{j-1}$ and $\{X_m^{j-1}\}_{m \in \mathbb{N}}$ well defined, choose $Y_j \in \mathcal{D}_j|_{Y_{j-1}}$ that satisfies $\text{card}(\{m \in \mathbb{N} : X_m^{j-1} \subseteq Y_j\}) = +\infty$. Enumerate $\{m \in \mathbb{N} : X_m^{j-1} \subseteq Y_j\} = \{i_1^j, i_2^j, \dots\}$, where $i_1^j < i_2^j < \cdots$ is a strictly increasing sequence of positive integers, and denote $X_m^j \coloneqq X_{i_m^{j-1}}^{j-1}$ for each $m \in \mathbb{N}$.

It is clear that $Y_j \subseteq Y_{j-1}$, and $\text{diam}(Y_j) > \delta$ for each $j \in \mathbb{N}_0$.

For each $j \in \mathbb{N}_0$, choose $u_j, v_j \in Y_j$ with $d(u_j, v_j) > \delta$. Since $(\mathcal{S}^n, d)$ is a compact metric space, consider a subsequence $\{u_{j_k}\}_{k \in \mathbb{N}}$ of $\{u_j\}_{j \in \mathbb{N}_0}$ such that $\lim_{k \to +\infty} u_{j_k} = u \in \mathcal{S}^n$. For each $k_0 \in \mathbb{N}$, if $k > k_0$, then we have $u_{j_k} \in Y_{j_{k_0}}$, and thus $u \in \overline{Y_{j_{k_0}}} = Y_{j_{k_0}}$. It follows that $u \in \bigcap_{j \in \mathbb{N}_0} Y_j$. Similarly, consider a subsequence $\{j_{k_l}\}_{l \in \mathbb{N}}$ of $\{j_k\}_{k \in \mathbb{N}}$, putting $i_l \coloneqq j_{k_l}$, and suppose that $\lim_{l \to +\infty} v_{i_l} = v \in \mathcal{S}^n$. Then we have $v \in \bigcap_{j \in \mathbb{N}_0} Y_j$ by similar arguments. Then $m_{f,\mathcal{D}_0}(u, v) = +\infty$. Also, since $d(u_{i_l}, v_{i_l}) > \delta$, we have $d(u, v) \geqslant \delta > 0$, and thus $u \neq v$. $\square$

Lemma 4.9. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov map and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence of f. Then the following statements hold:*

(i) *For all $x, y \in \mathcal{S}^n$, we have $m_{f,\mathcal{D}_0}(f(x), f(y)) \geqslant m_{f,\mathcal{D}_0}(x, y) - 1$.*

(ii) *If $x, y \in \text{int}_\circ(Z)$ for some $Z \in \mathcal{D}_l^{[n]}$, then $m_{f,\mathcal{D}_0}(f^r(x), f^r(y)) = m_{f,\mathcal{D}_0}(x, y) - r$ for each $r \in \mathbb{N}$ with $r \leqslant l$.*

Proof. To be concise, we denote $m(x, y) \coloneqq m_{f,\mathcal{D}_0}(x, y)$ for all $x, y \in \mathcal{S}^n$.

(i) Fix arbitrary $x, y \in \mathcal{S}^n$. It is clear that the claim holds if $m(x, y) = 0$. Assume now that $m(x, y) \in \mathbb{N} \cup \{+\infty\}$. If $m \in \mathbb{N}$ and X, Y are non-disjoint level-m rooms with $x \in X$ and $y \in Y$, then

$f(X)$ and $f(Y)$ are non-disjoint level-$(m-1)$ rooms containing $f(x)$ and $f(y)$, respectively. Therefore $m(f(x), f(y)) \geqslant m(x, y) - 1$.

(ii) Fix arbitrary $x, y \in \mathcal{S}^n$ and $Z \in \mathcal{D}_l^{[n]}$ such that $x, y \in \mathrm{int}_\circ(Z)$. Let $r \in \mathbb{N}$, $r \leqslant l$ be arbitrary. Then it follows from (i) that $m(f^r(x), f^r(y)) \geqslant m(x, y) - r$.

Put $Z' := f^r(Z) \in \mathcal{D}_{l-r}^{[n]}$. Then $f^r(x), f^r(y) \in \mathrm{int}_\circ(Z')$. It is clear that $m(f^r(x), f^r(y)) \geqslant l - r$. Suppose that $m \in \mathbb{N}_0$, $m \geqslant l - r$, and X', Y' are non-disjoint level-m rooms with $f^r(x) \in X'$ and $f^r(y) \in Y'$. Let X'', Y'' be level-$(l-r)$ rooms with $X' \subseteq X''$ and $Y' \subseteq Y''$. Then we have $X'' \cap \mathrm{int}_\circ(Z') \neq \emptyset$, and $Y'' \cap \mathrm{int}_\circ(Z') \neq \emptyset$. Since $\dim(Z') = \dim(X') = \dim(Y')$, it follows from Lemma 2.6 that $Z' = X'' = Y''$, Hence X', $Y' \in \mathcal{D}_m|_{Z'}$, and $X := (f^r|_Z)^{-1}(X')$ and $Y := (f^r|_Z)^{-1}(Y')$ are non-disjoint level-$(m+r)$ rooms containing x and y, respectively. Therefore $m(f^r(x), f^r(y)) \leqslant m(x, y) - r$ and the proof is complete. $\square$

Lemma 4.10. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov map and $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence of f. Then for all $k \in \mathbb{N}$, and $x, y \in \mathcal{S}^n$,*

$$k \cdot m_{f^k, \mathcal{D}_0}(x, y) \leqslant m_{f, \mathcal{D}_0}(x, y) \leqslant k \cdot (m_{f^k, \mathcal{D}_0}(x, y) + 1).$$

Proof. Fix arbitrary $k \in \mathbb{N}$ and $x, y \in \mathcal{S}^n$.

It is clear that $k \cdot m_{f^k, \mathcal{D}_0}(x, y) \leqslant m_{f, \mathcal{D}_0}(x, y)$. For the second inequality, suppose that $m \in \mathbb{N}_0$ and $X, Y \in \mathcal{D}_m^{[n]}$ satisfy $x \in X$, $y \in Y$, and $X \cap Y \neq \emptyset$. Let $m = ks + r$, where $s, r \in \mathbb{N}_0$, and $r \in \{0, 1, \ldots, k-1\}$. Then there exists $X', Y' \in \mathcal{D}_{ks}^{[n]}$ such that $X \subseteq X'$ and $Y \subseteq Y'$. Therefore $m_{f^k, \mathcal{D}_0}(x, y) \geqslant s$, and $k \cdot (m_{f^k, \mathcal{D}_0}(x, y) + 1) \geqslant ks + k \geqslant ks + r = m$. Therefore $m_{f, \mathcal{D}_0}(x, y) \leqslant k \cdot (m_{f^k, \mathcal{D}_0}(x, y) + 1)$. $\square$

For expanding cellular Markov branched covers, we have the following.

Lemma 4.11. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover. Let $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ and $\{\mathcal{C}_m\}_{m\in\mathbb{N}_0}$ be essential sequences of f. Then the following are true:*

(i) *There exists a number $k_0 \in \mathbb{N}$, depending on $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$, such that*

$$\min\{m_{f, \mathcal{D}_0}(x, z), \, m_{f, \mathcal{D}_0}(z, y)\} \leqslant m_{f, \mathcal{D}_0}(x, y) + k_0$$

for all $x, y, z \in \mathcal{S}^n$.

(ii) *There exists $k_1 \in \mathbb{N}$, depending on $\{\mathcal{C}_m\}_{m\in\mathbb{N}_0}$ and $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$, such that*

$$m_{f, \mathcal{D}_0}(x, y) - k_1 \leqslant m_{f, \mathcal{C}_0}(x, y) \leqslant m_{f, \mathcal{D}_0}(x, y) + k_1$$

for all $x, y \in \mathcal{S}^n$.

Proof. (i) Denote $m(x, y) \coloneqq m_{f, \mathcal{D}_0}(x, y)$ for all $x, y \in \mathcal{S}^n$. By Lemma 4.6, fix $k \in \mathbb{N}$ such that $J(\mathcal{D}_k, \mathcal{D}_0) > 10$. Suppose that there are $x, y, z \in \mathcal{S}^n$ such that

$$\min\{m(x, z), m(z, y)\} > m(x, y) + k.$$

To be concise, we denote $m \coloneqq m(x, y) \in \mathbb{N}_0$. Let X' and Y' be level-$(m + 1)$ rooms containing x and y, respectively. By the definition of $m(x, y)$, we have $X' \cap Y' = \emptyset$. Since $m(x, z) > m + k$, we can choose non-disjoint level-$(m + k + 1)$ rooms X, Z_1 with $x \in X$, $z \in Z_1$. Likewise, choose non-disjoint level-$(m + k + 1)$ rooms Y, Z_2 with $y \in Y$, $z \in Z_2$.

Let $A \coloneqq X \cup Y \cup Z_1 \cup Z_2$. Clearly A is connected. Since A intersects disjoint level-$(m + 1)$ rooms X' and Y', it follows from Corollary 2.11 that A cannot be contained in any level-$(m + 1)$ flower. Then by Corollary 3.17, we have that $f^{m+1}(A)$ cannot be contained in any level-0 flower. It follows from Lemma 2.13 that $f^{m+1}(A)$ joins opposite sides of $\mathcal{D}_0$. Since

$$f^{m+1}(A) = f^{m+1}(X) \cup f^{m+1}(Y) \cup f^{m+1}(Z_1) \cup f^{m+1}(Z_2),$$

we have $J(\mathcal{D}_k, \mathcal{D}_0) \leqslant 4$, which contradicts the assumption that $J(\mathcal{D}_k, \mathcal{D}_0) > 10$. Therefore

$$\min\{m(x, z), m(z, y)\} \leqslant m(x, y) + k$$

for all $x, y, z \in \mathcal{S}^n$. Put $k_0 \coloneqq k$, then the proof of (i) is complete.

(ii) Fix arbitrary $x, y \in \mathcal{S}^n$. Suppose that $\widetilde{m} \in \mathbb{N}_0$ and $\widetilde{X}, \widetilde{Y} \in \mathcal{C}_{\widetilde{m}}^{[n]}$ satisfy $x \in \widetilde{X}$, $y \in \widetilde{Y}$, and $\widetilde{X} \cap \widetilde{Y} \neq \emptyset$. By Proposition 3.20, pick covers $\mathcal{F}(\widetilde{X})$ and $\mathcal{F}(\widetilde{Y})$ of $\widetilde{X}$ and $\widetilde{Y}$, respectively, by elements in $\mathcal{F}(\mathcal{D}_{\widetilde{m}})$ with $\operatorname{card}(\mathcal{F}(\widetilde{X})) \leqslant M$ and $\operatorname{card}(\mathcal{F}(\widetilde{Y})) \leqslant M$. Here $M \in \mathbb{N}$ is a constant independent of $\widetilde{m}$, $\widetilde{X}$, $\widetilde{Y}$. Then there exists a chain $X_1, \ldots, X_{4M} \in \mathcal{D}_{\widetilde{m}}^{[n]}$ satisfying $x \in X_1$, $y \in X_{4M}$, and $X_i \cap X_{i+1} \neq \emptyset$ for each $i \in \{1, \ldots, 4M - 1\}$. Pick $x_i \in X_i$ for each $i \in \{2, \ldots, 4M - 1\}$, and put $x_1 \coloneqq x$, $x_{4M} \coloneqq y$. Then $m_{f, \mathcal{D}_0}(x_i, x_{i+1}) \geqslant \widetilde{m}$ for each $i \in \{1, \ldots, 4M - 1\}$.

Now we show that $\widetilde{m} \leqslant m_{f, \mathcal{D}_0}(x_1, x_i) + (i - 2)k_0$ for each $i \in \{2, \ldots, 4M\}$ by induction. Here k_0 is the constant in (i). It is clear that $\widetilde{m} \leqslant m_{f, \mathcal{D}_0}(x_1, x_2)$. Fix arbitrary $i \in \{3, \ldots, 4M\}$, and make the induction hypothesis that $\widetilde{m} \leqslant m_{f, \mathcal{D}_0}(x_1, x_{i-1}) + (i - 3)k_0$. Then, by (i) we have

$$\widetilde{m} - (i-3)k_0 \leqslant \min\{m_{f, \mathcal{D}_0}(x_1, x_{i-1}), m_{f, \mathcal{D}_0}(x_{i-1}, x_i)\} \leqslant m_{f, \mathcal{D}_0}(x_1, x_i) + k_0,$$

and thus $\widetilde{m} \leqslant m_{f, \mathcal{D}_0}(x_1, x_i) + (i - 2)k_0$.

Therefore

$$\widetilde{m} \leqslant m_{f, \mathcal{D}_0}(x_1, x_{4M}) + (4M - 2)k_0 = m_{f, \mathcal{D}_0}(x, y) + (4M - 2)k_0.$$

It follows that $m_{f,\mathcal{C}_0}(x,y) \leqslant m_{f,\mathcal{D}_0}(x,y) + (4M-2)k_0$. By the same argument, there exists $M' \in \mathbb{N}$, independent of x and y, such that $m_{f,\mathcal{D}_0}(x,y) \leqslant m_{f,\mathcal{C}_0}(x,y) + (4M'-2)k_0$.

Put $k_1 := \max\{(4M-2)k_0, (4M'-2)k_0\}$, then the proof is complete. $\square$

Now we give the definition of visual metrics for expanding cellular Markov maps.

Definition 4.12 (Visual metrics). Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov map. A metric ϱ on $\mathcal{S}^n$ is a *visual metric* for f if there exist essential sequence $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$, and constants $C > 1$ and $\Lambda > 1$ such that

$$(4.3) \qquad C^{-1}\Lambda^{-m_{f,\mathcal{D}_0}(x,y)} \leqslant \varrho(x,y) \leqslant C\Lambda^{-m_{f,\mathcal{D}_0}(x,y)}$$

for all $x, y \in \mathcal{S}^n$, where $m_{f,\mathcal{D}_0}(x,y)$ is defined by (4.2). We call the constant Λ an *expansion factor* for ϱ, and the metric space $(\mathcal{S}^n, \varrho)$ the *visual sphere*.

Remark. It follows from Lemma 4.8 that $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ is expanding if there exist $C > 1$ and $\Lambda > 1$ such that (4.3) holds. Therefore, visual metrics are defined only for expanding cellular Markov maps.

Lemma 4.13. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov map and ϱ be a visual metric for f. Then the following statements are true:*

  (i) *ϱ induces the given topology on $\mathcal{S}^n$.*
  (ii) *$f \colon (\mathcal{S}^n, \varrho) \to (\mathcal{S}^n, \varrho)$ is a Lipschitz map.*

Proof. Let $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence and $C > 1$, $\Lambda > 1$ be constants such that

$$C^{-1}\Lambda^{-m_{f,\mathcal{D}_0}(x,y)} \leqslant \varrho(x,y) \leqslant C\Lambda^{-m_{f,\mathcal{D}_0}(x,y)}$$

for all $x, y \in \mathcal{S}^n$. To be concise, we denote $m(x,y) := m_{f,\mathcal{D}_0}(x,y)$ for all $x, y \in \mathcal{S}^n$.

(i) Let d be a metric on $\mathcal{S}^n$ compatible with the topology. We show that $\mathrm{id} \colon (\mathcal{S}^n, \varrho) \to (\mathcal{S}^n, d)$ is a homeomorphism. Let x be a point, and $\{x_i\}_{i \in \mathbb{N}}$ be a sequence of points in $\mathcal{S}^n$.

Assume first that $\varrho(x_i, x) \to 0$ as $i \to +\infty$, which is equivalent to $m_i := m(x_i, x) \to +\infty$. Then for each $i \in \mathbb{N}$ there exist non-disjoint $X_i, Y_i \in \mathcal{D}_{m_i}^{[n]}$, with $x_i \in X_i$, $x \in Y_i$. Clearly $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ is expanding, then we have $d(x_i, x) \leqslant 2\,\mathrm{mesh}(\mathcal{D}_{m_i}) \to 0$ as $i \to +\infty$.

Conversely, suppose that $d(x_i, x) \to 0$ as $i \to +\infty$. Fix an arbitrary $m \in \mathbb{N}$. Then there is a level-m flower $U_m \in \mathcal{F}(\mathcal{D}_m)$ containing x. Since U_m is open, there exists $i_m \in \mathbb{N}$, such that $x_i \in U_m$ for each $i \in \mathbb{N}$

with $i > i_m$. Then for each $i \in \mathbb{N}$ with $i > i_m$, we can find non-disjoint X_i, $Y_i \in \mathcal{D}_m^{[n]}$, with $x_i \in X_i$, $x \in Y_i$, which implies $m(x_i, x) \geqslant m$. Therefore $m_i \to +\infty$, and hence $\varrho(x_i, x) \to 0$ as $i \to +\infty$.

(ii) By Lemma 4.9, we have

$$\varrho(f(x), f(y)) \leqslant C\Lambda^{-m(f(x),f(y))} \leqslant C\Lambda^{-m(x,y)+1} \leqslant \Lambda C^2 \varrho(x,y),$$

for all x, $y \in \mathcal{S}^n$. Hence f is ΛC^2-Lipschitz with respect to ϱ.   □

Moreover, for expanding cellular Markov branched covers, we establish the existence of visual metrics.

Theorem 4.14 (Existence of visual metric). *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover. Then there exists a visual metric for f.*

For the proof of Theorem 4.14, recall (see e.g. [He01, Section 14.1]) that a function $q \colon X \times X \to [0, +\infty)$ is a *quasimetric* if it has the following properties:

(i) $q(x,y) = q(y,x)$ for all x, $y \in X$.

(ii) $q(x,y) = 0$ if and only if $x = y$.

(iii) For a constant $K \geqslant 1$, the inequality

(4.4) $q(x,y) \leqslant K(q(x,z) + q(z,y))$

holds for all $x, y, z \in X$.

Recall also that a sufficient "snowflaking" of a quasimetric is comparable to some metric.

Lemma 4.15 ([He01, Proposition 14.5]). *Let q be a quasimetric on a set X. Then there is $\epsilon_0 > 0$ depending only on K in (4.4) such that $q_\epsilon(x,y) := (q(x,y))^\epsilon$ is bi-Lipschitz equivalent to a metric for each $0 < \epsilon < \epsilon_0$,.*

Now we are ready to prove Theorem 4.14.

Proof of Theorem 4.14. Let $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence of f. To be concise, we denote $m(x,y) := m_{f,\mathcal{D}_0}(x,y)$ for all x, $y \in \mathcal{S}^n$.

Fix $\lambda > 1$, and set $q(x,y) := \lambda^{-m(x,y)}$. By Lemma 4.11 (i), we have

$$q(x,y) \leqslant \lambda^{k_0}(q(x,z) + q(z,y))$$

for all x, y, $z \in \mathcal{S}^n$. It can be checked that $q(x,y)$ is a quasimetric on $\mathcal{S}^n$. Then, by Lemma 4.15, there exist constants $\epsilon > 0$ and $C > 1$, and a metric ϱ on $\mathcal{S}^n$ satisfying

$$C^{-1}(q(x,y))^\epsilon < \varrho(x,y) < C(q(x,y))^\epsilon$$

for all x, $y \in \mathcal{S}^n$. Put $\Lambda := \lambda^\epsilon > 1$, then

$$C^{-1}\Lambda^{-m(x,y)} < \varrho(x,y) < C\Lambda^{-m(x,y)}$$

for all x, $y \in \mathcal{S}^n$, and thus ϱ is a visual metric. $\square$

In what follows, we give some elementary properties of visual metrics.

First, we show that, for cellular Markov branched covers, the definition of visual metrics is independent of the choice of essential sequence, and that the expansion factor of a visual metric is unique.

Lemma 4.16. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover. Let ϱ be a visual metric for f, and $\Lambda > 1$ be an expansion factor of ϱ. Then, for each essential sequence $\{\mathcal{C}_m\}_{m\in\mathbb{N}_0}$ for f, there exists $\widetilde{C} > 1$ such that*

$$\widetilde{C}^{-1}\Lambda^{-m_{f,\mathcal{C}_0}(x,y)} \leqslant \varrho(x,y) \leqslant \widetilde{C}\Lambda^{-m_{f,\mathcal{C}_0}(x,y)}$$

for all x, $y \in \mathcal{S}^n$.

Proof. Let $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence for f, and $C > 1$ be a constant satisfying

$$C^{-1}\Lambda^{-m_{f,\mathcal{D}_0}(x,y)} \leqslant \varrho(x,y) \leqslant C\Lambda^{-m_{f,\mathcal{D}_0}(x,y)}$$

for all x, $y \in \mathcal{S}^n$. Let also $\{\mathcal{C}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence.

For all x, $y \in \mathcal{S}^n$, by Lemma 4.11 (ii), we have that

$$m_{f,\mathcal{C}_0}(x,y) - k_1 \leqslant m_{f,\mathcal{D}_0}(x,y) \leqslant m_{f,\mathcal{C}_0}(x,y) + k_1,$$

and it follows that

$$\left(C\Lambda^{k_1}\right)^{-1}\Lambda^{-m_{f,\mathcal{C}_0}(x,y)} \leqslant \varrho(x,y) \leqslant \left(C\Lambda^{k_1}\right)\Lambda^{-m_{f,\mathcal{C}_0}(x,y)}.$$

Put $\widetilde{C} := C\Lambda^{k_1}$, then the proof is complete. $\square$

Corollary 4.17. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover. Let ϱ be a visual metric for f. Then the expansion factor of ϱ is unique.*

Proof. Suppose that $\Lambda > 1$ and $\widetilde{\Lambda} > 1$ are expansion factors of ϱ. Let $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence for f, and denote $m(x,y) := m_{f,\mathcal{D}_0}(x,y)$ for all x, $y \in \mathcal{S}^n$. By Lemma 4.16, there are constants $C > 1$, $\widetilde{C} > 1$ such that

$$C^{-1}\Lambda^{-m(x,y)} \leqslant \varrho(x,y) \leqslant C\Lambda^{-m(x,y)},$$
$$\widetilde{C}^{-1}\widetilde{\Lambda}^{-m(x,y)} \leqslant \varrho(x,y) \leqslant \widetilde{C}\widetilde{\Lambda}^{-m(x,y)}$$

for all x, $y \in \mathcal{S}^n$. Thus, we have that

$$\left(C\widetilde{C}\right)^{-1} \leqslant \left(\widetilde{\Lambda}/\Lambda\right)^{-m(x,y)} \leqslant C\widetilde{C}$$

for all x, $y \in \mathcal{S}^n$ with $x \neq y$. Now we show $\widetilde{\Lambda} = \Lambda$. Suppose on the contrary that $\widetilde{\Lambda}/\Lambda \neq 1$, and choose $m \in \mathbb{N}$ with $m \cdot \left|\log\big(\widetilde{\Lambda}/\Lambda\big)\right| > \log\big(C\widetilde{C}\big)$. Pick $Z \in \mathcal{D}_m^{[n]}$, and x, $y \in Z$ with $x \neq y$. Then $m(x,y) \geqslant m$, and $\left|\log\big(\widetilde{\Lambda}/\Lambda\big)^{-m(x,y)}\right| > \log\big(C\widetilde{C}\big)$, which is a contradiction. $\qquad\square$

A cellular Markov branched cover f and its iterations have the same visual metrics.

Lemma 4.18. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover and ϱ be a metric on $\mathcal{S}^n$. Let $k \in \mathbb{N}$ be arbitrary. Then, ϱ is a visual metric of f if and only ϱ is a visual metric of f^k.*

Proof. Let $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence. To be concise, we denote $m(x,y) \coloneqq m_{f,\mathcal{D}_0}(x,y)$ and $m_k(x,y) \coloneqq m_{f^k,\mathcal{D}_0}(x,y)$ for all x, $y \in \mathcal{S}^n$.

Suppose first that ϱ is a visual metric of f, and let $\Lambda > 1$ be the expansion factor of ϱ. By Lemma 4.16, there exists a constant $C > 1$ such that

$$C^{-1}\Lambda^{-m(x,y)} \leqslant \varrho(x,y) \leqslant C\Lambda^{-m(x,y)}$$

for all x, $y \in \mathcal{S}^n$. Then by Lemma 4.10,

$$(C\Lambda)^{-1}(\Lambda^k)^{-m_k(x,y)} \leqslant \varrho(x,y) \leqslant C(\Lambda^k)^{-m_k(x,y)},$$

and thus ϱ is a visual metric for f^k.

For the other direction, suppose now that ϱ is a visual metric of f^k, and let $\Lambda_k > 1$ be the expansion factor of ϱ. By Lemma 4.16, let $C_k > 1$ be a constant such that

$$C_k^{-1}\Lambda_k^{-m_k(x,y)} \leqslant \varrho(x,y) \leqslant C_k\Lambda_k^{-m_k(x,y)}$$

for all x, $y \in \mathcal{S}^n$. Then by Lemma 4.10,

$$C_k^{-1}\big(\Lambda_k^{1/k}\big)^{-m(x,y)} \leqslant \varrho(x,y) \leqslant C_k\Lambda_k\big(\Lambda_k^{1/k}\big)^{-m(x,y)}$$

and thus ϱ is a visual metric for f. $\qquad\square$

Recall that two metrics d and $\widetilde{d}$ on a topological space X are *snowflake equivalent* if there exist constants $L > 1$ and $\alpha > 0$ such that

$$L^{-1}d(x,y)^\alpha \leqslant \widetilde{d}(x,y) \leqslant Ld(x,y)^\alpha$$

for all x, $y \in X$. Clearly, snowflake equivalence implies quasisymmetric equivalence.

The following lemma shows that the family of visual metrics agrees with the snowflake equivalence class of metrics that includes at least one visual metric.

Lemma 4.19. *Let $f\colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover, and ϱ be a visual metric of f. Let $\widetilde{\varrho}$ be a metric on $\mathcal{S}^n$, then $\widetilde{\varrho}$ is a visual metric if and only if ϱ and $\widetilde{\varrho}$ are snowflake equivalent.*

Proof. Let $\Lambda > 1$ be the expansion factor of ϱ. Let $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence of f, and $C > 1$ be a constant for which

$$C^{-1}\Lambda^{-m_{f,\mathcal{D}_0}(x,y)} \leqslant \varrho(x,y) \leqslant C\Lambda^{-m_{f,\mathcal{D}_0}(x,y)}$$

holds for all x, $y \in \mathcal{S}^n$.

In what follows, we denote $m(x,y) := m_{f,\mathcal{D}_0}(x,y)$ for all x, $y \in \mathcal{S}^n$.

First, suppose that $\widetilde{\varrho}$ is a visual metric, and let $\widetilde{\Lambda} > 1$ be the expansion factor of $\widetilde{\varrho}$. By Lemma 4.16, there exists a constant $\widetilde{C} > 1$ satisfying

$$\widetilde{C}^{-1}\widetilde{\Lambda}^{-m(x,y)} \leqslant \widetilde{\varrho}(x,y) \leqslant \widetilde{C}\widetilde{\Lambda}^{-m(x,y)}$$

for all x, $y \in \mathcal{S}^n$. Then it follows that

$$(\widetilde{C}C^\alpha)^{-1}\varrho(x,y)^\alpha \leqslant \widetilde{\varrho}(x,y) \leqslant (\widetilde{C}C^\alpha)\varrho(x,y)^\alpha$$

for all x, $y \in \mathcal{S}^n$, where $\alpha := \log\widetilde{\Lambda}/\log\Lambda > 0$.

Conversely, suppose that ϱ and $\widetilde{\varrho}$ are snowflake equivalent. Let $L > 1$ and $\alpha > 0$ satisfy

$$L^{-1}\varrho(x,y)^\alpha \leqslant \widetilde{\varrho}(x,y) \leqslant L\varrho(x,y)^\alpha$$

for all x, $y \in \mathcal{S}^n$. Then it follows that

$$(LC^\alpha)^{-1}(\Lambda^\alpha)^{-m(x,y)} \leqslant \widetilde{\varrho}(x,y) \leqslant LC^\alpha(\Lambda^\alpha)^{-m(x,y)}$$

for all x, $y \in \mathcal{S}^n$. Hence $\widetilde{\varrho}$ is a visual metric with expansion factor Λ^α. $\qquad\square$

Lemma 4.20. *Let $f\colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover. Equip $\mathcal{S}^n$ with a visual metric ϱ for f, and let $\Lambda > 1$ be the expansion factor for ϱ. Then for each essential sequence $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$, the following statements hold:*

(i) *There exists $C' > 1$ such that for each $m \in \mathbb{N}_0$ and each $X \in \mathcal{D}_m^{[n]}$ we have*

$$(1/C')\Lambda^{-m} \leqslant \operatorname{diam}(X) \leqslant C'\Lambda^{-m}.$$

(ii) *There exists $C'' > 0$ such that for each $m \in \mathbb{N}_0$ and all $X, Y \in \mathcal{D}_m^{[n]}$ with $X \cap Y = \emptyset$ we have*

$$\operatorname{dist}(X,Y) \geqslant C''\Lambda^{-m}.$$

Proof. Let $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence. To be concise, we denote $m(x,y) \coloneqq m_{f,\mathcal{D}_0}(x,y)$ for all x, $y \in \mathcal{S}^n$. By Lemma 4.16, there exists $C > 1$ such that for all x, $y \in \mathcal{S}^n$ we have

$$C^{-1}\Lambda^{-m(x,y)} \leqslant \varrho(x,y) \leqslant C\Lambda^{-m(x,y)}.$$

(i) Let $k \in \mathbb{N}_0$ and $X \in \mathcal{D}_k^{[n]}$ be arbitrary. For all x, $y \in X$, we have $m(x,y) \geqslant k$, and then $\varrho(x,y) \leqslant C\Lambda^{-k}$. It follows that $\operatorname{diam}(X) \leqslant C\Lambda^{-k}$. Consider now x, $y \in \operatorname{int}_\circ(X)$. By Lemma 4.9 (ii), we have $m(f^k(x), f^k(y)) = m(x,y) - k$. Then

$$\varrho(x,y) \geqslant C^{-1}\Lambda^{-m(x,y)} = C^{-1}\Lambda^{-m(f^k(x),f^k(y))-k}$$
$$\geqslant C^{-2}\Lambda^{-k}\varrho(f^k(x), f^k(y)).$$

Since $f^k|_X \colon X \to f^k(X)$ is a bijection, and $f^k(\operatorname{int}_\circ(X)) = \operatorname{int}_\circ(f^k(X))$, we have that

$$\operatorname{diam}(X) \geqslant \operatorname{diam}(\operatorname{int}_\circ(X)) \geqslant bC^{-2}\Lambda^{-k}$$

where $b \coloneqq \min\{\operatorname{diam}(\operatorname{int}_\circ(X_0)) : X_0 \in \mathcal{D}_0^{[n]}\} > 0$.

Put $C' \coloneqq \max\{C, C^2 b^{-1}\}$, then the proof is complete.

(ii) Let $m \in \mathbb{N}_0$ and $X, Y \in \mathcal{D}_m^{[n]}$ satisfy $X \cap Y = \emptyset$. Let $x \in X$, $y \in Y$ be arbitrary. Choose $k \in \mathbb{N}$ for which $J(\mathcal{D}_k, \mathcal{D}_0) > 2$. Then, for each pair $\widetilde{X}, \widetilde{Y} \in \mathcal{D}_{m+k}^{[n]}$ with $x \in \widetilde{X}$, $y \in \widetilde{Y}$, we have $\widetilde{X} \cap \widetilde{Y} = \emptyset$. Hence $m(x,y) \leqslant m + k$, and $\varrho(x,y) \geqslant C^{-1}\Lambda^{-m-k}$. Since $x \in X$ and $y \in Y$ are arbitrary, we have $\operatorname{dist}(X,Y) \geqslant C^{-1}\Lambda^{-k}\Lambda^{-m}$. Put $C'' \coloneqq C^{-1}\Lambda^{-k}$, then the proof is complete. $\qquad\square$

Recall that a metric space $(\mathfrak{X}, d)$ is *doubling* if there exists a number $N \in \mathbb{N}$ such that every ball in $\mathfrak{X}$ of radius $r > 0$ can be covered by N balls of radius $r/2$.

Lemma 4.21. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover and let ϱ be a visual metric for f. Then $(\mathcal{S}^n, \varrho)$ is doubling if*

$$\sup\{N_{\operatorname{loc}}(f^m, x) : m \in \mathbb{N}, x \in \mathcal{S}^n\} < +\infty.$$

Proof. Let $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence. To be concise, denote $m(x,y) \coloneqq m_{f,\mathcal{D}_0}(x,y)$ for all x, $y \in \mathcal{S}^n$.

Let $\Lambda > 1$ be the expansion factor of ϱ. By Corollary 3.23, there is $M_0 \in \mathbb{N}$ such that $\operatorname{card}(\{X \in \mathcal{D}_m^{[n]} : x \in X\}) \leqslant M_0$ for each $m \in \mathbb{N}$ and each $x \in \mathcal{S}^n$.

Let $C' > 1$ and $C'' > 0$ be the constants given by Lemma 4.20 (i) and (ii), respectively. Set $C_0 \coloneqq \max\{2, 2\operatorname{diam}(\mathcal{S}^n)\}$, $C \coloneqq \max\{C_0, \Lambda/C_0\}$,

and

$$k := \max\left\{ 1, \left\lceil \log_\Lambda \frac{C}{4C'\Lambda} \right\rceil, \left\lceil \log_\Lambda \frac{C}{C''} \right\rceil \right\}.$$

For each $r \in (0, 2\operatorname{diam}(\mathcal{S}^n)]$, let $m(r) \in \mathbb{N}_0$ be the number for which $r \in (C_0\Lambda^{-m(r)-1}, C_0\Lambda^{-m(r)}]$, then

   (i) $C^{-1}\Lambda^{-m(r)} \leqslant r \leqslant C\Lambda^{-m(r)}$.

   (ii) $\operatorname{diam}(X) \leqslant r/4$ for each $X \in \mathcal{D}^{[n]}_{m(r)+k}$.

   (iii) If $m(r) - k \geqslant 0$, then $\operatorname{dist}(X, Y) \geqslant r$ holds for all disjoint $X, Y \in \mathcal{D}^{[n]}_{m(r)-k}$.

Fix arbitrary $x \in \mathcal{S}^n$, $r \in (0, 2\operatorname{diam}(\mathcal{S}^n)]$, and denote $m := m(r)$. Let $\mathcal{C} := \left\{ X \in \mathcal{D}^{[n]}_{m+k} : X \cap B(x, r) \neq \emptyset \right\}$. Then $\mathcal{C}$ is a cover of $B(x, r)$ by sets of diameters no more than $r/4$. It suffices to find an upper bound for $\operatorname{card}(\mathcal{C})$ independent of x and r.

If $m \leqslant k$, then it is clear that $\operatorname{card}(\mathcal{C}) \leqslant \operatorname{card}(\mathcal{D}_{2k})$. Suppose that $m > k$ in what follows. Pick $X \in \mathcal{D}^{[n]}_{m-k}$ with $x \in X$. For each $Z \in \mathcal{C}$ there exists $Y \in \mathcal{D}^{[n]}_{m-k}$ such that $Z \subseteq Y$, then it follows from $\operatorname{dist}(X, Y) < r$ that $X \cap Y \neq \emptyset$.

Consider the set $\mathcal{U}(X) := \left\{ Y \in \mathcal{D}^{[n]}_{m-k} : X \cap Y \neq \emptyset \right\}$. Then we have

$$\mathcal{C} \subseteq \bigcup_{Y \in \mathcal{U}(X)} \left\{ Z \in \mathcal{D}^{[n]}_{m+k} : Z \subseteq Y \right\}.$$

For each $Y \in \mathcal{D}_{m-k}$, we have $\operatorname{card}(\mathcal{D}_{m+k}|_Y) \leqslant \operatorname{card}(\mathcal{D}_{2k})$, and thus $\operatorname{card}(\mathcal{C}) \leqslant \operatorname{card}(\mathcal{D}_{2k}) \cdot \operatorname{card}(\mathcal{U}(X))$. Since for each $Y \in \mathcal{U}(X)$, the intersection $X \cap Y$ is a collection of cells in $\mathcal{D}_{m-k}|_X$, there exists a level-$(m-k)$ vertex in $X \cap Y$. It follows that $\mathcal{U}(X) \subseteq \bigcup_p \{ Y \in \mathcal{D}^{[n]}_{m-k} : p \in Y \}$, where p is taken over all vertices in $\mathcal{D}_{m-k}|_X$. Since $\operatorname{card}(\mathcal{D}_{m-k}|_X) \leqslant \operatorname{card}(\mathcal{D}_0)$, we have $\operatorname{card}(\mathcal{U}(X)) \leqslant M_0 \operatorname{card}(\mathcal{D}_0)$.

Thus $\operatorname{card}(\mathcal{C}) \leqslant M_0 \operatorname{card}(\mathcal{D}_0) \operatorname{card}(\mathcal{D}_{2k})$. This upper bound does not depend on the choice of x and r. Therefore $(\mathcal{S}^n, \varrho)$ is doubling. $\qquad\square$

5. Branched quasisymmetries and quasiregular maps

In this section we give a brief introduction to *branched quasisymmetric maps* and *quasiregular maps*, along with a characterization of quasiregular maps between Riemannian manifolds. We also introduce branched quasisymmetric maps with uniformity. For more details, we refer the reader to [LP19] and [GW16]. This section serves as a preliminary for the study of the quasisymmetric uniformization problem in Sections 6 and 7.

5.1. **Branched quasisymmetric maps.** Guo and Williams introduced branched quasisymmetric maps in [GW16]. In what follows, we shall adopt the formulation in [LP19].

Recall that a subset $E \subseteq X$ is a *continuum* in the topological space X, if E is compact, connected, and contains at least two points.

Definition 5.1 (Branched quasisymmetric maps). A continuous map $f \colon X \to Y$ between metric spaces X and Y is a *branched η-quasisymmetric map (abbr: η-BQS)* for a homeomorphism $\eta \colon [0, +\infty) \to [0, +\infty)$ if the distortion inequality

$$\operatorname{diam}(f(E')) \leqslant \eta\left(\frac{\operatorname{diam} E'}{\operatorname{diam} E}\right) \operatorname{diam}(f(E))$$

holds for all continua E and E' in X with $E' \cap E \neq \emptyset$. If there exists η for which f is an η-BQS map, we simply say that f is a BQS map.

Note that the definition above is slightly different from [GW16, Definition 6.45], where f is assumed, in addition, to be discrete, open, and of bounded local multiplicity.

We record some elementary properties of branched quasisymmetries. First, the composition of two branched quasisymmetric maps is branched quasisymmetric, quantitatively.

Lemma 5.2 ([LP19, Lemma 3.1]). *If $f \colon X \to Y$ and $g \colon Y \to Z$ are branched quasisymmetric maps for homeomorphisms η_f and η_g, respectively, then the composition $h \coloneqq g \circ f$ is a branched $\eta_g \circ \eta_f$-quasisymmetric map.*

A quasisymmetric map is a branched quasisymmetric map, quantitatively.

Lemma 5.3 ([LP19, Lemma 3.2]). *If $f \colon X \to Y$ is an η-quasisymmetric map for $\eta \colon [0, +\infty) \to [0, +\infty)$, then f is a branched η'-quasisymmetric map for $\eta' \colon t \mapsto 2\eta(2t)$.*

Definition 5.4 (Local branched quasisymmetric maps). A continuous map $f \colon X \to Y$ between metric spaces is a *local η-branched quasisymmetric map* for a homeomorphism $\eta \colon [0, +\infty) \to [0, +\infty)$ if there exists $r > 0$ such that $f|_{B(x,r)}$ is an η-BQS map for each $x \in X$. In this case, we call r the *locality scale of f*. We simply say f is a local BQS map if there exists η for which f is local η-BQS.

A discrete and open local branched quasisymmetry on a bounded turning space self-improves to a branched quasisymmetry. Recall that a metric space (X, d) has *bounded turning* if there is a constant $\lambda \geqslant 1$

such that, for all points $x, y \in X$, there exists a continuum $E \subseteq X$ with $x, y \in E$ and $\operatorname{diam}_d(E) \leqslant \lambda d(x, y)$.

Lemma 5.5 ([LP19, Theorem 13.12]). *Let $f \colon X \to Y$ be a discrete and open local branched quasisymmetry between compact metric space X and Y, where X has bounded turning. Then f is a branched quasisymmetry.*

5.2. **Quasiregular maps.** Recall that a continuous map $f \colon M \to N$ between oriented Riemannian n-manifolds is *quasiregular* if $f \in W_{\mathrm{loc}}^{1,n}(M, N)$ and satisfies the distortion inequality

$$(5.1) \qquad \|Df\|^n \leqslant K J_f \quad \text{a.e. in } M$$

for some constant $K \geqslant 1$, where $\|Df\|$ is the operator norm of Df and J_f is the Jacobian determinant. The infimal K satisfying (5.1) is the *outer dilatation* of f, denoted by $K_O(f)$.

For a quasiregular map f, it is also true that there exists $K' \geqslant 1$ such that

$$(5.2) \qquad J_f \leqslant K'(l(Df))^n \quad \text{a.e. in } M,$$

where $l(Df) := \inf_{\|x\|=1} \|Df(x)\|$. The infimal K' satisfying (5.2) is the *inner dilatation* of f, denoted by $K_I(f)$. We have inequalities $K_O(f) \leqslant K_I(f)^{n-1}$ and $K_I(f) \leqslant K_O(f)^{n-1}$. The *(maximal) dilatation* of f is $K(f) := \max\{K_O(f), K_I(f)\}$. A quasiregular map f is K-quasiregular if $K(f) \leqslant K$.

We record a geometric characterization for quasiregular maps on a domain $G \subseteq \mathbb{R}^n$. For the characterization, we first introduce *moduli of curve families*; see e.g. [Vu88, Ri93] for detailed discussions.

Given a topological space $\mathfrak{X}$, a continuous map $\gamma \colon I \to \mathfrak{X}$, where $I \subseteq \mathbb{R}$ is an interval, is called a *curve* in $\mathfrak{X}$. For subsets $E, F, G \subseteq \mathfrak{X}$, we denote by $\Delta(E, F; G)$ the family of non-constant closed curves joining E, F in G, i.e., curves $\gamma \colon [a, b] \to \mathfrak{X}$ on closed interval $[a, b]$ with $\gamma(a) \in E$, $\gamma(b) \in F$, and $\gamma((a, b)) \subseteq G$. In particular, when $G = \mathfrak{X}$, we write $\Delta(E, F) := \Delta(E, F; G)$. For curves $\gamma \colon I \to \mathfrak{X}$ and $\gamma' \colon I' \to \mathfrak{X}$, we call γ' a *subcurve* of γ if $I' \subseteq I$ and $\gamma' = \gamma|_{I'}$. We shall later refer to the following fact regarding subcurves.

Lemma 5.6. *Let G be an open subset of a topological space $\mathfrak{X}$, and D be a subset of G. Let $\gamma \colon [0, 1] \to \mathfrak{X}$ be a curve in $\Delta(D, \mathfrak{X} \smallsetminus G)$. Then $\gamma' := \gamma|_{[0,a]}$, where $a := \inf\{\gamma^{-1}(\mathfrak{X} \smallsetminus G)\}$, is a subcurve of γ in $\Delta(D, \partial G; G)$.*

For a curve γ in $\mathbb{R}^n$, the length of γ is defined in the usual way and denoted by $\operatorname{length}(\gamma)$. We call a curve γ *rectifiable* if $\operatorname{length}(\gamma) < +\infty$.

Let Γ be a family of curves in $\mathbb{R}^n$. A Borel function $\rho\colon \mathbb{R}^n \to [0, +\infty]$ is called *admissible* for Γ if $\int_\gamma \rho\, ds \geqslant 1$ for each rectifiable curve $\gamma \in \Gamma$. For each $p \in [1, +\infty)$, the *p-modulus* of Γ is defined as

$$\operatorname{Mod}_p(\Gamma) := \inf_\rho \int_{\mathbb{R}^n} \rho^p\, dm,$$

where ρ ranges over admissible Borel functions for Γ. Here and in what follows m is the n-dimensional Lebesgue measure on $\mathbb{R}^n$ With the dimension n understood, the n-modulus $\operatorname{Mod}_n(\Gamma)$ of Γ is usually abbreviated to *modulus* and denoted by $\operatorname{Mod}(\Gamma)$.

We record some elementary properties for modulus.

Lemma 5.7 ([Vu88, Lemma 5.2 (ii)]). *Let Γ_1, Γ_2 be families of curves in $\mathbb{R}^n$. If $\Gamma_1 \subseteq \Gamma_2$, then $\operatorname{Mod}(\Gamma_1) \leqslant \operatorname{Mod}(\Gamma_2)$.*

Lemma 5.8 ([Vu88, Lemma 5.3]). *Let Γ, Γ' be families of curves in $\mathbb{R}^n$. If for each $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma'$ such that γ' is a subcurve of γ, then $\operatorname{Mod}(\Gamma) \leqslant \operatorname{Mod}(\Gamma')$.*

Lemma 5.9 ([Vu88, Lemma 5.5]). *Let $G \subseteq \mathbb{R}^n$ be a Borel set and denote $\Gamma_r(G) := \{\gamma : \gamma$ is a curve in G with $\operatorname{length}(\gamma) \geqslant r\}$ for each $r > 0$. Then $\operatorname{Mod}(\Gamma_r(G)) \leqslant m(G)r^{-n}$.*

Lemma 5.10 ([Vu88, Lemma 7.38]). *Let E, $F \subseteq \mathbb{R}^n$ be disjoint continua with $\operatorname{diam}(E) > 0$, $\operatorname{diam}(F) > 0$. Then*

$$\operatorname{Mod}(\Delta(E,F)) \geqslant c_n \log\left(1 + \frac{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}}{\operatorname{dist}(E,F)} \right),$$

where $c_n > 0$ is a constant depending only on n.

Moreover, Lemma 5.10 can be strengthened to a two-sided bound of $\operatorname{Mod}(\Delta(E,F))$. In fact, the upper bound of $\operatorname{Mod}(\Delta(E,F))$ is valid even if E, F are disconnected.

Lemma 5.11. *Let $G \subseteq \mathbb{R}^n$ be a domain and E, $F \subseteq G$ be non-empty disjoint compact sets satisfying $\operatorname{dist}(E,F) \leqslant \operatorname{dist}(E, \mathbb{R}^n \smallsetminus G)$. Then*

$$\operatorname{Mod}(\Delta(E,F;G)) \leqslant \alpha_n \left(1 + \frac{\operatorname{diam}(E)}{\operatorname{dist}(E,F)} \right)^n,$$

where α_n denotes the n-dimensional Lebesgue measure of the unit ball $\mathbb{B}^n$.

Proof. Denote $d := \operatorname{diam}(E)$, $r := \operatorname{dist}(E,F)$, and consider the open set

$$D := \bigcup_{x \in E} \mathbb{B}^n(x, r).$$

It is clear that $E \subseteq D \subseteq G \smallsetminus F$, and that $m(\overline{D}) \leqslant \alpha_n(d+r)^n$.

For each $\gamma \in \Delta(E, F; G)$, by Lemma 5.6, there is a subcurve γ' of γ such that $\gamma' \in \Delta(E, \partial D; D)$. Since

$$\Delta(E, \partial D; D) \subseteq \{\gamma' : \gamma' \text{ is a curve in } \overline{D} \text{ with } \operatorname{length}(\gamma') \geqslant r\},$$

it follows from Lemmas 5.8 and 5.9 that

$$\operatorname{Mod}(\Delta(E, F; G)) \leqslant \alpha_n(d+r)^n \cdot r^{-n},$$

and the proof is complete. $\hfill\square$

Remark. In particular, if we set $G = \mathbb{R}^n$ in Lemma 5.11, then

$$\operatorname{Mod}(\Delta(E, F)) \leqslant \alpha_n\left(1 + \frac{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}}{\operatorname{dist}(E, F)}\right)^n.$$

Recall the notation $N(f, A)$ in (3.2). Then we have the following characterization of quasiregular maps in terms of moduli of curve families.

Theorem 5.12 ([Ri93, Theorem II.6.7])**.** *Let $f \colon G \to \mathbb{R}^n$ be a non-constant map on a domain $G \subseteq \mathbb{R}^n$ and let $1 \leqslant K < +\infty$. Then f is a quasiregular map with $K_O(f) \leqslant K$ if and only if the following hold:*

(i) *f is sense-preserving, discrete, and open.*
(ii) *For each Borel set $A \subseteq G$ with $N(f, A) < +\infty$ and each curve family Γ in A, we have*

$$\operatorname{Mod}(\Gamma) \leqslant K \cdot N(f, A) \cdot \operatorname{Mod}(f\Gamma).$$

We also have the following property about moduli of curve families in a normal domain of a quasiregular map. Recall that for a map $f \colon G \to \mathbb{R}^n$ on a domain G, a *normal domain* U is a domain for which $\overline{U} \subseteq G$ and $f(\partial U) = \partial f(U)$.

Lemma 5.13 ([Ri93, Corollary II.9.2])**.** *Let $f \colon G \to \mathbb{R}^n$ be a non-constant quasiregular map on a domain $G \subseteq \mathbb{R}^n$, and D be a normal domain for f. Let Γ be a family of curves in $f(D)$ and $\widetilde{\Gamma} := \{\gamma : \gamma$ is a curve in D such that $f \circ \gamma \in \Gamma\}$, then*

$$\operatorname{Mod}(\Gamma) \leqslant \frac{K_I(f)}{N(f, D)} \operatorname{Mod}(\widetilde{\Gamma}),$$

In [GW16], Guo and Williams study the quasiregularity of maps between metric measure spaces and show that weakly metrically quasiregular maps are BQS maps. In [LP19], Lindquist and the second-named author give a characterization of quasiregular maps between closed Riemannian manifolds in terms of BQS maps; see [LP19, Theorem 1.1].

We record a quantitative version of [LP19, Theorem 1.1].

Theorem 5.14 ([LP19, Theorem A.1])**.** *Let M and N be closed oriented Riemannian n-manifolds and $f\colon M \to N$ be a K-quasiregular map. Then there exists $\eta\colon [0, +\infty) \to [0, +\infty)$ depending only on K, such that f is a local η-BQS map.*

Note that Theorem 5.14 is quantitative in the sense that the distortion function η depends only on K, but the locality scale may depend also on f, M, and N.

Theorem 5.15 ([LP19, Theorem A.10])**.** *Let M and N be oriented Riemannian n-manifolds and $f\colon M \to N$ a discrete, open, and sense-preserving local η-BQS map. Then f is K-quasiregular for some $K \geqslant 1$ depending only on η.*

5.3. **Branched quasisymmetric maps with uniformity.** To study the relation between uniformly quasiregular maps and BQS maps, we consider branched quasisymmetric maps with uniformity. First, we introduce *local uniform branched quasisymmetric maps*.

Definition 5.16. A continuous map $f\colon X \to X$ on a metric space X is a *local uniform η-branched quasisymmetric map (abbr: local η-UBQS)* for a homeomorphism $\eta\colon [0, +\infty) \to [0, +\infty)$ if $f^k\colon X \to X$, $f^k := f \circ f^{k-1}$, is a local η-BQS map for each $k \in \mathbb{N}$. We simply say that f is a local UBQS map if there exists a homeomorphism η for which f is a local η-UBQS map.

Note that in Definition 5.16 we have no restriction on the locality scale. Therefore, we furthermore introduce *uniformly local UBQS maps* with respect to open covers. For a continuous map $f\colon X \to X$ on a locally connected compact metric space X, and a cover $\mathcal{U}$ of X by connected open sets, we denote

$$(f^*)^k \mathcal{U} := \big\{ \widetilde{U} : \widetilde{U} \text{ is a connected component of } f^{-k}(U) \text{ for some } U \in \mathcal{U} \big\}.$$

Definition 5.17 (Uniformly local UBQS maps)**.** A continuous map $f\colon X \to X$ on a locally connected compact metric space X is a *uniformly local η-UBQS map* for a homeomorphism $\eta\colon [0, +\infty) \to [0, +\infty)$ if there exists a finite cover $\mathcal{U}$ of X by connected open sets, such that $f^k|_{\widetilde{U}}$ is an η-BQS for all $k \in \mathbb{N}$ and $\widetilde{U} \in (f^*)^k \mathcal{U}$. We simply say that f is a *uniformly local UBQS map* if there exists a homeomorphism η for which f is uniformly local η-UBQS map.

Recall that a K-quasiregular map $f\colon M \to M$ on an oriented Riemannian n-manifold M is *uniform* if $f^k\colon M \to M$ is K-quasiregular for each $k \in \mathbb{N}$. We have the following characterization of uniformly quasiregular maps.

Theorem 5.18. *Let M be a closed and oriented Riemannian n-manifold and $f\colon M \to M$ a non-constant continuous map. Then f is uniformly quasiregular if and only if f is a discrete, open, and sense-preserving local UBQS map.*

Proof. Assume that $f\colon M \to M$ is a discrete, open, and sense-preserving local η-UBQS map, that is, f^k is a discrete, open, and sense-preserving local η-BQS map for each $k \in \mathbb{N}$. By Theorem 5.15, there exists $K \geqslant 1$ depending only on η such that for each $k \in \mathbb{N}$, the map f^k is K-quasiregular. Hence f is uniformly quasiregular.

Consider the converse implication. Assume that f is uniformly K-quasiregular for some $K \geqslant 1$. By Theorem 5.14, there exists $\eta\colon [0, +\infty) \to [0, +\infty)$ depending only on K such that for each $k \in \mathbb{N}$, the map f^k is a discrete, open, and sense-preserving local η-BQS map. Therefore f is a local η-UBQS map. $\qquad\square$

6. Quasisymmetric uniformization: branched quasisymmetries

In this section, we discuss the quasisymmetric uniformization of visual metrics in terms of BQS maps, and prove the metric part of Theorem 1.1. Precisely, we establish the following theorem.

Theorem 6.1. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be an expanding cellular Markov branched cover. Let σ be the spherical metric on $\mathbb{S}^n$ and let ϱ be a visual metric for f. Then $\mathrm{id}\colon (\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \varrho)$ is a quasisymmetry if and only if $f\colon (\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \sigma)$ is a uniformly local UBQS map.*

6.1. **Cellular neighborhoods.** Before we get down to the quasisymmetric uniformization problem, we introduce a class of neighborhoods constructed from cell decompositions, which plays an important role in this section and Section 7.

Let $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence of a cellular Markov map $f\colon \mathcal{S}^n \to \mathcal{S}^n$. For each pair $m, k \in \mathbb{N}_0$ and each $X \in \mathcal{D}_m^{[n]}$, we denote

$$(6.1) \qquad \begin{aligned} \mathcal{U}_{m,k}^1(X) &:= \big\{ Y \in \mathcal{D}_{m+k}^{[n]} : X \cap Y \neq \emptyset \big\}, \\ U_{m,k}^1(X) &:= \mathrm{int}(\textstyle\bigcup \mathcal{U}_{m,k}^1(X)), \end{aligned}$$

and

$$(6.2) \qquad \begin{aligned} \mathcal{U}_{m,k}^2(X) &:= \big\{ Y \in \mathcal{D}_{m+k}^{[n]} : Y \cap \textstyle\bigcup \mathcal{U}_{m,k}^1(X) \neq \emptyset \big\}, \\ U_{m,k}^2(X) &:= \mathrm{int}(\textstyle\bigcup \mathcal{U}_{m,k}^2(X)). \end{aligned}$$

Furthermore, for all m, k, $l \in \mathbb{N}_0$, and each $X \in \mathcal{D}_m^{[n]}$, we write
(6.3)
$$\mathcal{W}_{m,k,l}(X) := \big\{ Z \in \mathcal{D}_{m+k+l}^{[n]} : Z \notin \mathcal{U}_{m,k+l}^1(X) \text{ and } Z \subseteq \bigcup \mathcal{U}_{m,k}^1(X) \big\},$$
$$W_{m,k,l}(X) := \bigcup \mathcal{W}_{m,k,l}(X).$$

Then we have the following properties.

Lemma 6.2. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be a cellular Markov map and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence. Then, for all m, $k \in \mathbb{N}_0$ and each $X \in \mathcal{D}_m^{[n]}$, the following statements are true:*

   (i) $\bigcup \mathcal{U}_{m,k}^1(X) = \overline{U_{m,k}^1(X)}$, *and* $\bigcup \mathcal{U}_{m,k}^2(X) = \overline{U_{m,k}^2(X)}$.

   (ii) $\partial U_{m,k}^1(X) \subseteq \mathcal{S}^n \smallsetminus U_{m,k}^1(X) = \bigcup \big(\mathcal{D}_{m+k}^{[n]} \smallsetminus \mathcal{U}_{m,k}^1(X)\big)$.

   (iii) $X \subseteq U_{m,k}^1(X) \subseteq \overline{U_{m,k}^1(X)} \subseteq U_{m,k}^2(X)$.

   (iv) $U_{m,k}^1(X)$ *and* $U_{m,k}^2(X)$ *are path-connected.*

   (v) $\partial U_{m,k}^1(X) = \bigcup\{ c \in \mathcal{D}_{m+k} : \mathrm{int}_\circ(c) \cap \partial U_{m,k}^1(X) \neq \emptyset \}$.

   (vi) $\mathrm{dist}(Z, X) \geqslant \mathrm{dist}(X, \partial U_{m,k}^1(X))$ *for each* $Z \in \mathcal{D}_{m+k}^{[n]} \smallsetminus \mathcal{U}_{m,k}^1(X)$. *Here $\mathcal{S}^n$ is equipped with a spherical metric.*

Proof. Let $m, k \in \mathbb{N}_0$ and $X \in \mathcal{D}_m^{[n]}$ be arbitrary.

   (i) For each $Y \in \mathcal{U}_{m,k}^1(X)$, we have $Y = \overline{\mathrm{int}_\circ(Y)} \subseteq \overline{U_{m,k}^1(X)}$, thus $\bigcup \mathcal{U}_{m,k}^1(X) \subseteq \overline{U_{m,k}^1(X)}$. Then $\bigcup \mathcal{U}_{m,k}^1(X) = \overline{U_{m,k}^1(X)}$ since $\bigcup \mathcal{U}_{m,k}^1(X)$ is closed. It follows from the same argument that $\bigcup \mathcal{U}_{m,k}^2(X) = \overline{U_{m,k}^2(X)}$

   (ii) Clearly $\partial U_{m,k}^1(X) \subseteq \mathcal{S}^n \smallsetminus U_{m,k}^1(X)$. We show that $\mathcal{S}^n \smallsetminus U_{m,k}^1(X) = \bigcup \big(\mathcal{D}_{m+k}^{[n]} \smallsetminus \mathcal{U}_{m,k}^1(X)\big)$.

   It follows from $U_{m,k}^1(X) = \mathrm{int}(\bigcup \mathcal{U}_{m,k}^1(X))$ that $\mathcal{S}^n \smallsetminus U_{m,k}^1(X) \subseteq \overline{\mathcal{S}^n \smallsetminus \bigcup \mathcal{U}_{m,k}^1(X)}$. By Lemma 2.8 (ii) we have $\mathcal{S}^n = \bigcup \mathcal{D}_{m+k}^{[n]}$, and thus $\mathcal{S}^n \smallsetminus \bigcup \mathcal{U}_{m,k}^1(X) \subseteq \bigcup \big(\mathcal{D}_{m+k}^{[n]} \smallsetminus \mathcal{U}_{m,k}^1(X)\big)$. Then since $\bigcup \big(\mathcal{D}_{m+k}^{[n]} \smallsetminus \mathcal{U}_{m,k}^1(X)\big)$ is closed, we have

$$\mathcal{S}^n \smallsetminus U_{m,k}^1(X) \subseteq \overline{\mathcal{S}^n \smallsetminus \bigcup \mathcal{U}_{m,k}^1(X)} \subseteq \bigcup \big(\mathcal{D}_{m+k}^{[n]} \smallsetminus \mathcal{U}_{m,k}^1(X)\big).$$

On the other hand, consider arbitrary $Z \in \mathcal{D}_{m+k}^{[n]} \smallsetminus \mathcal{U}_{m,k}^1(X)$. Suppose on the contrary that $Z \cap U_{m,k}^1(X) \neq \emptyset$. Then, since $U_{m,k}^1(X)$ is open, and $Z = \overline{\mathrm{int}_\circ(Z)}$, we have that $\mathrm{int}_\circ(Z) \cap U_{m,k}^1(X) \neq \emptyset$, and thus $\mathrm{int}_\circ(Z) \cap Y \neq \emptyset$ for some $Y \in \mathcal{U}_{m,k}^1(X)$. Since $\dim(Z) = n$, it follows from Lemma 2.2 that $\mathrm{int}_\circ(Z)$ is open. Together with $Y = \overline{\mathrm{int}_\circ(Y)}$, we have that $\mathrm{int}_\circ(Z) \cap \mathrm{int}_\circ(Y) \neq \emptyset$, which implies $Z = Y$. This yields a contradiction. It follows that $\bigcup \big(\mathcal{D}_{m+k}^{[n]} \smallsetminus \mathcal{U}_{m,k}^1(X)\big) \subseteq \mathcal{S}^n \smallsetminus U_{m,k}^1(X)$.

(iii) First we show that $X \subseteq U^1_{m,k}(X)$. Assume on the contrary that $X \smallsetminus U^1_{m,k}(X) \neq \emptyset$, then, by (ii), there exists $Z \in \mathcal{D}^{[n]}_{m+k} \smallsetminus \mathcal{U}^1_{m,k}(X)$ for which $X \cap Z \neq \emptyset$, which yields a contradiction.

Then we show that $\overline{U^1_{m,k}(X)} \subseteq U^2_{m,k}(X)$. Let $x \in \overline{U^1_{m,k}(X)} = \bigcup \mathcal{U}^1_{m,k}(X)$ be arbitrary, and suppose that $x \in Y$ for some $Y \in \mathcal{U}^1_{m,k}(X)$. Suppose also that $x \in \mathrm{int}_\circ(c)$ for some $c \in \mathcal{D}_{m+k}$ with $c \subseteq Y$. Consider a 0-cell $\{p\} \in \mathcal{D}^{[0]}_{m+k}$ satisfying $p \in c$, then $x \in \mathrm{Fl}_{m+k}(p)$. For each $Z \in \mathcal{D}^{[n]}_{m+k}$ with $p \in Z$, it is clear that $Z \cap Y \neq \emptyset$, and thus $Z \in \mathcal{U}^2_{m,k}(X)$. Then it follows from Lemma 2.10 that $\overline{\mathrm{Fl}_{m+k}(p)} \subseteq \bigcup \mathcal{U}^2_{m,k}(X)$, and thus $\mathrm{Fl}_{m+k}(p) \subseteq \mathrm{int}(\bigcup \mathcal{U}^2_{m,k}(X)) = U^2_{m,k}(X)$. Since x is arbitrary, the claim follows.

(iv) Let $x \in U^1_{m,k}(X)$ be arbitrary, and let $Y \in \mathcal{U}^1_{m,k}(X)$ satisfies $x \in Y$. Consider $y \in Y \cap X$. It is clear that there exists a curve $\gamma \colon [0,1] \to Y$ with $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma((0,1)) \subseteq \mathrm{int}_\circ(Y)$. By (iii), we have that $y \in U^1_{m,k}(X)$. Also $\gamma((0,1)) \subseteq U^1_{m,k}(X)$ since $\mathrm{int}_\circ(Y)$ is an open subset of $\bigcup \mathcal{U}^1_{m,k}(X)$. Therefore, $\gamma([0,1]) \subseteq U^1_{m,k}(X)$. Since x is arbitrary, and X is a path-connected subset of $U^1_{m,k}(X)$, it follows that $U^1_{m,k}(X)$ is path-connected.

It follows from similar arguments that $U^2_{m,k}(X)$ is path-connected.

(v) It is clear that $\partial U^1_{m,k}(X) \subseteq \bigcup \{c \in \mathcal{D}_{m+k} : \mathrm{int}_\circ(c) \cap \partial U^1_{m,k}(X) \neq \emptyset\}$.

Consider now the other direction. Let $c \in \mathcal{D}_{m+k}$ satisfies $\mathrm{int}_\circ(c) \cap \partial U^1_{m,k}(X) \neq \emptyset$. Since we have by (ii) that

$$\partial U^1_{m,k}(X) = \overline{U^1_{m,k}(X)} \smallsetminus U^1_{m,k}(X) = \bigcup \mathcal{U}^1_{m,k}(X) \cap \bigcup (\mathcal{D}^{[n]}_{m+k} \smallsetminus \mathcal{U}^1_{m,k}(X)),$$

there exists $Y \in \mathcal{U}^1_{m,k}(X)$ and $Y' \in \mathcal{D}^{[n]}_{m+k} \smallsetminus \mathcal{U}^1_{m,k}(X)$ such that $\mathrm{int}_\circ(c) \cap Y \cap Y' \neq \emptyset$. Then it follows from Lemma 2.6 (ii) that $c \subseteq Y \cap Y' \subseteq \partial U^1_{m,k}(X)$.

(vi) Consider $Z \in \mathcal{D}^{[n]}_{m+k}$ with $Z \cap X = \emptyset$. It follows from (ii) that $Z \subseteq \mathcal{S}^n \smallsetminus U^1_{m,k}(X)$, and thus $\mathrm{dist}(X, Z) \geqslant \mathrm{dist}(X, \mathcal{S}^n \smallsetminus U^1_{m,k}(X)) = \mathrm{dist}(X, \partial U^1_{m,k}(X))$. $\square$

Corollary 6.3. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$, $n \geqslant 2$, be a cellular Markov branched cover and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence. Then for all m, $k \in \mathbb{N}_0$, and each $X \in \mathcal{D}^{[n]}_m$, the set $U^1_{m,k}(X) \smallsetminus X$ is path-connected.*

Proof. Let m, $k \in \mathbb{N}_0$ and $X \in \mathcal{D}^{[n]}_m$ be arbitrary. Set $A := \mathcal{S}^n \smallsetminus X$ and $B := U^1_{m,k}(X)$. It follows from the Jordan–Brouwer separation theorem

that A is path-connected. Also, by Lemma 6.2 (iv) we have that B is path-connected. It is clear that $A \cup B = \mathcal{S}^n$, and $A \cap B = U^1_{m,k}(X) \smallsetminus X$.

Consider the following Mayer–Vietoris sequence

$$\cdots \to \widetilde{H}_{q+1}(\mathcal{S}^n) \to \widetilde{H}_q(A \cap B) \to \widetilde{H}_q(A) \oplus \widetilde{H}_q(B) \to \widetilde{H}_q(\mathcal{S}^n) \to \cdots$$

of reduced homology groups. Since $\widetilde{H}_1(\mathcal{S}^n) = 0$, and $\widetilde{H}_0(A) = \widetilde{H}_0(B) = 0$, we have that $\widetilde{H}_0(A \cap B) = 0$, hence $A \cap B$ is connected. $\qquad\square$

For more on homology groups and Mayer–Vietoris sequences, see e.g. [Sp66].

Note that, the above are elementary topological properties that are valid even if we do not assume f to be a branched cover. For expanding cellular Markov branched covers, we have the following.

Lemma 6.4. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence. Then there exists $l_0 \in \mathbb{N}$ such that for all m, k, $l \in \mathbb{N}_0$ with $l \geqslant l_0$ and each $X \in \mathcal{D}^{[n]}_m$, we have $\overline{U^1_{m,k+l}(X)} \subseteq U^1_{m,k}(X)$.*

Proof. Since f is expanding, there exists $l_0 \in \mathbb{N}$ such that no cell in $\mathcal{D}_l$ with $l \geqslant l_0$ joins opposite sides of $\mathcal{D}_0$. Fix arbitrary m, k, $l \in \mathbb{N}_0$ with $l \geqslant l_0$ and $X \in \mathcal{D}^{[n]}_m$.

Suppose on the contrary that $\overline{U^1_{m,k+l}(X)} \nsubseteq U^1_{m,k}(X)$ and consider $x \in \overline{U^1_{m,k+l}(X)} \smallsetminus U^1_{m,k}(X)$. By Lemma 6.2 (i) and (ii), suppose $Z \in \mathcal{U}^1_{m,k+l}(X)$ and $Y \in \mathcal{D}^{[n]}_{m+k} \smallsetminus \mathcal{U}^1_{m,k}(X)$ satisfies $Z \cap Y \neq \emptyset$. Suppose also $X' \in \mathcal{D}^{[n]}_{m+k}$ satisfies $X' \subseteq X$ and $Z \cap X \neq \emptyset$. It follows from Corollary 2.11, that Z cannot be contained in a level-$(m+k)$ flower. Then, by Corollary 3.17, the cell $f^{m+k}(Z) \in \mathcal{D}^{[n]}_l$ cannot be contained in a level-0 flower. However, by the choice of l_0, each level-l room is contained in some level-0 flower (see Lemma 2.13), which leads to a contradiction. $\qquad\square$

As a consequence, when $l \in \mathbb{N}$ is sufficiently large, the structure of a subset in the form $W_{m,k,l}(X)$ is clear.

Lemma 6.5. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence. Then, there exists $l_0 \in \mathbb{N}$ such that for all m, k, $l \in \mathbb{N}_0$ with $l \geqslant l_0$ and $X \in \mathcal{D}^{[n]}_m$, we have $W_{m,k,l}(X) = \overline{U^1_{m,k}(X)} \smallsetminus U^1_{m,k+l}(X)$.*

Proof. Let $l_0 \in \mathbb{N}$ be the constant in Lemma 6.4. Fix arbitrary m, k, $l \in \mathbb{N}_0$ and $X \in \mathcal{D}^{[n]}_m$, where $l \geqslant l_0$.

First, we prove that $\overline{U^1_{m,k}(X)} \smallsetminus U^1_{m,k+l}(X) \subseteq W_{m,k,l}(X)$. It follows from Lemma 6.2 (i) and (ii) that

$$\overline{U^1_{m,k}(X)} \smallsetminus U^1_{m,k+l}(X) = \bigcup \mathcal{U}^1_{m,k}(X) \cap \bigcup (\mathcal{D}^{[n]}_{m+k+l} \smallsetminus \mathcal{U}^1_{m,k+l}(X)).$$

Also, it is clear that

$$\bigcup \mathcal{U}^1_{m,k}(X) = \bigcup_{Y \in \mathcal{U}^1_{m,k}(X)} \bigcup \{ Z \in \mathcal{D}^{[n]}_{m+k+l} : Z \subseteq Y \}.$$

Consider arbitrary $Y \in \mathcal{U}^1_{m,k}(X)$, $Z \in \mathcal{D}^{[n]}_{m+k+l}$ and $W \in \mathcal{D}^{[n]}_{m+k+l} \smallsetminus \mathcal{U}^1_{m,k+l}(X)$ with $Z \subseteq Y$, $Z \cap W \neq \emptyset$. Suppose that $Y' \in \mathcal{D}^{[n]}_{m+k}$ satisfies $W \subseteq Y'$. If $Z \notin \mathcal{U}^1_{m,k+l}(X)$, then we have $Z \cap W \subseteq Z \subseteq W_{m,k,l}(X)$. Suppose now that $Z \in \mathcal{U}^1_{m,k+l}(X)$. It follows from Lemma 6.4 that $Z \cap W \subseteq U^1_{m,k}(X)$, and thus $Y' \cap U^1_{m,k}(X) \neq \emptyset$. Then by Lemma 6.2 (ii) we have $Y' \in \mathcal{U}^1_{m,k}(X)$, and thus $Z \cap W \subseteq W \subseteq W_{m,k,l}(X)$. In conclusion, we have that $\overline{U^1_{m,k}(X)} \smallsetminus U^1_{m,k+l}(X) \subseteq W_{m,k,l}(X)$. On the other hand, it is easy to see from the definition that $W_{m,k,l}(X) \subseteq \overline{U^1_{m,k}(X)} \smallsetminus U^1_{m,k+l}(X)$, and thus the proof is complete. $\square$

Also, we have that the boundary of the subset $U^1_{m,k}(X)$ is contained in $W_{m,k,l}(X)$, whenever $l \in \mathbb{N}$ is sufficiently large.

Lemma 6.6. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence. Then there exists $l_0 \in \mathbb{N}$ such that for each m, k, $l \in \mathbb{N}_0$ with $l \geqslant l_0$, and each $X \in \mathcal{D}^{[n]}_m$, we have $\partial U^1_{m,k}(X) \subseteq W_{m,k,l}(X)$. In particular, if we equip $\mathcal{S}^n$ with a metric compatible with its topology, then*

$$\mathrm{dist}\big(X, \partial U^1_{m,k}(X)\big) \geqslant \mathrm{dist}(X, W_{m,k,l}(X)) > 0.$$

Proof. Let $l_0 \in \mathbb{N}$ be the constant in Lemma 6.5. Fix arbitrary m, k, $l \in \mathbb{N}_0$ and $X \in \mathcal{D}^{[n]}_m$, where $l \geqslant l_0$. Then, it follows from Lemma 6.5 that

$$\partial U^1_{m,k}(X) = \overline{U^1_{m,k}(X)} \smallsetminus U^1_{m,k}(X) \subseteq \overline{U^1_{m,k}(X)} \smallsetminus U^1_{m,k+l}(X) = W_{m,k,l}(X).$$

Clearly $X \cap W_{m,k,l}(X) = \emptyset$, then $\mathrm{dist}(X, W_{m,k,l}(X)) > 0$, and the proof is complete. $\square$

6.2. **BQS properties with respect to visual metrics.** First, we show that a cellular Markov branched cover is a uniformly local UBQS map with respect to each visual metric.

Lemma 6.7. *Let $f \colon \mathcal{S}^n \to \mathcal{S}^n$ be an expanding cellular Markov branched cover, and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence of f. Let* var *be a visual metric for f on $\mathcal{S}^n$. Then, for each $m \in \mathbb{N}$ and each $U \in \mathcal{F}(\mathcal{D}_m)$, the*

map $f^m|_U$ is an η-BQS map (with respect to ϱ) for some η independent of m and U.

Proof. To be concise, denote $m(x,y) \coloneqq m_{f,\mathcal{D}_0}(x,y)$ for all x, $y \in \mathcal{S}^n$. Let $\Lambda > 1$ be the expansion factor for ϱ, and let $C > 1$ be the constant for which

$$C^{-1}\Lambda^{-m(x,y)} \leqslant \varrho(x,y) \leqslant C\Lambda^{-m(x,y)}$$

holds for all x, $y \in \mathcal{S}^n$. By Lemma 4.20 (i), there exists a constant $C' > 1$ for which

$$C'^{-1}\Lambda^{-m} \leqslant \mathrm{diam}(X) \leqslant C'\Lambda^{-m}$$

holds for all $m \in \mathbb{N}$ and $X \in \mathcal{D}_m^{[n]}$.

Denote for each $m \in \mathbb{N}$ and each $x \in \mathcal{S}^n$

$$U_m(x) \coloneqq \bigcup \{Y \in \mathcal{D}_m : \text{there exists } X \in \mathcal{D}_m \text{ with } x \in X, \ X \cap Y \neq \emptyset\}.$$

It is clear that $U_m(x)$ is a connected set with $U_m(x) \subseteq B(x, 4C'\Lambda^{-m})$.

Fix $k_0 \in \mathbb{N}$ such that $8C'\Lambda^{-k_0}$ is strictly less than the Lebesgue number of the open cover $\mathcal{F}(\mathcal{D}_0)$. Then, for each $m \in \mathbb{N}$ and $x \in \mathcal{S}^n$, since $f^m(U_{m+k_0}(x)) \subseteq U_{k_0}(f^m(x))$, and $U_{k_0}(f^m(x))$ is contained in a level-0 flower, it follows from Corollary 3.17 that $U_{m+k_0}(x)$ is contained in a level-m flower.

Fix arbitrary $m \in \mathbb{N}$, level-m vertex p, and a continuum $A \subseteq \mathrm{Fl}_m(p)$. Put

$$l \coloneqq \max\{l' \in \mathbb{N} : \text{there exists a level-}l' \text{ flower that contains } A\}.$$

It is clear that $m \leqslant l < +\infty$.

Now we give an estimation of $\mathrm{diam}(A)$ and $\mathrm{diam}(f^m(A))$. First, it is clear that $\mathrm{diam}(A) \leqslant 2C'\Lambda^{-l}$. We then show that $\mathrm{diam}(A) \geqslant C^{-1}\Lambda^{-(l+k_0+1)}$. Suppose on the contrary $\mathrm{diam}(A) < C^{-1}\Lambda^{-(l+k_0+1)}$. Fix $x \in A$, then for each $y \in A$, we have

$$C^{-1}\Lambda^{-m(x,y)} \leqslant \varrho(x,y) < C^{-1}\Lambda^{-(l+k_0+1)},$$

and thus $m(x,y) > l + k_0 + 1$. Then we have $A \subseteq U_{l+k_0+1}(x)$, and thus A is contained in a level-$(l+1)$ flower, which contradicts the choice of l. Therefore, we have

$$C^{-1}\Lambda^{-(l+k_0+1)} \leqslant \mathrm{diam}(A) \leqslant 2C'\Lambda^{-l}.$$

Clearly $f^m(A)$ is also a continuum. It follows from Corollary 3.17 that $f^m(A)$ is contained in a level-$(l-m)$ flower, and that $f^m(A)$ cannot be contained in a level-$(l'-m)$ flower for any $l' > l$. By the same argument as above, we have

$$C^{-1}\Lambda^{-(l+k_0+1)+m} \leqslant \mathrm{diam}(f^m(A)) \leqslant 2C'\Lambda^{-l+m}.$$

Then it follows that

$$(2C'C\Lambda^{k_0+1})^{-1}\Lambda^m \leqslant \operatorname{diam}(f^m(A))/\operatorname{diam}(A) \leqslant 2C'C\Lambda^{k_0+1}\Lambda^m.$$

Let E and E' be two continua contained in $\operatorname{Fl}_m(p)$, then we have

$$\frac{\operatorname{diam}(f^m(E))}{\operatorname{diam}(f^m(E'))} \leqslant (2C'C\Lambda^{k_0+1})^2 \frac{\operatorname{diam}(E)}{\operatorname{diam}(E')}.$$

Define $\eta\colon [0,+\infty) \to [0,+\infty), \ t \mapsto (2C'C\Lambda^{k_0+1})^2 t$, then the proof is complete. $\qquad\square$

By Lemma 3.14, we have $\mathcal{F}(\mathcal{D}_m) = (f^*)^k(\mathcal{F}(\mathcal{D}_0))$. Hence the following corollary is a straightforward consequence of Lemma 6.7.

Corollary 6.8. *An expanding cellular Markov branched covering map $f\colon \mathcal{S}^n \to \mathcal{S}^n$ is a uniformly local UBQS map with respect to each visual metric ϱ.*

Corollary 6.9. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be an expanding cellular Markov branched cover, $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence, and ϱ be a visual metric of f. Let σ be the spherical metric on $\mathbb{S}^n$. If $\mathrm{id}\colon (\mathbb{S}^n, \varrho) \to (\mathbb{S}^n, \sigma)$ is a quasisymmetry, then for each $m \in \mathbb{N}$ and each $U \in \mathcal{F}(\mathcal{D}_m)$, the restriction $f^m|_U$ is an η-BQS map (with respect to σ) for some η independent of m and U. In particular, f is uniformly local UBQS (with respect to σ).*

Proof. Suppose that $\mathrm{id}\colon (\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \varrho)$ is a quasisymmetry. By Lemma 6.7, for each $m \in \mathbb{N}_0$ and each $U \in \mathcal{F}(\mathcal{D}_m)$, the map $f^m|_U$ is an η'-BQS map (with respect to ϱ) for some η' independent of m and U. Then it follows from Lemmas 5.2 and 5.3 that $f^m|_U$ is an η-BQS map (with respect to σ) for some η independent of $m \in \mathbb{N}_0$ and $U \in \mathcal{F}(\mathcal{D}_m)$. $\qquad\square$

6.3. **BQS with uniformity implies quasisymmetric equivalence.** In this subsection, we consider the converse of Corollary 6.9. We show that if an expanding cellular Markov branched cover on $\mathbb{S}^n$ is uniformly local UBQS with respect to the spherical metric σ, then its visual metrics are quasisymmetric equivalent to σ.

In the following lemmas, we show some metric properties with respect to the spherical metric σ, for essential sequences of cellular Markov branched covers on $\mathbb{S}^n$ with a uniformly local UBQS condition. These metric properties allow us to establish the quasisymmetric equivalence between the spherical metric and visual metrics. See [BM17, Chapter 18] for a similar discussion on Thurston maps.

Lemma 6.10. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be an expanding cellular Markov branched cover, and $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence. Let σ be the spherical metric on $\mathbb{S}^n$. If for each $m \in \mathbb{N}$ and each $U \in \mathcal{F}(\mathcal{D}_m)$, the restriction $f^m|_U$ is an η-BQS map (with respect to σ) for some η independent of m and U, then the following statements hold:*

(i) *There exists $C_1 > 1$ such that for each $m \in \mathbb{N}_0$ and all $Z, Z' \in \mathcal{D}_m^{[n]}$ with $Z \cap Z' \neq \emptyset$, we have*

$$C_1^{-1} \leqslant \frac{\operatorname{diam}(Z)}{\operatorname{diam}(Z')} \leqslant C_1.$$

(ii) *There exists $C_2 > 0$ such that for all $m, k \in \mathbb{N}_0$ and all $Z \in \mathcal{D}_m^{[n]}$, $W \in \mathcal{D}_{m+k}^{[n]}$ with $Z \cap W \neq \emptyset$, we have*

$$\operatorname{diam}(Z) \geqslant C_2 \operatorname{diam}(W).$$

(iii) *For each $k \in \mathbb{N}$, there exists $C_3(k) > 0$ such that for each $m \in \mathbb{N}_0$ and all $Z \in \mathcal{D}_m^{[n]}$, $W \in \mathcal{D}_{m+k}^{[n]}$ with $Z \cap W \neq \emptyset$, we have*

$$\operatorname{diam}(W) \geqslant C_3(k) \operatorname{diam}(Z).$$

Proof. (i) Fix $m_1 \in \mathbb{N}$ such that $\operatorname{mesh}(\mathcal{D}_m) < \delta_0/2$ for all $m \geqslant m_1$, where δ_0 is the Lebesgue number of the open cover $\mathcal{F}(\mathcal{D}_0)$. Such m_1 exists because f is expanding.

The set $\left\{\operatorname{diam}(Z')/\operatorname{diam}(Z) : Z', Z \in \mathcal{D}_k^{[n]}, \ k \leqslant m_1\right\}$ is a finite set of positive real numbers, and we set $a \geqslant 1$ to be its maximum. Then, when $m \leqslant m_1$, we have $a^{-1} \leqslant \operatorname{diam}(Z)/\operatorname{diam}(Z') \leqslant a$ for all $Z, Z' \in \mathcal{D}_m^{[n]}$ with $Z \cap Z' \neq \emptyset$.

Suppose now that $m > m_1$. Denote $g := f^{m-m_1}$. Fix arbitrary $Z, Z' \in \mathcal{D}_m^{[n]}$ with $Z \cap Z' \neq \emptyset$. It is clear that $Z \cup Z'$ and $g(Z \cup Z')$ are connected sets. Since $g(Z \cup Z') = g(Z) \cup g(Z')$ is a union of two level-m_1 rooms, we have $\operatorname{diam}(g(Z \cup Z')) \leqslant 2 \cdot \operatorname{mesh}_\sigma(\mathcal{D}_{m_1}) < \delta_0$, and thus $g(Z) \cup g(Z')$ is contained in a level-0 flower. Then, by Corollary 3.17, there exists $U \in \mathcal{F}(\mathcal{D}_{m-m_1})$ such that $Z \cup Z' \subseteq U$. Since $g|_U$ is η-BQS, we have

$$\operatorname{diam}(g(Z))/\operatorname{diam}(g(Z')) \leqslant \eta(\operatorname{diam}(Z)/\operatorname{diam}(Z')),$$

and thus

$$\operatorname{diam}(Z)/\operatorname{diam}(Z') \geqslant \eta^{-1}(\operatorname{diam}(g(Z))/\operatorname{diam}(g(Z'))).$$

The set $\left\{\eta^{-1}(\operatorname{diam}(Y)/\operatorname{diam}(X)) : Y, X \in \mathcal{D}_{m_1}^{[n]}\right\}$ is a finite set of positive numbers. Set $b < 1$ to be one of its lower bounds. Then we have $\operatorname{diam}(Z)/\operatorname{diam}(Z') \geqslant b$, and, by the same argument, we have $\operatorname{diam}(Z')/\operatorname{diam}(Z) \geqslant b$.

Let $C_1 := \max\{a,\, b^{-1}\}$, then we have

$$C_1^{-1} \leqslant \operatorname{diam}(Z)/\operatorname{diam}(Z') \leqslant C_1,$$

and the proof of (i) is complete.

(ii) Consider $m,\, k \in \mathbb{N}_0$, and $Z \in \mathcal{D}_m^{[n]}$, $W \in \mathcal{D}_{m+k}^{[n]}$ with $Z \cap W \neq \emptyset$. Suppose that $Z' \in \mathcal{D}_m^{[n]}$ satisfies $W \subseteq Z'$. By (i) we have

$$\operatorname{diam}(Z) \geqslant C_1^{-1}\operatorname{diam}(Z') \geqslant C_1^{-1}\operatorname{diam}(W).$$

It suffices to put $C_2 := C_1^{-1}$.

(iii) Fix an arbitrary $k \in \mathbb{N}$. Let $m_1 \in \mathbb{N}$ be the same as in the proof of (i).

Set $a_k := \min\big\{\operatorname{diam}(Y)/\operatorname{diam}(X) : Y \in \mathcal{D}_{l+k}^{[n]},\ X \in \mathcal{D}_l^{[n]} : l \leqslant m_1\big\}$. Then we have $\operatorname{diam}(W)/\operatorname{diam}(Z) \geqslant a_k$ for all $Z \in \mathcal{D}_m^{[n]}$ and $W \in \mathcal{D}_{m+k}^{[n]}$ whenever $m \leqslant m_1$.

Suppose now that $m > m_1$ and denote $g := f^{m-m_1}$ for simplicity. Consider $Z \in \mathcal{D}_m^{[n]}$ and $W \in \mathcal{D}_{m+k}^{[n]}$ with $Z \cap W \neq \emptyset$. Then $Z \cup W$ is contained in a level-$(m - m_1)$ flower, and thus by the same arguments in the proof of (i), we have

$$\operatorname{diam}(W)/\operatorname{diam}(Z) \geqslant \eta^{-1}(\operatorname{diam}(g(W))/\operatorname{diam}(g(Z))) \geqslant b_k,$$

where $b_k := \min\big\{\eta^{-1}(\operatorname{diam}(P)/\operatorname{diam}(Z)) : P \in \mathcal{D}_{m_1+k}^{[n]},\ Z \in \mathcal{D}_{m_1}^{[n]}\big\}$. Set $C_3(k) := \max\{a_k,\, b_k\}$, then we have $\operatorname{diam}(W) \geqslant C_3(k)\operatorname{diam}(Z)$, and the proof of (iii) is complete. $\qquad\square$

Lemma 6.11. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be an expanding cellular Markov branched cover, and $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence. Let σ be the spherical metric on $\mathbb{S}^n$. If for each $m \in \mathbb{N}$ and each $U \in \mathcal{F}(\mathcal{D}_m)$, the restriction $f^m|_U$ is an η-BQS map (with respect to σ) for some η independent of m and U, then the following statements hold:*

(i) *There exists $C_4 > 0$ such that for each $m \in \mathbb{N}$ and all $Z, W \in \mathcal{D}_m^{[n]}$ with $Z \cap W = \emptyset$, we have*

$$\operatorname{dist}(Z, W) \geqslant C_4 \operatorname{diam}(Z).$$

(ii) *There exists $C_5 > 1$ such that for all $x,\, y \in \mathbb{S}^n$, $m := m_{f,\mathcal{D}_0}(x, y)$, and each $Z \in \mathcal{D}_m^{[n]}$ containing x, we have*

$$C_5^{-1} \leqslant \frac{\sigma(x, y)}{\operatorname{diam}(Z)} \leqslant C_5.$$

Proof. (i) Fix $m \in \mathbb{N}$, and $Z, W \in \mathcal{D}_m^{[n]}$ satisfying $Z \cap W = \emptyset$. We denote $\mathcal{U}(Z) := \mathcal{U}_{m,0}^1(Z)$ and $U(Z) := U_{m,0}^1(Z)$ for simplicity.

Same as the proof of Lemma 6.10 (i), we may fix sufficiently large $m_2 \in \mathbb{N}$ for which $\mathrm{mesh}_\sigma(\mathcal{D}_{m_2}) < \delta_0/3$, where δ_0 is the Lebesgue number of the open cover $\mathcal{F}(\mathcal{D}_0)$.

Put

$$a := \min\{\mathrm{dist}(X, Y)/\mathrm{diam}(X) : X, Y \in \mathcal{D}_m^{[n]}, \ X \cap Y = \emptyset, \ m \leqslant m_2\}.$$

Then $\mathrm{dist}(Z, W) \geqslant a \, \mathrm{diam}\, Z$ if $m \leqslant m_2$.

Suppose now that $m > m_2$. Denote $g := f^{m-m_2}$. It is clear that $\bigcup \mathcal{U}(Z)$ and $g(\bigcup \mathcal{U}(Z))$ are connected sets. Since

$$g\Big(\bigcup \mathcal{U}(Z)\Big) = \bigcup\{g(Z') : Z' \in \mathcal{U}(Z)\}$$

is a union of level-m_2 rooms, and $\mathrm{diam}(g(\bigcup \mathcal{U}(Z))) \leqslant 3 \, \mathrm{mesh}_\sigma(\mathcal{D}_{m_2}) < \delta_0$, we have that $g(\bigcup \mathcal{U}(Z))$ is contained in a level-0 flower. Then, by Corollary 3.17, $\bigcup \mathcal{U}(Z)$ is contained in a level-$(m - m_2)$ flower.

Suppose that $U(Z) \subseteq \bigcup \mathcal{U}(Z) \subseteq \mathrm{Fl}_{m-m_2}(p)$ for some level-$(m - m_2)$ vertex p. Fix an arbitrary $x \in Z$. Let $y \in \partial U(Z)$ satisfies $\sigma(x, y) = \mathrm{dist}(x, \partial U(Z))$. Consider $B := \{z \in \mathbb{S}^n : \sigma(x, z) \leqslant \sigma(x, y)\}$, then B is a continuum with $B \subseteq \bigcup \mathcal{U}(Z) \subseteq \mathrm{Fl}_{m-m_2}(p)$. By Lemma 6.2 (ii), the set B joins two distinct level-m rooms. Thus, by Corollary 2.11, the connected set B cannot be contained in a level-m flower. It follows from Corollary 3.17 that $g(B)$ cannot be contained in any level-m_2 flower. Then $\mathrm{diam}(g(B)) \geqslant \delta_{m_2}$, where δ_{m_2} is the Lebesgue number of the open cover $\mathcal{F}(\mathcal{D}_{m_2})$.

Since $g|_{\mathrm{Fl}_{m-m_2}(p)}$ is η-BQS, and B, Z are continua in $\mathrm{Fl}_{m-m_2}(p)$, we have

$$\frac{\sigma(x, y)}{\mathrm{diam}(Z)} \geqslant \frac{\mathrm{diam}(B)}{2\,\mathrm{diam}(Z)} \geqslant \frac{1}{2}\eta^{-1}\!\left(\frac{\mathrm{diam}(g(B))}{\mathrm{diam}(g(Z))}\right) \geqslant \frac{1}{2}\eta^{-1}\!\left(\frac{\delta_{m_2}}{\mathrm{diam}(g(Z))}\right).$$

Put $b := \min\{(1/2)\eta^{-1}(\delta_{m_2}/\mathrm{diam}(X)) : X \in \mathcal{D}_{m_2}^{[n]}\}$. Then it follows from Lemma 6.2 (vi) that

$$\mathrm{dist}(Z, W) \geqslant \mathrm{dist}(Z, \partial U(Z)) = \inf\{\mathrm{dist}(x, \partial U(Z)) : x \in Z\} \geqslant b \, \mathrm{diam}(Z).$$

Therefore $C_4 := \min\{a, b\}$ is the desired constant. The proof of (iv) is now complete.

(ii) Let $x, y \in \mathbb{S}^n$ be arbitrary. Denote $m := m_{f,\mathcal{D}_0}(x, y)$ and suppose $Z \in \mathcal{D}_m^{[n]}$ satisfies $x \in Z$.

Suppose that $Z', Z'' \in \mathcal{D}_m$ satisfy $x \in Z'$, $y \in Z''$ and $Z' \cap Z'' \neq \emptyset$. Let $W, W' \in \mathcal{D}_{m+1}$ satisfy $x \in W$, $y \in W'$. The definition of $m_{f,\mathcal{D}_0}(x, y)$ implies $W \cap W' = \emptyset$.

By Lemma 6.10 (i), we have

$$\sigma(x, y) \leqslant \mathrm{diam}(Z') + \mathrm{diam}(Z'') \leqslant (1 + C_1)\,\mathrm{diam}(Z') \leqslant C_1(1 + C_1)\,\mathrm{diam}(Z).$$

Then, by (i) and by Lemma 6.10 (iii), we have that

$$\sigma(x,y) \geqslant \text{dist}(W,W') \geqslant C_4 \, \text{diam}(W) \geqslant C_4 C_3(1) \, \text{diam}(Z).$$

Let $C_5 := \max\{C_1(1+C_1), \, C_4 C_3(1)\}$, then we have

$$C_5^{-1} \leqslant \sigma(x,y)/\text{diam}(Z) \leqslant C_5.$$

The proof is complete. $\square$

Theorem 6.12. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be an expanding cellular Markov branched cover, $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence, and ϱ be a visual metric for f. Let σ be the spherical metric on $\mathbb{S}^n$. If for each $m \in \mathbb{N}$ and each $U \in \mathcal{F}(\mathcal{D}_m)$, the restriction $f^m|_U$ is an η-BQS map (with respect to σ) for some η independent of m and U, then $\text{id}\colon (\mathbb{S}^n,\sigma) \to (\mathbb{S}^n,\varrho)$ is a quasisymmetry.*

Remark. Note that, *quasi-visual approximation* introduced in [BM22] is a powerful tool for so-called quasisymmetric uniformization problems. For Theorem 6.12, one may verify that $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ is a quasi-visual approximation of both $(\mathbb{S}^n,\sigma)$ and $(\mathbb{S}^n,\varrho)$, then it follows that $\text{id}\colon (\mathbb{S}^n,\sigma) \to (\mathbb{S}^n,\varrho)$ is a quasisymmetry. For the definition and a detailed discussion on quasi-visual approximation, see e.g. [BM22, Section 2]. However, for the convenience of the reader, we include a straightforward proof of Theorem 6.12 as follows.

Proof of Theorem 6.12. Denote $m(x,y) := m_{f,\mathcal{D}_0}(x,y)$ for all $x,\, y \in \mathbb{S}^n$.

Let $\Lambda > 1$ be the expansion factor of ϱ, and $C > 0$ be the constant for which

(6.4) $$C^{-1}\Lambda^{-m(x,y)} \leqslant \varrho(x,y) \leqslant C\Lambda^{-m(x,y)}$$

holds for all $x,\, y \in \mathbb{S}^n$.

Fix arbitrary $x,\, y,\, z \in \mathbb{S}^n$. Denote $m_1 := m(x,y)$ and $m_2 := m(x,z)$. Let $X_1 \in \mathcal{D}_{m_1}^{[n]}$, $X_2 \in \mathcal{D}_{m_2}^{[n]}$ be n-cells containing x. Then, by Lemma 6.11 (ii), we have that

$$\sigma(x,y)/\sigma(x,z) \leqslant C_5^2 \, \text{diam}_\sigma(X_1)/\text{diam}_\sigma(X_2),$$

where the constant $C_5 > 1$ comes from Lemma 6.11 (ii). Consider the following cases according to m_1 and m_2.

Case 1. Suppose $m_1 \leqslant m_2$, and denote $k := m_2 - m_1$. Fix an arbitrary contant $\lambda > \max\{1, C_1, 1/C_3(1)\}$. By Lemma 6.10 (iii),

$$\text{diam}_\sigma(X_1)/\text{diam}_\sigma(X_2) \leqslant \lambda^{k+1} = (\Lambda^{k+1})^\alpha,$$

where $\alpha := \log \lambda / \log \Lambda > 0$. Then it follows from (6.4) that $\Lambda^{-k} \leqslant C^2 \varrho(x,y)/\varrho(x,z)$, and thus

$$\frac{\mathrm{diam}_\sigma(X_1)}{diam_\sigma(X_2)} \leqslant \left(C^2 \Lambda \frac{\varrho(x,y)}{\varrho(x,z)} \right)^\alpha.$$

Case 2. Suppose $m_1 > m_2$, and denote $k := m_1 - m_2$. Fix $m_0 \in \mathbb{N}$ such that $\mathrm{mesh}(\mathcal{D}_m) < \delta_0/2$ for each $m \geqslant m_0$, where δ_0 is the Lebesgue number of $\mathcal{F}(\mathcal{D}_0)$. Put

$$a_l := \max_{m \in \mathbb{N}_0, \, m \leqslant m_0} \left\{ \mathrm{diam}_\sigma(X) / \mathrm{diam}_\sigma(Y) : X \in \mathcal{D}_{m+l}^{[n]}, \; Y \in \mathcal{D}_m^{[n]} \right\}$$

for each $l \in \mathbb{N}$. If $m_2 \leqslant m_0$, then $\mathrm{diam}_\sigma(X_1) / \mathrm{diam}_\sigma(X_2) \leqslant a_k$.

Assume now that $m_2 > m_0$, then there exists a flower $U \in \mathcal{F}(\mathcal{D}_{m_2 - m_0})$ with $X_1 \cup X_2 \subseteq U$. Since the restriction of $f^{m_2 - m_0}|_U$ is η-BQS, we have

$$\frac{\mathrm{diam}_\sigma(X_1)}{\mathrm{diam}_\sigma(X_2)} \leqslant \frac{1}{\eta^{-1}\left(\frac{\mathrm{diam}_\sigma(Y_2)}{\mathrm{diam}_\sigma(Y_1)}\right)} \leqslant \frac{1}{\eta^{-1}(1/a_k)},$$

where $Y_1 := f^{m_2 - m_0}(X_1)$ and $Y_2 := f^{m_2 - m_0}(X_2)$.

Since $\mathrm{mesh}(\mathcal{D}_l) \to 0$ as $l \to +\infty$, we have $a_l \to 0$ and $\frac{1}{\eta^{-1}(1/a_l)} \to 0$ as $l \to +\infty$. Let $h : \mathbb{R} \to (0, +\infty)$ be a continuous function with the following properties:

(1) h is strictly decreasing.

(2) $h(t) \to 0$ as $t \to +\infty$; $h(t) \to +\infty$ as $t \to -\infty$.

(3) $h(l) > \max\left\{ a_l, \frac{1}{\eta^{-1}(1/a_l)} \right\}$ for each $l \in \mathbb{N}$.

Then we have that $\mathrm{diam}_\sigma(X_1) / \mathrm{diam}_\sigma(X_2) \leqslant h(k)$. Moreover, since $\Lambda^{-k} \leqslant C^2 \varrho(x,y)/\varrho(x,z)$, we have that

$$\frac{\mathrm{diam}_\sigma(X_1)}{\mathrm{diam}_\sigma(X_2)} \leqslant h\left(-\log_\Lambda \left(C^2 \frac{\varrho(x,y)}{\varrho(x,z)} \right) \right).$$

Define $\theta_i : [0, +\infty) \to [0, +\infty)$, $i \in \{1, 2\}$, by $\theta_1(t) := C_5^2 (C^2 \Lambda t)^{\frac{\log \lambda}{\log \Lambda}}$ and $\theta_2(t) := C_5^2 h(-\log_\Lambda C^2 t)$. Then both $\theta_i : [0, +\infty) \to [0, +\infty)$, $i \in \{1, 2\}$, are strictly increasing continuous functions with $\theta_i(t) \to +\infty$ as $t \to +\infty$.

Define $\theta : [0, +\infty) \to [0, +\infty)$ by $\theta := \max\{\theta_1, \theta_2\}$. Then $\theta : [0, +\infty) \to [0, +\infty)$ is a strictly increasing continuous function with $\theta(0) = 0$, and $\theta(t) \to +\infty$ as $t \to +\infty$, which implies that θ is a homeomorphism. It follows from the discussion above that

$$\frac{\sigma(x,y)}{\sigma(x,z)} \leqslant \theta\left(\frac{\varrho(x,y)}{\varrho(x,z)} \right).$$

Therefore id: $(\mathbb{S}^n, \varrho) \to (\mathbb{S}^n, \sigma)$ is a quasisymmetry, so is the inverse id: $(\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \varrho)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We finish the section with a proof of Theorem 6.1.

Proof of Theorem 6.1. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be an expanding cellular Markov branched cover, $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence, and ϱ be a visual metric for f. Let σ be the spherical metric on $\mathbb{S}^n$.

First, if id : $(\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \varrho)$ is a quasisymmetry, then by Corollary 6.9, $f\colon (\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \sigma)$ is a uniformly local UBQS map.

For the converse, suppose now that $f\colon (\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \sigma)$ is a uniformly local UBQS map. Let $\mathcal{U}$ be a finite cover of $\mathbb{S}^n$ by connected open sets, and $\eta\colon [0, +\infty) \to [0, +\infty)$ be a homeomorphism, such that $f^k|_{\widetilde{U}}$ is an η-BQS for all $k \in \mathbb{N}$ and $\widetilde{U} \in (f^*)^k\mathcal{U}$. Since f is expanding, let $m_0 \in \mathbb{N}$ be a constant such that $\mathrm{mesh}_\sigma(\mathcal{D}_{m_0})$ is strictly less than the Lebesgue number of $\mathcal{U}$. Since $(\mathbb{S}^n, \sigma)$ has bounded turning, it follows from Lemma 5.5 that f is a BQS map (with respect to σ). Assume that f is ξ-BQS. It is clear that $f^m|_{\mathrm{Fl}_m(p)}$ is a ξ^m-BQS map for each $m \leqslant m_0$ and each level-m vertex p. Consider now the case where $m > m_0$. Let p be an arbitrary level-m vertex. By the choice of m_0, we have that $f^{m-m_0}(\mathrm{Fl}_m(p))$ is contained in some $U \in \mathcal{U}$, and thus $\mathrm{Fl}_m(p)$ is contained in some $\widetilde{U} \in (f^*)^{m-m_0}(\mathcal{U})$. Consequently $f^{m-m_0}|_{\mathrm{Fl}_m(p)}$ is η-BQS, and $f^m|_{\mathrm{Fl}_m(p)}$ is $(\xi^{m_0} \circ \eta)$-BQS. Let $\theta := \max\{\xi, \xi^2, \ldots, \xi^{m_0}, \xi^{m_0} \circ \eta\}$, then $\theta\colon [0, +\infty) \to [0, +\infty)$ is surjective, strictly increasing, and satisfies $\theta(0) = 0$. Clearly $f^m|_{\mathrm{Fl}_m(p)}$ is θ-BQS for each $m \in \mathbb{N}$ and each level-m vertex p. By Theorem 6.12, id: $(\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \varrho)$ is a quasisymmetry. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7. Quasisymmetric uniformization: uniformly quasiregular maps

In this section, we discuss the quasisymmetric uniformization problem of visual metrics in terms of uniformly quasiregular maps, and prove the quasiregular part of Theorem 1.1. Precisely, we prove the following theorem.

Theorem 7.1. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be an expanding cellular Markov branched cover, and let ϱ be a visual metric for f. Then* id: $(\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \varrho)$ *is a quasisymmetry if and only if f is uniformly quasiregular, and*

$$\sup\{N_{\mathrm{loc}}(f^m, x) : m \in \mathbb{N}, \ x \in \mathbb{S}^n\} < +\infty.$$

7.1. **Quasisymmetric equivalence implies UQR.** Given an expanding cellular Markov branched cover $f\colon \mathbb{S}^n \to \mathbb{S}^n$ and a visual metric ϱ of f, we show that if ϱ is quasisymmetrically equivalent to the

spherical metric, then f is uniformly quasiregular, and of uniformly bounded local multiplicity.

First, as a consequence of Corollary 6.8, f is uniformly quasiregular.

Corollary 7.2. *Let* $f\colon \mathcal{S}^n \to \mathcal{S}^n$ *be an expanding cellular Markov branched cover, and* ϱ *be a visual metric for* f *on* $\mathcal{S}^n$. *Let* $\varphi\colon (\mathcal{S}^n, \varrho) \to (\mathbb{S}^n, \sigma)$ *be a quasisymmetric homeomorphism, then the map* $\widetilde{f} := \varphi \circ f \circ \varphi^{-1}$ *is uniformly quasiregular.*

Proof. By Corollary 6.8, we know that $f\colon (\mathcal{S}^n, \varrho) \to (\mathcal{S}^n, \varrho)$ is uniformly local UBQS. Let $\eta\colon [0, +\infty) \to [0, +\infty)$ be a homeomorphism, and $\mathcal{U}$ be an opencover of $\mathcal{S}^n$ by connected open subset of $\mathcal{S}^n$ such that $f^k|_U$ is η-BQS for all $k \in \mathbb{N}$ and $U \in (f^*)^k \mathcal{U}$.

Since φ is quasisymmetric, also φ^{-1} is quasisymmetric. By Lemma 5.3, the maps φ and φ^{-1} are BQS. Suppose that φ is a ξ_1-BQS map, and φ^{-1} is a ξ_2-BQS map for homeomorphisms ξ_1, ξ_2 respectively. Put $\varphi_* \mathcal{U} := \{\varphi(U) : U \in \mathcal{U}\}$, then $\varphi_* \mathcal{U}$ is a finite open cover of $\mathbb{S}^n$ by connected open sets. Put also $\widetilde{\eta} := \xi_1 \circ \eta \circ \xi_2$, then for all $k \in \mathbb{N}$ and $\widetilde{U} \in (f^*)^k(\varphi_* \mathcal{U})$, the restricted map $\widetilde{f}^k|_{\widetilde{U}}$ is $\widetilde{\eta}$-BQS. Then $\widetilde{f}$ is uniformly local UBQS with respect to σ.

It is clear that $\widetilde{f}$ is discrete and open, and sense-preserving. By Theorem 5.18, we obtain that $\widetilde{f}$ is uniformly quasiregular. $\square$

Moreover, if $\mathrm{id}\colon (\mathbb{S}^n, \varrho) \to (\mathbb{S}^n, \sigma)$ is a quasisymmetry, then there is a uniform upper bound for the local multiplicities of the family $\{f^m\}_{m \in \mathbb{N}}$.

Theorem 7.3. *Let* $f\colon \mathbb{S}^n \to \mathbb{S}^n$ *be an expanding cellular Markov branched cover, and* ϱ *be a visual metric for* f. *Let* σ *be the spherical metric on* $\mathbb{S}^n$. *If* $\mathrm{id}\colon (\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \varrho)$ *is a quasisymmetry, then*

$$\sup\{N_{\mathrm{loc}}(f^m, x) : m \in \mathbb{N},\ x \in \mathcal{S}^n\} < +\infty.$$

Proof. Suppose that $\mathrm{id}\colon (\mathbb{S}^n, \sigma) \to (\mathbb{S}^n, \varrho)$ is a quasisymmetry. By Corollary 7.2, we may assume that f is uniformly K-quasiregular for $K \geqslant 1$. Also by Corollary 6.9 there exists a homeomorphism $\eta\colon [0, +\infty) \to [0, +\infty)$ such that $f^m|_U$ is a η-BQS map (with respect to σ) for all $m \in \mathbb{N}_0$ and $U \in \mathcal{F}(\mathcal{D}_m)$.

Let $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence of f. By Corollary 3.23, It suffices to prove $\sup\{N(\mathcal{D}_m, p) : m \in \mathbb{N},\ p \text{ is a vertex of } \mathcal{D}_m\} < +\infty$.

Fix $m_0 \in \mathbb{N}_0$ such that for each $m \in \mathbb{N}_0$ with $m \geqslant m_0$ and each $\mathrm{Fl}_m \in \mathcal{F}(\mathcal{D}_m)$, the closure $\overline{\mathrm{Fl}_m}$ is contained in a hemisphere. By Lemma 4.6, fix $l_0 \in \mathbb{N}$ such that $J(\mathcal{D}_l, \mathcal{D}_0) > 4$ for each $l \in \mathbb{N}_0$ with $l \geqslant l_0$. Then for each $m \in \mathbb{N}_0$ and each level-m vertex p, we have $\overline{\mathrm{Fl}_{m+l_0}(p)} \subseteq \mathrm{Fl}_m(p)$.

For each $m \in \mathbb{N}_0$ with $m \geqslant m_0$, and each level-m vertex p, fix a stereographic projection $\phi_{m,p} \colon \mathbb{S}^n \smallsetminus \{a_p\} \to \mathbb{R}^n$ with the following properties:

(i) $\phi_{m,p}(p) = 0$, and $\phi_{m,p}\big(\overline{\mathrm{Fl}_m(p)}\big) \subseteq \mathbb{B}^n(0,2)$.

(ii) $\phi_{m,p}|_{\overline{\mathrm{Fl}_m(p)}}$ is L-bilipschitz for some $L \in [1,2]$.

We may assume $L = 2$, without loss of generality.

Let q be an arbitrary level-m_0 vertex. Denote $G := \phi_{m_0,q}(\mathrm{Fl}_{m_0}(q))$, $F := \phi_{m_0,q}(\mathrm{Fl}_{m_0+2l_0}(q))$, and $E := \phi_{m_0,q}(\mathrm{Fl}_{m_0+3l_0}(q))$. Clearly $\overline{E} \subseteq F \subseteq G \subseteq \mathbb{B}^n(0,2)$, and we pick a, $b > 0$ such that

$$\mathbb{B}^n(0,2a) \subseteq \overline{E} \subseteq F \subseteq \mathbb{B}^n(0,b/2).$$

Consider $M_q := \mathrm{Mod}(\Delta(\overline{E}, \partial F))$. Then, by Lemmas 5.6 and 5.8,

$$M_q \geqslant \mathrm{Mod}\big(\Delta(\partial\mathbb{B}^n(0,a), \partial\mathbb{B}^n(0,b); \overline{\mathbb{B}^n(0,b)} \smallsetminus \mathbb{B}^n(0,a)\big) > 0.$$

Put $M_0 := \inf\{M_q : q \text{ is a level-}m_0 \text{ vertex.}\}$. Since M_0 is the infimum of a finite collection of positive numbers, we have $M_0 > 0$.

Now fix arbitrary $m \in \mathbb{N}_0$ with $m \geqslant m_0 + l_0$, and level-m vertex p. Consider the domain $G := \phi_{m,p}(\mathrm{Fl}_m(p))$ in $\mathbb{R}^n$, and put

$$D := \phi_{m,p}(\mathrm{Fl}_{m+l_0}(p)), \ \ F := \phi_{m,p}(\mathrm{Fl}_{m+2l_0}(p)), \ \ E := \phi_{m,p}(\mathrm{Fl}_{m+3l_0}(p)).$$

Denote $q := f^{m-m_0}(p)$, and put

$$D' := \phi_{m_0+l_0,q}(\mathrm{Fl}_{m_0}(q)), \ F' := \phi_{m_0,q}(\mathrm{Fl}_{m_0+2l_0}(q)), \ E' := \phi_{m_0,q}(\mathrm{Fl}_{m_0+3l_0}(q)).$$

Since $\phi_{m,p}$ and $\phi_{m_0,q}$ are L-bilipschitz, the map $\widetilde{f} \colon G \to \mathbb{R}^n$, $\widetilde{f} := \phi_{m_0,q} \circ f^{m-m_0} \circ \phi_{m,p}^{-1}$ is KL^{4n}-quasiregular.

It is clear that $\widetilde{f}(D) = D'$. By Corollary 3.18, D is a normal domain for $\widetilde{f}$. Consider $\Gamma' := \Delta(\overline{E'}, \partial F'; D')$, and $\Gamma := \Delta(\overline{E}, \partial F; D)$. We shall prove

$$\Gamma = \big\{\gamma : \gamma \text{ is a curve in } D \text{ and } \widetilde{f} \circ \gamma \in \Gamma'\big\}.$$

Indeed, let $\gamma \colon [0,1] \to D$ be a curve with $\widetilde{f} \circ \gamma \in \Gamma'$, and denote $\alpha := \phi_{m,p}^{-1} \circ \gamma$. Then $\widetilde{f}(\gamma(0)) \in \overline{E'}$, and $\widetilde{f}(\gamma(1)) \in \partial F'$, implying $f^{m-m_0}(\alpha(0)) \in \overline{\mathrm{Fl}_{m_0+3l_0}(q)}$ and $f^{m-m_0}(\alpha(1)) \in \partial\mathrm{Fl}_{m_0+2l_0}(q)$. It follows from Lemma 3.19 that $\alpha(0) \in \overline{\mathrm{Fl}_{m+3l_0}(p)}$ and $\alpha(1) \in \overline{\mathrm{Fl}_{m+2l_0}(p)}$. Moreover, it follows from Proposition 3.14 (i) and Corollary 3.18 that $\alpha(1) \in \partial\mathrm{Fl}_{m+2l_0}(p)$. Hence $\gamma \in \Delta(\overline{E}, \partial F; D)$.

Then, by Lemma 5.13,

$$N\big(\widetilde{f}, D\big) \leqslant KL^{4n}\mathrm{Mod}(\Gamma)/\mathrm{Mod}(\Gamma') \leqslant KL^{4n}M_0^{-1}\,\mathrm{Mod}(\Gamma).$$

The second inequality above follows from Lemmas 5.6 and 5.8.

By the choice of l_0, we have $\overline{E} \subseteq F \subseteq \overline{F} \subseteq D$, which implies $\mathrm{dist}(\overline{E}, \partial F) \leqslant \mathrm{dist}(\overline{E}, \mathbb{R}^n \smallsetminus D)$. Then it follows from Lemma 5.11 that

$$N\big(\widetilde{f}, D\big) \leqslant \frac{\alpha_n K L^{4n}}{M_0} \left(1 + \frac{\mathrm{diam}(\overline{E})}{\mathrm{dist}(\overline{E}, \partial F)} \right)^n.$$

It suffices to give an estimation of $\mathrm{diam}(\overline{E}) / \mathrm{dist}(\overline{E}, \partial F)$. Let $x \in \overline{\mathrm{Fl}_{m+3l_0}(p)}$, $y \in \partial \mathrm{Fl}_{m+2l_0}(p)$ be arbitrary. Suppose that $x \in X$, $y \in c$ for $X \in \mathcal{D}^{[n]}_{m+3l_0}$ with $p \in X$, and $c \in \mathcal{D}_{m+2l_0}$ with $p \notin c$. Suppose $y \in Y$ for some $Y \in \mathcal{D}^{[n]}_{m+3l_0}$. By the choice of l_0, we have $X \cap Y = \emptyset$. Then it follows from Lemma 6.11 (i) that $\sigma(x, y) \geqslant \mathrm{dist}_\sigma(X, Y) \geqslant C_4 \, \mathrm{diam}_\sigma(X)$. Moreover, it follows from Lemma 6.10 (i) that $\mathrm{diam}_\sigma\big(\overline{\mathrm{Fl}_{m+3l_0}(p)}\big) \leqslant (1 + C_1) \, \mathrm{diam}_\sigma(X)$, thus

$$\sigma(x, y) \geqslant C_4 (1 + C_1)^{-1} \, \mathrm{diam}_\sigma\big(\overline{\mathrm{Fl}_{m+3l_0}(p)}\big).$$

Since x, y are arbitrary,

$$\mathrm{dist}_\sigma\big(\overline{\mathrm{Fl}_{m+3l_0}(p)}, \partial \mathrm{Fl}_{m+2l_0}(p)\big) \geqslant C_4 (1 + C_1)^{-1} \, \mathrm{diam}_\sigma\big(\overline{\mathrm{Fl}_{m+3l_0}(p)}\big).$$

Since $\phi_{m,p}$ is L-bilipschitz on $\mathrm{Fl}_m(p)$, we have $\mathrm{diam}(\overline{E}) / \mathrm{dist}(\overline{E}, \partial F) \leqslant L^2 C_4^{-1} (1 + C_1)$. Therefore,

$$N\big(\widetilde{f}, D\big) \leqslant \frac{\alpha_n K L^{4n}}{M_0} \left(1 + \frac{L^2 (1 + C_1)}{C_4} \right)^n.$$

Clearly $N(f^{m-m_0}, \mathrm{Fl}_{m+l_0}(p)) = N\big(\widetilde{f}, D\big)$, and it follows from Lemma 3.21 that

$$\begin{aligned}
N(\mathcal{D}_{m-m_0}, p) &\leqslant \mathrm{card}(\mathcal{D}_0) \cdot N_{\mathrm{loc}}(f^{m-m_0}, p) \\
&\leqslant \mathrm{card}(\mathcal{D}_0) \cdot N\big(\widetilde{f}, D\big) \\
&\leqslant \mathrm{card}(\mathcal{D}_0) \frac{\alpha_n K L^{4n}}{M_0} \left(1 + \frac{L^2 (1 + C_1)}{C_4} \right)^n.
\end{aligned}$$

Since $m \geqslant m_0 + l_0$ and level-m vertex p are arbitrary, it follows that $\sup\{N(\mathcal{D}_{m-m_0}, p) : m \in \mathbb{N}, \ m - m_0 \geqslant l_0, \ p \text{ is a vertex of } \mathcal{D}_m\} < +\infty$. Since each level-$(m - m_0)$ vertex p' is also a level-m vertex, it follows that

$\sup\{N(\mathcal{D}_{m-m_0}, p') : m \in \mathbb{N}, \ m - m_0 \geqslant l_0, \ p' \text{ is a vertex of } \mathcal{D}_{m-m_0}\} < +\infty$,

which is equivalent to

$\sup\{N(\mathcal{D}_m, p') : m \in \mathbb{N}, \ m \geqslant l_0, \ p' \text{ is a vertex of } \mathcal{D}_m\} < +\infty$.

It follows immediately that

$$\sup\{N(\mathcal{D}_m, p) : m \in \mathbb{N}, \ p \text{ is a vertex of } \mathcal{D}_m\} < +\infty,$$

and then the proof is complete. $\qquad\square$

7.2. **UQR implies quasisymmetric equivalence.** Now we show that if an expanding cellular Markov branched cover $f\colon \mathbb{S}^n \to \mathbb{S}^n$ is uniformly quasiregular with uniformly bounded local multiplicity, then the visual metrics of f are quasisymmetrically equivalent to the spherical metric.

Given an expanding cellular Markov branched cover $f\colon \mathcal{S}^n \to \mathcal{S}^n$ and arbitrary $k \in \mathbb{N}$, recall that visual metrics of f and f^k coincide (see Lemma 4.18), and that f^k is uniformly quasiregular if f is uniformly quasiregular. Hence it suffices to prove the claim for f^k. Since f is expanding, no cell in $\mathcal{D}_k$ joins opposite sides of $\mathcal{D}_0$ if $k \in \mathbb{N}$ is sufficiently large. Therefore, we may first assume that no cell in $\mathcal{D}_1$ joins opposite sides of $\mathcal{D}_0$. To make our proof concise, we give the following technical definition.

Definition 7.4. An essential sequence $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ of a cellular Markov branched cover $f\colon \mathcal{S}^n \to \mathcal{S}^n$ is *combinatorially bounded* if the following conditions are satisfied:

    (i) $\sup\{N(\mathcal{D}_m, x) : m \in \mathbb{N},\ x \in \mathcal{S}^n\} < +\infty$.

    (ii) No cell in $\mathcal{D}_1$ joins opposite sides of $\mathcal{D}_0$.

Remark. Condition (i) can be understood as follows:

    (a) The existence of an essential sequence with condition (i) is equivalent to the statement

$$\sup\{N_{\mathrm{loc}}(f^m, x) : m \in \mathbb{N},\ x \in \mathcal{S}^n\} < +\infty.$$

    (b) For dimension $n = 2$, consider expanding Thurston maps. If an expanding Thurston map $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ is a rational map, then f has no periodic critical point, and thus admits an essential sequence with condition (i).

In what follows, we assume $f\colon \mathcal{S}^n \to \mathcal{S}^n$ to be an expanding cellular Markov branched cover, and $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be a combinatorially bounded essential sequence. We show some elementary properties of f and $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$, then we obtain two lemmas about metric properties of cells, similarly as Lemmas 6.10 and 6.11.

First, the following topological fact is clear.

Lemma 7.5. *Let $m,\, k \in \mathbb{N}_0$, $Y \in \mathcal{D}_m^{[n]}$, and $Y_1, Y_2 \in \mathcal{U}_{m,k}^1(Y) \smallsetminus (\mathcal{D}_{m+k}|_Y)^{[n]}$ be arbitrary. Then, there are $X_1, \ldots, X_l$ in $\mathcal{U}_{m,k}^1(Y) \smallsetminus (\mathcal{D}_{m+k}|_Y)^{[n]}$ with the following properties:*

    (i) $X_1 = Y_1$, and $X_l = Y_2$.

(ii) *For each $i \in \{1, \ldots, l-1\}$, we have $X_i \cap X_{i+1} \neq \emptyset$ and $X_i \cap X_{i+1} \nsubseteq Y$.*

Proof. Denote $D = U^1_{m,k}(Y)$, then it follows from Corollary 6.3 that $D \smallsetminus Y$ is connected.

For each $X \in \mathcal{U}^1_{m,k}(Y) \smallsetminus (\mathcal{D}_{m+k}|_Y)^{[n]}$, it is clear that $\mathrm{int}_\circ(X) \cap D \neq \emptyset$ and $\mathrm{int}_\circ(X) \cap Y = \emptyset$, then we have $\mathrm{int}_\circ(X) \cap (D \smallsetminus Y) \neq \emptyset$.

Put $\Omega := \big\{ X \cap (D \smallsetminus Y) : X \in \mathcal{U}^1_{m,k}(Y) \smallsetminus (\mathcal{D}_{m+k}|_Y)^{[n]} \big\}$. It is clear that Ω is a finite cover of $D \smallsetminus Y$ by non-empty closed subsets of $D \smallsetminus Y$. Lemma 4.5 implies that there exist $D_1, \ldots, D_l \in \Omega$ with the following properties:

(i) $D_1 = Y_1 \cap (D \smallsetminus Y)$, $D_l = Y_2 \cap (D \smallsetminus Y)$.

(ii) $D_i \cap D_{i+1} \neq \emptyset$ for each $i \in \{1, \ldots, l-1\}$.

For each $i \in \{2, \ldots, l-1\}$, suppose that $D_i = X_i \cap (D \smallsetminus Y)$ for $X_i \in \mathcal{U}^1_{m,k}(Y) \smallsetminus (\mathcal{D}_{m+k}|_Y)^{[n]}$. Then we obtain $X_1, \ldots, X_l$ as desired. □

The following lemma is a consequence of condition (i) in Definition 7.4.

Lemma 7.6. *For each $k \in \mathbb{N}_0$, there exist constants $N_1(k)$, $N_2(k) \in \mathbb{N}$ such that*

$$\mathrm{card}\big(\mathcal{U}^1_{m,k}(X)\big) \leqslant N_1(k) \quad and \quad \mathrm{card}\big(\mathcal{U}^2_{m,k}(X)\big) \leqslant N_2(k)$$

hold for each $m \in \mathbb{N}_0$ and each $X \in \mathcal{D}^{[n]}_m$.

Proof. By condition (i) of Definition 7.4, set

$$N := \sup_{m \in \mathbb{N}_0,\, x \in \mathcal{S}^n} \mathrm{card}\big(\{ X \in \mathcal{D}^{[n]}_m : x \in X \}\big) < +\infty.$$

Fix arbitrary m, $k \in \mathbb{N}_0$ and $X \in \mathcal{D}^{[n]}_m$. Suppose that $Y \in \mathcal{D}^{[n]}_{m+k}$, $X \cap Y \neq \emptyset$, then there exists $X' \in (\mathcal{D}_{m+k}|_X)^{[n]}$ with $X' \cap Y \neq \emptyset$. Then $X' \cap Y$ contains a level-$(m+k)$ vertex since $X' \cap Y$ is a union of cells in $\mathcal{D}_{m+k}$. It follows $\mathcal{U}^1_{m,k}(X) = \bigcup_p \{ Y \in \mathcal{D}^{[n]}_{m+k} : p \in Y \}$, where p ranges over all level-$(m+k)$ vertices contained in X. Since $\mathcal{D}_{m+k}|_X = (f^m|_X)^*(\mathcal{D}_k|_{f^m(X)})$, it is clear that $\mathrm{card}\big((\mathcal{D}_{m+k}|_X)^{[0]}\big) \leqslant \mathrm{card}(\mathcal{D}_k)$. Then we have $\mathrm{card}(\mathcal{U}^1_{m,k}(X)) \leqslant N \cdot \mathrm{card}(\mathcal{D}_k)$. It suffices to define $N_1(l) := N \cdot \mathrm{card}(\mathcal{D}_l)$ for each $l \in \mathbb{N}_0$.

Consider now $\mathcal{U}^2_{m,k}(X)$. It is clear that $\mathcal{U}^2_{m,k}(X) = \bigcup \{ \mathcal{U}^1_{m+k,0}(Y) : Y \in \mathcal{U}^1_{m,k}(X) \}$. Then $\mathrm{card}(\mathcal{U}^2_{m,k}(X)) \leqslant N_1(0) \cdot N_1(k)$. It suffices to define $N_2(k) := N_1(0) \cdot N_1(k)$. □

The following Lemmas 7.7 through 7.10 are consequences of condition (ii) in Definition 7.4.

Lemma 7.7. *The following statements hold:*

    (i) *For each $m \in \mathbb{N}_0$ and each $c \in \mathcal{D}_m$ with $\dim(c) \geqslant 1$, there exists $\sigma \in \mathcal{D}_{m+1}$ with $\dim(\sigma) < \dim(c)$ and $\mathrm{int}_\circ(\sigma) \subseteq \mathrm{int}_\circ(c)$.*

    (ii) *For each $m \in \mathbb{N}_0$ and each $c \in \mathcal{D}_m$, there exists a level-$(m + \dim(c))$ vertex p with $p \in \mathrm{int}_\circ(c)$.*

Proof. (i) Since $\mathcal{D}_{m+1}|_\sigma = (f^m|_c)^*(\mathcal{D}_1|_{f^m(c)})$, it suffices to proof the claim for each $c \in \mathcal{D}_0$ with $\dim(c) \geqslant 1$.

Let $c \in \mathcal{D}_0$ with $\dim(c) = d \geqslant 1$ be arbitrary. For each $x \in \mathrm{int}_\circ(c)$, consider $\tau \in \mathcal{D}_1$ with $x \in \mathrm{int}_\circ(\tau)$ and $\sigma \in \mathcal{D}_0$ with $\mathrm{int}_\circ(\tau) \subseteq \mathrm{int}_\circ(\sigma)$ (see Lemma 3.13), then $\sigma = c$. It follows that $\mathrm{int}_\circ(c) = \bigcup\{\mathrm{int}_\circ(\tau) : \tau \in \mathcal{D}_1, \ \mathrm{int}_\circ(\tau) \subseteq \mathrm{int}_\circ(c)\}$.

Suppose on the contrary that $\dim(\tau) = d$ for each $\tau \in \mathcal{D}_1$ with $\mathrm{int}_\circ(\tau) \subseteq \mathrm{int}_\circ(c)$, then each $\mathrm{int}_\circ(\tau)$ is an open subset of $\mathrm{int}_\circ(c)$. Since $\mathrm{int}_\circ(c)$ is connected and the cell interiors of cells in $\mathcal{D}_1$ are pairwise disjoint, we have $\mathrm{card}(\{\tau \in \mathcal{D}_1 : \mathrm{int}_\circ(\tau) \subseteq \mathrm{int}_\circ(c)\}) = 1$. It follows that $\mathrm{int}_\circ(c) = \mathrm{int}_\circ(\tau)$ and thus $c = \tau$ for some $\tau \in \mathcal{D}_1$, which contradicts the assumption that no cell in $\mathcal{D}_1$ joins opposite sides of $\mathcal{D}_0$. Therefore, there exists $\sigma \in \mathcal{D}_1$ with $\dim(\sigma) < \dim(c)$ and $\mathrm{int}_\circ(\sigma) \subseteq \mathrm{int}_\circ(c)$.

(ii) For each $m, k \in \mathbb{N}_0$, and each $c \in \mathcal{D}_m$, we have $\mathcal{D}_{m+k}|_c = (f^m|_c)^*(\mathcal{D}_k|_{f^m(c)})$. Thus, it suffices to prove the claim for $m = 0$ and each $c \in \mathcal{D}_0$.

We give a proof by induction on $\dim(c)$. Let $c \in \mathcal{D}_0$ be arbitrary. If $\dim(c) = 0$, then $c = \mathrm{int}_\circ(c)$ consists of a single level-0 vertex p. Suppose now $\dim(c) = d \geqslant 1$.

We make the induction hypothesis that, for each $\sigma \in \mathcal{D}_0$ with $\dim(\sigma) < d$, there exists a level-$\dim(\sigma)$ vertex p with $p \in \mathrm{int}_\circ(\sigma)$. By (i), there exists $\sigma \in \mathcal{D}_1$ with $\dim(\sigma) < \dim(c)$ and $\mathrm{int}_\circ(\sigma) \subseteq \mathrm{int}_\circ(c)$. Denote $b := \dim(\sigma)$. By the fact that $\mathcal{D}_{1+b}|_\sigma = f^*(\mathcal{D}_b|_{f(\sigma)})$, and by the induction hypothesis, there exists a level-$(1 + b)$ vertex p with $p \in \mathrm{int}_\circ(\sigma) \subseteq \mathrm{int}_\circ(c)$. Since $b < d$, we have $1 + b \leqslant d$ and it follows that p is a level-d vertex. By induction, the proof is complete. $\quad\square$

Corollary 7.8. *For all $m \in \mathbb{N}_0$ and $c \in \mathcal{D}_m$, there is a level-$(m + n)$ vertex p with $p \in \mathrm{int}_\circ(c)$.*

Proof. The claim follows immediately from Lemma 7.7 (ii), because $\dim(c) \leqslant n$ implies that each vertex of level-$(m + \dim(c))$ is also a vertex of level-$(m + n)$. $\quad\square$

Lemma 7.9. *Let $X \in \mathcal{D}_0^{[n]}$ and p_1, p_2 be level-n vertices with $p_1, p_2 \in X$. Then there exists $l(X, p_1, p_2) \in \mathbb{N}$ such that for each $l \in \mathbb{N}$ with $l \geqslant l(X, p_1, p_2)$, there exists a path-connected subset $A \subseteq X$ with the following properties:*

(i) A is a union of cells in $(\mathcal{D}_l|_X)^{[n]}$, and A joins p_1, p_2.

(ii) For each $c \in \mathcal{D}_0|_X$, if $p_1 \notin c$ and $p_2 \notin c$, then $A \cap c = \emptyset$.

Proof. Pick Y_1, $Y_2 \in (\mathcal{D}_{n+1}|_X)^{[n]}$ with $p_1 \in Y_1$, $p_2 \in Y_2$.

It is clear that $\text{int}_\circ(Y_1)$ and $\text{int}_\circ(Y_2)$ are subsets of $\text{int}_\circ(X)$. Fix a curve $\gamma \colon [0,1] \to \text{int}_\circ(X)$ with $\gamma(0) \in \text{int}_\circ(Y_1)$ and $\gamma(1) \in \text{int}_\circ(Y_2)$. Sicne γ and $\partial_\circ X$ are disjoint continua, $\text{dist}(\gamma, \partial_\circ X) > 0$. Equip $\mathcal{S}^n$ with a metric compatible with the given topology. Since f is expanding, we may pick $l(X, p_1, p_2) \in \mathbb{N}$ such that $l(X, p_1, p_2) \geqslant n+1$ and $\text{mesh}(\mathcal{D}_l) < \text{dist}(\gamma, \partial_\circ X)$ for each $l \geqslant l(X, p_1, p_2)$.

Now we consider arbitrary $l \geqslant l(X, p_1, p_2)$.

Denote

$$A := Y_1 \cup Y_2 \cup \bigcup \{Z \in (\mathcal{D}_l|_X)^{[n]} : Z \cap \gamma \neq \emptyset\}.$$

We show that A satisfies the properties in the claim. Since γ, Y_1, Y_2 and every $Z \in \mathcal{D}_l^{[n]}$ with $Z \cap \gamma \neq \emptyset$ are path-connected, also A is connected. It is clear that A is union of elements in $\mathcal{D}_l^{[n]}$.

Let $c \in \mathcal{D}_0|_X$ satisfies $p_1 \notin c$, $p_2 \notin c$. Clearly $c \neq X$, and thus $c \subseteq \partial_\circ X$. Then, since $\text{mesh}(\mathcal{D}_l) < \text{dist}(\gamma, \partial_\circ X)$, we have $Z \cap c = \emptyset$ for each $Z \in \mathcal{D}_l^{[n]}$ with $\gamma \cap Z \neq \emptyset$. It suffices to show $Y_1 \cap c = Y_2 \cap c = \emptyset$.

Suppose $Y_1 \cap c \neq \emptyset$ on the contrary. Let $X_1 \in (\mathcal{D}_n|_X)^{[n]}$ satisfies $Y_1 \subseteq X_1$. Since $c \cap X_1$ is either an empty set, or a union of cells in $\mathcal{D}_n|_{X_1}$, there exists $c_1 \in \mathcal{D}_n$ such that $c_1 \subseteq X_1 \cap c$, and $Y_1 \cap c_1 \neq \emptyset$. Then the cell $f^n(Y_1) \in \mathcal{D}_1$ meets both $f^n(\{p_1\})$ and $f^n(c_1)$, which are disjoint cells in $\mathcal{D}_0|_{f^n(X_1)}$. This contradicts the assumption that no cell in $\mathcal{D}_1$ joins opposite sides in $\mathcal{D}_0$.

Therefore, we have $Y_1 \cap c = \emptyset$, and we also have $Y_2 \cap c = \emptyset$ by the same argument. □

Lemma 7.10. *There exists $l_0 \in \mathbb{N}$ such that for each pair m, $l \in \mathbb{N}_0$ with $l \geqslant l_0$, each $X \in \mathcal{D}_m^{[n]}$, and each pair of level-$(m+n)$ vertices p_1, p_2 with p_1, $p_2 \in X$, there exists a path-connected subset $A \subseteq X$ with the following properties:*

(i) *A is a union of elements in $(\mathcal{D}_{m+l}|_X)^{[n]}$ and A joins p_1, p_2.*

(ii) *For each $c \in \mathcal{D}_m|_X$, if $p_1 \notin c$ and $p_2 \notin c$, then $A \cap c = \emptyset$.*

Proof. Denote

$$l_0 := \sup\{l(Y, q_1, q_2) : Y \in \mathcal{D}_0^{[n]}, \text{ and } q_1, q_2 \in Y \text{ are level-}n \text{ vertices}\},$$

where $l(Y, q_1, q_2)$ is the constant in Lemma 7.9. Since l_0 is the supremum of a finite collection of positive numbers, we have $l_0 < +\infty$.

Fix arbitrary numbers m, $l \in \mathbb{N}_0$ with $l \geqslant l_0$. Fix also a cell $X \in \mathcal{D}_m^{[n]}$, and level-$(m+n)$ vertices p_1, p_2 with p_1, $p_2 \in X$. Then $Y := f^m(X) \in \mathcal{D}_0^{[n]}$, and $q_1 := f^m(p_1)$, $q_2 := f^m(p_2)$ are level-n vertices in Y.

It follows from Lemma 7.9 that there exists a path-connected subset $A(Y, q_1, q_2) \subseteq Y$ with the following properties:

   (i) $A(Y, q_1, q_2)$ is a union of elements in $\mathcal{D}_l^{[n]}$, and $A(Y, q_1, q_2)$ joins q_1, q_2.

   (ii) For each $c \in \mathcal{D}_0|_X$, if $q_1 \notin c$ and $q_2 \notin c$, then $A(Y, q_1, q_2) \cap c = \emptyset$.

Since $\mathcal{D}_m|_X = (f^m|_X)^*(\mathcal{D}_0|_Y)$ and $\mathcal{D}_{m+l}|_X = (f^m|_X)^*(\mathcal{D}_l|_Y)$, it can be checked that $A := (f^m|_X)^{-1}(A(Y, q_1, q_2))$ is the subset with the properties desired. $\qquad\square$

Lemma 7.11. *There exists $l_0 \in \mathbb{N}$ with the following property: for all m, k, $l \in \mathbb{N}_0$ with $l \geqslant l_0$ and each $X \in \mathcal{D}_m^{[n]}$, there exists a connected component $W_{m,k,l}^*(X)$ of $W_{m,k,l}(X)$ such that $\partial U_{m,k}^1(X) \subseteq W_{m,k,l}^*(X)$.*

Proof. By Lemmas 6.6 and 7.10, pick $l_0 \in \mathbb{N}$ such that the following statements hold for all m, k, $l \in \mathbb{N}_0$ with $l \geqslant l_0$:

   (1) For each $X \in \mathcal{D}_m^{[n]}$, we have $\partial U_{m,k}^1(X) \subseteq W_{m,k,l}(X)$.

   (2) For each $Y \in \mathcal{D}_{m+k}^{[n]}$ and each pair p_1, p_2 of level-$(m+k+n)$ vertices contained in Y, there exists a path-connected subset $A \subseteq Y$ joining p_1, p_2, such that A is a union of elements in $(\mathcal{D}_{m+k+l}|_Y)^{[n]}$, and $A \cap c = \emptyset$ for each $c \in \mathcal{D}_{m+k}|_Y$ with $p_1 \notin c$, $p_2 \notin c$.

Fix arbitrary m, k, $l \in \mathbb{N}_0$ with $l \geqslant l_0$ and $X \in \mathcal{D}_m^{[n]}$. Let x_1, $x_2 \in \partial U_{m,k}^1(X)$ be arbitrary. We show that there is a chain consisting of elements in $\mathcal{W}_{m,k,l}(X)$ that joins x_1 and x_2.

Suppose $x_1 \in Y_1$, $x_2 \in Y_2$ for some Y_1, $Y_2 \in \mathcal{U}_{m,k}^1(X) \smallsetminus (\mathcal{D}_{m+k}|_X)^{[n]}$. By Lemma 7.5, consider X_1, $\ldots$, X_l in $\mathcal{U}_{m,k}^1(X) \smallsetminus (\mathcal{D}_{m+k}|_X)^{[n]}$ with the following properties:

   (i) $X_1 = Y_1$, $X_l = Y_2$.

   (ii) For each $i \in \{1, \ldots, l-1\}$, we have $X_i \cap X_{i+1} \neq \emptyset$ and $X_i \cap X_{i+1} \nsubseteq X$.

Suppose that $x_1 \in \mathrm{int}_\circ(\sigma_1)$ for some $\sigma_1 \in \mathcal{D}_{m+k}$. It follows from Lemma 6.2 (v) that $\sigma_1 \subseteq \partial U_{m,k}^1(X)$, hence $\sigma_1 \cap X = \emptyset$. Likewise, let $\sigma_2 \in \mathcal{D}_{m+k}$ satisfies $x_2 \in \mathrm{int}_\circ(\sigma_2)$ and $\sigma_2 \cap X = \emptyset$. Since for each $i \in \{1, \ldots, l-1\}$, the intersection $X_i \cap X_{i+1}$ is a union of cells in $\mathcal{D}_{m+k}$

(see Lemma 2.6), we may fix $c_i \in \mathcal{D}_{m+k}$ such that $c_i \subseteq X_i \cap X_{i+1}$ and $c_i \nsubseteq X$.

By Corollary 7.8, fix level-$(m + k + n)$ vertices $p_0, p_1, \ldots, p_{l-1}, p_l$ such that $p_0 \in \mathrm{int}_\circ(\sigma_1)$, $p_l \in \mathrm{int}_\circ(\sigma_2)$, and $p_i \in \mathrm{int}_\circ(c_i)$ for each $i \in \{1, \ldots, l-1\}$. Since $c_i \nsubseteq X$ for each $i \in \{1, \ldots, l-1\}$, we have $p_0, p_1, \ldots, p_l \notin X$.

Put $B_i := \bigcup \{Z \in \mathcal{W}_{m,k,l}(X) : Z \cap \sigma_i \neq \emptyset\}$, $i \in \{1, 2\}$. By Lemma 6.6, we have $\sigma_i \subseteq B_i$, and B_i is connected for $i \in \{1, 2\}$. Clearly $B_i \subseteq W_{m,k,l}(X)$.

For each $i \in \{1, \ldots, l\}$, the intersection $X \cap X_i$ is a union of cells in $\mathcal{D}_{m+k}|_{X_i}$ that does not meet $\{p_{i-1}, p_i\}$. Then, by the choice of $l \geqslant l_0$, there exists a path-connected subset $A_i \subseteq X_i$ joining p_{i-1}, p_i, such that A_i is a union of elements in $\mathcal{D}_{m+k+l}^{[n]}$, and that $A_i \cap X = \emptyset$. Clearly $A_i \subseteq W_{m,k,l}(X)$.

Put $E := B_1 \cup A_1 \cup \cdots \cup A_l \cup B_2$, then E is a path-connected subset of $W_{m,k,l}(X)$ that joins x_1, x_2. Since $x_1, x_2 \in \partial U_{m,k}^1(X)$ are arbitrary, $\partial U_{m,k}^1(X)$ is contained in a connected component of $W_{m,k,l}(X)$. □

The following lemmas regarding metric properties of n-cells in a combinatorially bounded essential sequence are analogous to Lemmas 6.10 and 6.11.

Lemma 7.12. *Let* $f \colon \mathbb{S}^n \to \mathbb{S}^n$ *be an expanding cellular Markov branched cover, and* $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ *be a combinatorially bounded essential sequence. Let* σ *be the spherical metric on* $\mathbb{S}^n$. *Suppose that* f *is a uniformly* K-*quasiregular map. Then, for each* $k \in \mathbb{N}_0$, *there exists* $\Lambda_1(k) > 0$ *such that for all* $m \in \mathbb{N}_0$ *and all* $Y \in \mathcal{D}_m^{[n]}$, $Z \in \mathcal{D}_{m+k}^{[n]}$ *with* $Y \cap Z = \emptyset$, *we have*

$$\mathrm{dist}_\sigma(Y, Z) \geqslant \Lambda_1(k) \, \mathrm{diam}_\sigma(Y).$$

Proof. Since f is expanding, we fix sufficiently large $m_0 \in \mathbb{N}$ such that $\overline{U_{m,k}^2(X)} = \bigcup \mathcal{U}_{m,k}^2(X)$ is contained in a hemisphere for all $m, k \in \mathbb{N}_0$ with $m \geqslant m_0$ and each $X \in \mathcal{D}_m^{[n]}$.

By Lemma 7.11, we fix sufficiently large $l_0 \in \mathbb{N}$ such that the following statement hold: for all $m, k, l \in \mathbb{N}_0$ with $l \geqslant l_0$, and each $X \in \mathcal{D}_m^{[n]}$, there is a connected component $W_{m,k,l}^*(X)$ of $W_{m,k,l}(X)$ with $\partial U_{m,k}^1(X) \subseteq W_{m,k,l}^*(X)$.

Fix arbitrary $m, k \in \mathbb{N}_0$ with $m \geqslant m_0$, and $Y \in \mathcal{D}_m^{[n]}$, $Z \in \mathcal{D}_{m+k}^{[n]}$ with $Y \cap Z = \emptyset$. Then it follows from Lemma 6.2 (vi) and $\partial U_{m,k}^1(Y) \subseteq W_{m,k,l_0}^*(Y)$ that

$$\mathrm{dist}_\sigma(Y, Z) \geqslant \mathrm{dist}_\sigma(Y, \partial U_{m,k}^1(Y)) \geqslant \mathrm{dist}_\sigma(Y, W_{m,k,l_0}^*(Y)).$$

Since $U^2_{m,0}(Y)$ and $U^2_{m_0,0}(f^{m-m_0}(Y))$ are contained in hemispheres, we fix stereographic projections $\phi_1\colon \mathbb{S}^n \smallsetminus \{p_1\} \to \mathbb{R}^n$ and $\phi_2\colon \mathbb{S}^n \smallsetminus \{p_2\} \to \mathbb{R}^n$ such that ϕ_1, ϕ_2 map $U^2_{m,0}(Y)$ and $U^2_{m_0,0}(f^{m-m_0}(Y))$, respectively, into $\mathbb{B}^n(0,2)$, and that ϕ_1, ϕ_2 are L-bilipschitz on $U^2_{m,0}(Y)$ and $U^2_{m_0,0}(f^{m-m_0}(Y))$, respectively, for some $L \in [1,2]$. Without loss of generality, we may assume $L = 2$ in what follows.

Consider the map

$$\widetilde{f}\colon \phi_1\big(U^2_{m,0}(Y)\big) \to \phi_2\big(U^2_{m_0,0}(f^{m-m_0}(Y))\big),$$
$$x \mapsto \phi_2 \circ f^{m-m_0} \circ \phi_1^{-1}(x).$$

Then, $\widetilde{f}$ is KL^{4n}-quasiregular, since f is uniformly K-quasiregular, and ϕ_1, ϕ_2 are L-bilipschitz.

Denote $U := \phi_1(Y)$, $V := \phi_1\big(U^1_{m,k}(Y)\big)$, $W := \phi_1\big(W^*_{m,k,l_0}(Y)\big)$ and $D := \phi_2\big(U^1_{m_0,0}(f^{m-m_0}(Y))\big)$. We shall give an estimation of $\mathrm{Mod}(\Delta(U,W;V))$ and $\mathrm{Mod}\big(\widetilde{f}(\Delta(U,W;V))\big)$.

Let $\widetilde{\gamma} \in \Delta(U,W;V)$ be arbitrary. It is clear that $\gamma := \phi_1^{-1}(\widetilde{\gamma})$ is a curve in $U^1_{m,k}(Y)$ that joins Y and $W^*_{m,k,l_0}(Y)$. Then, there exists Z', $Z'' \in \mathcal{D}^{[n]}_{m+k+l_0}$, such that $Z' \subseteq Y$, $Z'' \cap Y = \emptyset$ with γ joining Z' and Z''. By Corollary 2.11, the connected set γ cannot be contained in any level-$(m+k+l_0)$ flower, and it follows from Lemma 3.17 that $f^{m-m_0}(\gamma)$ cannot be contained in a level-(m_0+k+l_0) flower. Hence $\mathrm{diam}_\sigma\big(f^{m-m_0}(\gamma)\big) \geqslant \delta_k$, where δ_k is the Lebesgue number of the open cover $\mathcal{F}(\mathcal{D}_{m_0+k+l_0})$. It is clear that

$$f^{m-m_0}(\gamma) \subseteq \overline{U^1_{m_0,0}(f^{m-m_0}(Y))} \subseteq U^2_{m_0,0}(f^{m-m_0}(Y)).$$

Then, since ϕ_2 is L-bilipschitz on $U^2_{m_0,0}(f^{m-m_0}(Y))$, we have

$$\mathrm{length}(\phi_2(f^{m-m_0}(\gamma))) \geqslant \mathrm{diam}(\phi_2(f^{m-m_0}(\gamma))) \geqslant L^{-1}\delta_k.$$

Therefore, we have

$$\widetilde{f}(\Delta(U,W;V)) \subseteq \big\{\gamma : \gamma \text{ is a curve in } \overline{D},\ \mathrm{length}(\gamma) \geqslant L^{-1}\delta_k\big\}.$$

Then, by Lemmas 5.7 and 5.9, we have

$$\mathrm{Mod}\big(\widetilde{f}(\Delta(U,W;V))\big) \leqslant m(\overline{D}) \cdot (L^{-1}\delta_k)^{-n} \leqslant 4\alpha_n \cdot (L^{-1}\delta_k)^{-n}.$$

Here $m(\cdot)$ stands for the n-dimensional Lebesgue measure, and $\alpha_n := m(\mathbb{B}^n(0,1))$. Note that $m(\overline{D}) \leqslant 4\alpha_n$ since $\overline{D} \subseteq \phi_2\big(U^2_{m_0,0}(f^{m-m_0}(Y))\big) \subseteq \mathbb{B}^n(0,2)$.

Since $\phi_1(U^2_{m,0}(Y))$ is a domain (see Lemma 6.2 (iv)) and $\widetilde{f}$ is a KL^{4n}-quasiregular map on $\phi_1(U^2_{m,0}(Y))$, it follows from Theorem 5.12 and

Lemma 7.6 that

$$\mathrm{Mod}(\Delta(U,W;V)) \leqslant N(\widetilde{f},\overline{V}) \cdot K_O(\widetilde{f}) \cdot \mathrm{Mod}(\widetilde{f}(\Delta(U,W;V)))$$
$$\leqslant N_1(k) \cdot KL^{4n} \cdot 4\alpha_n \cdot (L^{-1}\delta_k)^{-n},$$

where $N(\widetilde{f},\overline{V})$ is defined in (3.2) and $N_1(k)$ is the constant given by Lemma 7.6.

For each $\gamma \in \Delta(U,W)$, either $\gamma \in \Delta(U,W;V)$, or we can find (by Lemma 5.6) a subcurve γ' of γ such that $\gamma' \in \Delta(U,\partial V;V)$, and thus $\gamma' \in \Delta(U,W;V)$ since $\partial V \subseteq W$. Then it follows from Lemma 5.8 that

$$\mathrm{Mod}(\Delta(U,W)) \leqslant \mathrm{Mod}(\Delta(U,W;V)).$$

Clearly $W^*_{m,k,l_0}(Y)$ is connected and closed, which implies that $W = \phi_1(W^*_{m,k,l_0}(Y))$ is a continuum. Since U, V, W are bounded subsets of $\mathbb{R}^n$ with $U \subseteq V$ and $\partial V \subseteq W$, we have $\mathrm{diam}(U) \leqslant \mathrm{diam}(\partial V) \leqslant \mathrm{diam}(W)$. Then, by Lemma 5.10, we have

$$c_n \log\left(1 + \frac{\mathrm{diam}(U)}{\mathrm{dist}(U,W)}\right) \leqslant \mathrm{Mod}(\Delta(U,W)) \leqslant \mathrm{Mod}(\Delta(U,W;V)),$$

where c_n is a positive real number depending on n. Then it follows that

$$\frac{\mathrm{diam}(U)}{\mathrm{dist}(U,W)} \leqslant \exp\left(4\alpha_n c_n^{-1} N_1(k) KL^{5n}\delta_k^{-n}\right) - 1.$$

Since ϕ_1 is L-bilipschitz on $U^2_{m,0}(Y)$, we have

$$\frac{\mathrm{dist}_\sigma(Y,Z)}{\mathrm{diam}_\sigma(Y)} \geqslant \frac{\mathrm{dist}_\sigma(Y,W^*_{m,k,l_0}(Y))}{\mathrm{diam}_\sigma(Y)} \geqslant \frac{L^{-2}}{\exp\left(4\alpha_n c_n^{-1} N_1(k) KL^{5n}\delta_k^{-n}\right) - 1}.$$

Put

$$a_k := \min\left\{\mathrm{dist}_\sigma(X,X')/\mathrm{diam}_\sigma(X) : X \in \mathcal{D}_m^{[n]},\ X' \in \mathcal{D}_{m+k}^{[n]},\ X \cap X' = \emptyset,\ m \leqslant m_0\right\},$$

and

$$\Lambda_1(k) := \min\left\{a_k,\ L^{-2}\left(\exp\left(4\alpha_n c_n^{-1} N_1(k) KL^{5n}\delta_k^{-n}\right) - 1\right)^{-1}\right\},$$

then the proof is complete. $\qquad\square$

Lemma 7.13. *Let $f \colon \mathbb{S}^n \to \mathbb{S}^n$ be an expanding cellular Markov branched cover, and $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be a combinatorially bounded essential sequence. Equip $\mathbb{S}^n$ with the spherical metric σ. Suppose that f is a uniformly K-quasiregular map. Then the following statements are true:*

  (i) *There exists $\Lambda_2 > 1$ such that for each $m \in \mathbb{N}_0$ and all $Y, Z \in \mathcal{D}_m^{[n]}$ with $Y \cap Z \neq \emptyset$, we have*

$$\frac{1}{\Lambda_2} \leqslant \frac{\mathrm{diam}(Y)}{\mathrm{diam}(Z)} \leqslant \Lambda_2.$$

(ii) *For each $k \in \mathbb{N}_0$, there exists $\Lambda_3(k) > 0$, such that for each $m \in \mathbb{N}_0$ and all $Y \in \mathcal{D}_m^{[n]}$, $Z \in \mathcal{D}_{m+k}^{[n]}$ with $Y \cap Z \neq \emptyset$, we have*

$$\operatorname{diam}(Z) \geqslant \Lambda_3(k) \operatorname{diam}(Y).$$

(iii) *There exists $\Lambda_4 > 0$, such that for all $m, k \in \mathbb{N}_0$ and all $Y \in \mathcal{D}_m^{[n]}$, $Z \in \mathcal{D}_{m+k}^{[n]}$ with $Y \cap Z \neq \emptyset$, we have*

$$\operatorname{diam}(Y) \geqslant \Lambda_4 \operatorname{diam}(Z).$$

(iv) *There exists $\Lambda_5 > 1$ such that for all x, $y \in \mathbb{S}^n$, $m := m_{f,\mathcal{D}_0}(x,y)$, and each $Y \in \mathcal{D}_m^{[n]}$ containing x, we have*

$$\frac{1}{\Lambda_5} \leqslant \frac{\sigma(x,y)}{\operatorname{diam}(Y)} \leqslant \Lambda_5.$$

Proof. Since f is expanding, we may fix $k_0 \in \mathbb{N}$ such that for each $X \in \mathcal{D}_0^{[n]}$ there exists $X' \in \mathcal{D}_{k_0}^{[n]}$ with $X' \subseteq X \smallsetminus \partial_\circ X$.

(i) By Lemma 7.12, we may fix a constant $0 < c < 1$, such that

$$\operatorname{dist}(X,Y) \geqslant c \operatorname{diam}(X)$$

for each $m \in \mathbb{N}$ and all $X \in \mathcal{D}_m^{[n]}$, $Y \in \mathcal{D}_{m+k_0}^{[n]}$ with $X \cap Y = \emptyset$.

Suppose that $m \in \mathbb{N}_0$, and that Y, $Z \in \mathcal{D}_m^{[n]}$, $Y \cap Z \neq \emptyset$. Suppose that $Y \neq Z$, then $Y \cap Z \subseteq \partial_\circ Z$. By the choice of k_0, there is $Z' \in \mathcal{D}_{m+k_0}^{[n]}$ satisfying $Z' \subseteq Z \smallsetminus \partial_\circ Z$. Consider $x \in Y \cap Z$ and $y \in Z'$. Then, we have

$$\operatorname{diam}(Z) \geqslant \sigma(x,y) \geqslant \operatorname{dist}(Y, Z') \geqslant c \operatorname{diam}(Y).$$

By similar arguments, we have $\operatorname{diam}(Y) \geqslant c \operatorname{diam}(Z)$. Put $\Lambda_2 := c^{-1}$, then the proof of (i) is complete.

(ii) Fix arbitrary $k \in \mathbb{N}$. By Lemma 7.12,

$$\operatorname{dist}(X,Y) \geqslant \Lambda_1(k + k_0) \operatorname{diam}(X)$$

for each $m \in \mathbb{N}$ and all $X \in \mathcal{D}_m^{[n]}$, $Y \in \mathcal{D}_{m+k+k_0}^{[n]}$ with $X \cap Y = \emptyset$.

Consider $m \in \mathbb{N}_0$, and $Y \in \mathcal{D}_m^{[n]}$, $Z \in \mathcal{D}_{m+k}^{[n]}$, with $Y \cap Z \neq \emptyset$.

First suppose that $Z \nsubseteq Y$, then $Y \cap Z \subseteq \partial_\circ Z$. Let $Z' \in \mathcal{D}_{m+k+k_0}^{[n]}$ satisfies $Z' \subseteq Z \smallsetminus \partial_\circ Z$. Consider $x \in Y \cap Z$ and $y \in Z'$, then we have

$$\operatorname{diam}(Z) \geqslant \sigma(x,y) \geqslant \operatorname{dist}(Y, Z') \geqslant c_k \operatorname{diam}(Y),$$

where $c_k := \Lambda_1(k_0 + k)$.

Suppose now that $Z \subseteq Y$. Then, there exists a collection $\{Z_1, Z_2, \ldots, Z_t\}$ of cells in $\mathcal{D}_{m+k}^{[n]}$ with the following properties:

(1) $Z_1 = Z$, $Z_t \nsubseteq Y$, and $Z_i \subseteq Y$ for $i \in \{2, \ldots, t-1\}$.

(2) $Z_i \cap Z_{i+1} \neq \emptyset$ for each $i \in \{1, \ldots, t-1\}$.

(3) $t \leqslant \operatorname{card}(\mathcal{D}_{m+k}|_Y) \leqslant \operatorname{card}(\mathcal{D}_k)$.

Then, by repeatedly applying (i), we have

$$\operatorname{diam}(Z) \geqslant c_k \Lambda_2^{-\operatorname{card}(\mathcal{D}_k)} \operatorname{diam}(Y).$$

Put $\Lambda_3(k) := c_k \Lambda_2^{-\operatorname{card}(\mathcal{D}_k)}$, then the proof of (ii) is complete.

(iii) Consider $m, k \in \mathbb{N}_0$, and $Y \in \mathcal{D}_m^{[n]}$, $Z \in \mathcal{D}_{m+k}^{[n]}$ with $Y \cap Z \neq \emptyset$. Let $Y' \in \mathcal{D}_m^{[n]}$ satisfies $Z \subseteq Y'$. Then $Y \cap Y' \neq \emptyset$. By (i) we have $\operatorname{diam}(Y) \geqslant \Lambda_2^{-1} \operatorname{diam}(Y') \geqslant \Lambda_2^{-1} \operatorname{diam}(Z)$. Put $\Lambda_4 := \Lambda_2^{-1}$, then the proof of (iii) is complete.

(iv) follows from (i) through (iii) and Lemma 7.12, by the same argument in the proof of Lemma 6.11 (ii). Now the proof is complete. □

Now we show that if an expanding cellular Markov branched cover with a combinatorially bounded essential sequence is a uniformly quasiregular map, then its visual spheres are quasi-spheres.

Proposition 7.14. *Let $f \colon \mathbb{S}^n \to \mathbb{S}^n$ be an expanding cellular Markov branched cover and $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be a combinatorially bounded essential sequence. Let σ be the spherical metric, and ϱ be a visual metric for f. If f is uniformly quasiregular, then the map $\operatorname{id} \colon (\mathbb{S}^n, \varrho) \to (\mathbb{S}^n, \sigma)$ is quasisymmetric.*

Proof. To be concise, we denote $m(x, y) := m_{f,\mathcal{D}_0}(x, y)$ for all $x, y \in \mathbb{S}^n$. Let $\Lambda > 1$ be the expansion factor for ϱ, and let $C > 1$ be the constant for which

$$C^{-1} \Lambda^{-m(x,y)} \leqslant \varrho(x, y) \leqslant C \Lambda^{-m(x,y)}$$

holds for all $x, y \in \mathbb{S}^n$.

It is clear that $(\mathbb{S}^n, \sigma)$ is connected and doubling, and that $(\mathbb{S}^n, \varrho)$ is connected. By Lemma 4.21, $(\mathbb{S}^n, \varrho)$ is doubling. Then, by [He01, Theorem 10.19], It suffices to prove that $\operatorname{id} \colon (\mathbb{S}^n, \varrho) \to (\mathbb{S}^n, \sigma)$ weakly quasisymmetric, that is, there exists a constant $H \geqslant 1$ such that for all $x, y, z \in \mathbb{S}^n$ the following implication holds:

$$\varrho(x, z) \leqslant \varrho(x, y) \Rightarrow \sigma(x, z) \leqslant H \sigma(x, y).$$

Suppose that $x, y, z \in \mathbb{S}^n$ satisfy $\varrho(x, z) \leqslant \varrho(x, y)$. Put $k := m(x, y)$ and $l := m(x, z)$. Then

$$C^{-1} \Lambda^{-l} \leqslant \varrho(x, z) \leqslant \varrho(x, y) \leqslant C \Lambda^{-k}.$$

Fix $l_0 \in \mathbb{N}_0$ such that $\Lambda^{l_0} > C^2$, then $k \leqslant l + l_0$.

Pick $X \in \mathcal{D}_l^{[n]}$, $X' \in \mathcal{D}_{l+l_0}^{[n]}$, and $X'' \in \mathcal{D}_k^{[n]}$ that all contain x. Then we have that

$$
\begin{aligned}
\sigma(x, z) &\leqslant \Lambda_5 \operatorname{diam}(X) && \text{by Lemma 7.13 (iv)} \\
&\leqslant \frac{\Lambda_5}{\Lambda_3(l_0)} \operatorname{diam}(X') && \text{by Lemma 7.13 (ii)} \\
&\leqslant \frac{\Lambda_5}{\Lambda_4 \Lambda_3(l_0)} \operatorname{diam}(X'') && \text{by Lemma 7.13 (iii)} \\
&\leqslant \frac{\Lambda_5^2}{\Lambda_4 \Lambda_3(l_0)} \sigma(x, y) && \text{by Lemma 7.13 (iv)}
\end{aligned}
$$

Now the proof is complete. $\qquad\square$

Theorem 7.1 follows immediately.

Proof of Theorem 7.1. If $\mathrm{id} \colon (\mathbb{S}^n, \varrho) \to (\mathbb{S}^n, \sigma)$ is a quasisymmetry, then it follows from Corollary 7.2 and Theorem 7.3 that f is uniformly quasiregular and

(7.1) $$\sup\{N_{\mathrm{loc}}(f^m, x) : m \in \mathbb{N}, \ x \in \mathbb{S}^n\} < +\infty.$$

For the converse, assume that f is uniformly quasiregular and (7.1) holds, and let $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence of f. Since f is expanding, there is $k \in \mathbb{N}$ such that no cell in $\mathcal{D}_k$ joins opposite sides of $\mathcal{D}_0$. Consider f^k. It is clear that f^k is an expanding cellular Markov branched cover with a combinatorially bounded essential sequence $\{\mathcal{D}_{km}\}_{m \in \mathbb{N}_0}$, and that f^k is uniformly quasiregular. By Lemma 4.18, ϱ is a visual metric of f^k. Then it follows from Proposition 7.14 that $\mathrm{id} \colon (\mathbb{S}^n, \varrho) \to (\mathbb{S}^n, \sigma)$ is a quasisymmetry. $\qquad\square$

7.3. **A subdivision condition for bounded local multiplicity.**
For dimension $n = 2$, an expanding Thurston map with no periodic critical point can be considered an example of cellular Markov maps having essential sequences satisfying condition (i) in Definition 7.4. For dimension $n \geqslant 3$, we give a condition, in terms of cell decompositions $\mathcal{D}_1$ and $\mathcal{D}_0$, to construct examples. We call this condition *transversal subdivision*.

Definition 7.15 (Transversal subdivision). A cell decomposition $\mathcal{D}_1$ is a *transversal subdivision* of a cell decomposition $\mathcal{D}_0$ of $\mathcal{S}^n$ if $\mathcal{D}_1$ is a refinement of $\mathcal{D}_0$ with the following property: for each $X \in \mathcal{D}_0^{[n]}$, and each $c \in \mathcal{D}_0|_X$, if $\sigma \in \mathcal{D}_1|_c$ and $\dim(\sigma) = \dim(c)$, then σ is contained in a unique $X_1 \in (\mathcal{D}_1|_X)^{[n]}$.

Remark. The condition can be understood intuitively as follows: the intersection of two cells in $(\mathcal{D}_1|_X)^{[n]}$ can only "transversally" intersect a cell in $\mathcal{D}_0|_X$.

In what follows, we consider a cellular Markov branched cover $f \colon \mathcal{S}^n \to \mathcal{S}^n$ and an essential sequence $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ such that $\mathcal{D}_1$ is a transversal subdivision of $\mathcal{D}_0$. We shall prove that

$$\sup\{N(\mathcal{D}_m, x) : m \in \mathbb{N}, x \in \mathcal{S}^n\} < +\infty.$$

First, we fix some notations. For each $x \in \mathcal{S}^n$ and each $m \in \mathbb{N}_0$, we denote

$$D(\mathcal{D}_m, x) := \min\{\dim(c) : c \in \mathcal{D}_m, \ x \in c\}.$$

Note that, by Lemma 2.5 we have $D(\mathcal{D}_m, x) = \dim(c(x))$, where $c(x)$ is the unique cell in $\mathcal{D}_m$ for which $x \in \mathrm{int}_\circ(c(x))$. In particular, $D(\mathcal{D}_m, x) = 0$ if and only if x is a level-m vertex.

Given $x \in \mathcal{S}^n$, the sequence $\{N(\mathcal{D}_m, x)\}_{m \in \mathbb{N}_0}$, where $N(\mathcal{D}_m, x)$ is defined by (3.4), is non-decreasing, and $\{D(\mathcal{D}_m, x)\}_{m \in \mathbb{N}_0}$ is non-increasing. Precisely, we have the following.

Lemma 7.16. *Let $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ be an essential sequence such that $\mathcal{D}_1$ is a transversal subdivision of $\mathcal{D}_0$. Then the following statements hold for each $x \in \mathcal{S}^n$ and each $m \in \mathbb{N}_0$:*

(i) $D(\mathcal{D}_{m+1}, x) \leqslant D(\mathcal{D}_m, x)$, *and* $N(\mathcal{D}_{m+1}, x) \geqslant N(\mathcal{D}_m, x)$.

(ii) *If x is a level-m vertex, then* $N(\mathcal{D}_{m+1}, x) = N(\mathcal{D}_m, x)$.

(iii) *If $N(\mathcal{D}_{m+1}, x) > N(\mathcal{D}_m, x)$, then* $D(\mathcal{D}_{m+1}, x) < D(\mathcal{D}_m, x)$.

Proof. Let $m \in \mathbb{N}_0$ and $x \in \mathcal{S}^n$ be arbitrary.

(i) Since $\mathcal{D}_{m+1}$ is a refinement of $\mathcal{D}_m$, for each $c \in \mathcal{D}_m$ with $x \in c$, there exists $\widetilde{c} \in \mathcal{D}_{m+1}$ with $x \in \widetilde{c}$ and $\widetilde{c} \subseteq c$. Clearly $\dim(\widetilde{c}) \leqslant \dim(c)$. Hence $D(\mathcal{D}_{m+1}, x) \leqslant D(\mathcal{D}_m, x)$.

Since $\mathcal{D}_{m+1}$ is a refinement of $\mathcal{D}_m$, we have

$$(7.2) \qquad \{X' \in \mathcal{D}_{m+1}^{[n]} : x \in X'\} = \bigcup_X \{X' \in (\mathcal{D}_{m+1}|_X)^{[n]} : x \in X'\},$$

where X ranges over all level-m rooms containing x. It is clear that $\mathrm{card}\big(\{X' \in \mathcal{D}_m^{[n]} : x \in X', \ X' \subseteq X\}\big) \geqslant 1$ for each $X \in \mathcal{D}_m^{[n]}$ containing x. Hence $N(\mathcal{D}_{m+1}, X) \geqslant N(\mathcal{D}_m, x)$.

(ii) Suppose that x is a level-m vertex. Then, for each level-m room X containing x, since $\mathcal{D}_{m+1}|_X = (f^m|_X)^*(\mathcal{D}_1|_{f^m(X)})$, it follows from Definition 7.15 that $\mathrm{card}\big(\{X' \in (\mathcal{D}_{m+1}|_X)^{[n]} : x \in X'\}\big) = 1$. Hence $N(\mathcal{D}_{m+1}, x) = N(\mathcal{D}_m, x)$ by (7.2).

(iii) Assume that $x \in \mathrm{int}_\circ(c)$, $c \in \mathcal{D}_m$, and $N(\mathcal{D}_{m+1}, x) > N(\mathcal{D}_m, x)$. By (ii) we have that x is not a level-m vertex, and thus $\dim(c) > 0$. Since $N(\mathcal{D}_{m+1}, x) > N(\mathcal{D}_m, x)$, by (7.2) there exists a $X \in \mathcal{D}_m^{[n]}$ with $\mathrm{card}(\{X' \in \mathcal{D}_{m+1}^{[n]} : x \in X', \ X' \subseteq X\}) > 1$. Suppose that $x \in X_1 \cap X_2$, where $X_1, X_2 \in (\mathcal{D}_{m+1}|_X)^{[n]}$, and that $x \in \mathrm{int}_\circ(c')$ for some $c' \in \mathcal{D}_{m+1}$. It can be checked that $c' \subseteq c \cap X_1 \cap X_2$. Since $\mathcal{D}_{m+1}|_X = (f^m|_X)^*(\mathcal{D}_1|_{f^m(X)})$, it follows from Definition 7.15 that $\dim(c') < \dim(c)$. Hence $D(\mathcal{D}_{m+1}, x) < D(\mathcal{D}_m, x)$. $\qquad\square$

Proposition 7.17. *Let $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ be an essential sequence such that $\mathcal{D}_1$ is a transversal subdivision of $\mathcal{D}_0$. Then there exists $M_0 \in \mathbb{N}$, depending only on $\mathcal{D}_0, \mathcal{D}_1$ and the dimension n, for which $N(\mathcal{D}_m, x) \leqslant M_0$ for each $x \in \mathcal{S}^n$ and each $m \in \mathbb{N}_0$.*

Proof. Fix an arbitrary $x \in \mathcal{S}^n$. Denote $N_m := N(\mathcal{D}_m, x)$ and $D_m := D(\mathcal{D}_m, x)$ for each $m \in \mathbb{N}_0$. Put $d_1 := \mathrm{card}(\mathcal{D}_1^{[n]})$. It is clear that $N_0 \leqslant N_1 \leqslant d_1$. For each $m \in \mathbb{N}_0$, since

$$\{X' \in \mathcal{D}_{m+1}^{[n]} : x \in X'\} = \bigcup_X \{X' \in (\mathcal{D}_{m+1}|_X)^{[n]} : x \in X'\}$$

where X ranges over all level-m rooms containing x, we have $N_{m+1} \leqslant d_1 N_m$.

Fix arbitrary $m \in \mathbb{N}$ and suppose $N_m > N_0$. Clearly there exists a sequence $\{i_j\}_{j\in\mathbb{N}}$ in $\mathbb{N}_0$ satisfying the following conditions:

(i) $i_j < i_{j+1}$ for each $j \in \mathbb{N}$.

(ii) $N_0 = N_{i_1} < \cdots < N_{i_s} = N_m$, and for each $t \in \{1, \ldots, s\}$, we have $N_j = N_{i_t}$ for each $j \in \{i_t, \ldots, i_{t+1} - 1\}$.

By Lemma 7.16, we have $n \geqslant D_{i_1} > \cdots > D_{i_s} = D_m \geqslant 0$. If $N_m > d_1^{n+1}$, then since $N_0 \leqslant d_1$, and $N_{i_j} \leqslant d_1 N_{i_j - 1} = d_1 N_{i_{j-1}}$ for each j, we have $s > n + 1$, which leads to a contradiction.

Therefore, $N_m \leqslant d_1^{n+1}$ for each $m \in \mathbb{N}$. Put $M_0 := d_1^{n+1}$, then the proof is complete. $\qquad\square$

Hence the following result is clear.

Proposition 7.18. *For a cellular Markov branched cover $f \colon \mathcal{S}^n \to \mathcal{S}^n$, if there exists an essential sequence $\{\mathcal{D}_m\}_{m\in\mathbb{N}_0}$ such that $\mathcal{D}_1$ is a transversal subdivision of $\mathcal{D}_0$, then*

$$\sup\{N_{\mathrm{loc}}(f^m, x) : m \in \mathbb{N}, \ x \in \mathcal{S}^n\} < +\infty.$$

8. Cubical maps

In this section, we introduce a class of cellular Markov maps called cubical maps. Then, we show that cubical maps are discrete and open, and have combinatorially bounded essential sequences. Cubical maps give intuitive examples of maps discussed in previous sections.

8.1. **Cubical structures and cubical decompositions.** First, we introduce the concept of *cubical structures*. We call, for each $n \in \mathbb{N}$, the cell decomposition $C_n := \{\{0\}, \{1\}, [0, 1]\}^n$ a *standard cubical structure on* $[0, 1]^n$; in addition, let $C_0 := \{\{0\}\}$. Given an n-cell Q, a collection $C(Q)$ of subsets of Q is called a *cubical structure on* Q if there exists a homeomorphism $\phi \colon Q \to [0, 1]^n$ such that $C(Q) = \phi^*(C_n)$. In this case, we call $C(Q)$ the cubical structure on Q induced by ϕ. To distinguish ϕ, we denote $C(Q)$ by $C_\phi(Q)$. The $(n-1)$-cubes in $C(Q)$ are called *faces* of $C(Q)$. Each face $q \in C(Q)$ has a unique *opposite face* $q^{\mathrm{op}} \in C(Q)$ satisfying $q^{\mathrm{op}} \cap q = \emptyset$.

Definition 8.1 (Cubical decomposition). A cell decomposition $\mathcal{D}$ of locally compact Hausdorff space $\mathfrak{X}$ is called a *cubical decomposition* if $\mathcal{D}|_\tau$ is a cubical structure on τ for each cell $\tau \in \mathcal{D}$. A k-cell in a cubical decomposition is called a k-*cube*.

Given a cubical decomposition $\mathcal{D}$, we say $q \in \mathcal{D}^{[n-1]}$ is *one-sided* if there exists a unique n-cube containing q. The boundary of $\mathcal{D}$ is defined as

$$\partial \mathcal{D} := \bigcup \left\{ \mathcal{D}|_\tau : \tau \in \mathcal{D}^{[n-1]} \text{ is one-sided} \right\}.$$

Here is an example of cubical decomposition of $[0, 1]^n$ that we call "k-decomposition".

Definition 8.2 (k-decomposition). Fix an arbitrary $k \in \mathbb{N}$, and let

$$A_k := \{\{h_0\}, \{h_1\}, \ldots, \{h_k\}, [h_0, h_1], \ldots, [h_{k-1}, h_k]\}^n,$$

where $h_i := i/k$ for $i \in \{0, \ldots, k\}$. We call the cubical decomposition A_k of $[0, 1]^n$ a k-*decomposition of* $[0, 1]^n$.

We can also define k-decompositions of general n-cells. For an n-cell Q, we call a cell decomposition of the form $\phi^*(A_k)$ a k-decomposition, where $\phi \colon Q \to [0, 1]^n$ is a homeomorphism.

We give another example of cubical decomposition of $[0, 1]^n$, which will be referred to as "bubble decomposition" later in this section.

Definition 8.3 (Bubble decomposition). Recall $C_n = \{\{0\}, \{1\}, [0, 1]\}^n$ and denote $s_0 := [0, 1]^{n-1} \times \{0\} \in C_n$. Let $\mathcal{S}$ be a non-empty collection of faces in C_n. Fix a homeomorphism $\phi_0 \colon s_0 \to \partial_\circ [0, 1]^n \smallsetminus \mathrm{int}_\circ(s_0)$ such

that $\phi_0|_{\partial_\circ s_0} = \mathrm{id}$. For each $s \in \mathcal{S}$, choose a homeomorphism $\tau_s \colon s \to s_0$ such that $C_n|_s = \tau_s^*(C_n|_{s_0})$, Denote $\phi_s \coloneqq \phi_0 \circ \tau_s$ and $\Phi \coloneqq \{\phi_s : s \in \mathcal{S}\}$.

Let $s \in \mathcal{S}$ be arbitrary. Let $T_s \colon [0,1]^n \to [0,1]^n$ be an Euclidean isometry for which $T_s(s) = s_0$, and $h_s \colon [0,1]^{n-1} \to \mathbb{R}$ be a continuous function with the following properties:

(i) $0 \leqslant h_s(x) < 1$, and $h_s^{-1}(0) = \partial_\circ [0,1]^{n-1}$.

(ii) $h_s(x_1, \ldots, x_{n-1}) < \min\{x_i, 1 - x_i\}$ for each $x_i \in (0,1)$, $i \in \{1, 2, \ldots, n-1\}$.

Then
$$Q_s \coloneqq \big\{T_s^{-1}(x', x_n) : x' \in [0,1]^{n-1}, \ x_n \leqslant h(x')\big\}$$
is an n-cell. We construct a cubical structure for Q_s as follows. Consider the graph
$$\Gamma_s \coloneqq \{T_s^{-1}(x', h_s(x')) : x' \in [0,1]^{n-1}\}.$$

Then the map $\sigma_s \colon \Gamma_s \to s$, $T_s^{-1}(x', h(x')) \mapsto T_s^{-1}(x', 0)$ is a homeomorphism with $\sigma_s|_{\partial_\circ s} = \mathrm{id}$, and $\psi_s \coloneqq \tau_s \circ \sigma_s$ is a homeomorphism satisfying $\psi_s|_{\partial_\circ s} = \tau_s|_{\partial_\circ s} = \phi_s|_{\partial_\circ s}$. By gluing ϕ_s and ψ_s, we obtain a homeomorphism $\rho_s \colon \partial_\circ Q_s \to \partial_\circ [0,1]^n$ defined by $\rho_s|_s = \phi_s$ and $\rho_s|_{\Gamma_s} = \psi_s$. By Lemma 2.3, we may extend ρ_s to a homeomorphism $\widetilde{\rho}_s \colon Q_s \to [0,1]^n$, and we obtain a cubical structure $C_{\widetilde{\rho}_s}(Q_s)$.

Set $Q_+ \coloneqq \overline{[0,1]^n \smallsetminus \bigcup_{s \in S} Q_s}$, then Q_+ is an n-cell, and $\partial_\circ Q_+$ is the union of all $r \notin \mathcal{S}$ and Γ_s, $s \in \mathcal{S}$. Then
$$B(\mathcal{S}, \Phi) \coloneqq \{Q_+\} \cup \bigcup_{s \in \mathcal{S}} C_{\widetilde{\rho}_s}(Q_s)$$
is a cubical decomposition. We call $B(\mathcal{S}, \Phi)$ a *bubble decomposition* of $[0,1]^n$, and Q_s a *bubble on the face* $s \in \mathcal{S}$.

Remarks. One may understand Definition 8.3 as follows:

(1) A bubble decomposition is actually a cell complex constructed by gluing cubes along faces; see e.g. FIGURE 8.1.

(2) Condition (ii) can be understood as: the bubble Q_s is contained in the "pyramid" with base s and tip $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$.

Likewise, we can define bubble decompositions for general n-cells. Let Q be an n-cell and $C_\varphi(Q)$ be a standard cubical structure (induced by φ) on Q. Let $\mathcal{S}$ be a non-empty collection of faces in $C_\varphi(Q)$. We denote $s_0 \coloneqq [0,1]^{n-1} \times \{0\}$, and fix a homeomorphism $\phi_0 \colon s_0 \to \partial_\circ [0,1]^n \smallsetminus \mathrm{int}_\circ(s_0)$ such that $\phi_0|_{\partial_\circ s_0} = \mathrm{id}$. For each $s \in \mathcal{S}$, choose a homeomorphism $\tau_s \colon s \to s_0$ which is cellular for $C_\varphi(Q)|_s = \tau_s^*(C_n|_{s_0})$. Let $\phi_s \coloneqq \phi_0 \circ \tau_s$ and denote $\Phi \coloneqq \{\phi_s : s \in \mathcal{S}\}$. Denote also
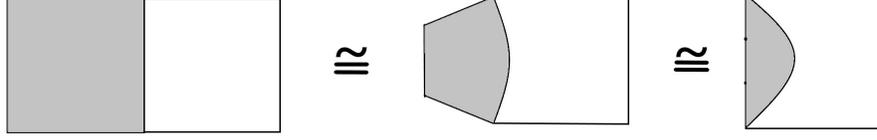
FIGURE 8.1. A bubble decomposition containing a single bubble.

$\varphi_* \mathcal{S} := \{\varphi(s) : s \in \mathcal{S}\}$, $\varphi_* \Phi := \{\phi_s \circ \varphi^{-1}|_{\varphi(s)} : \phi_s \in \Phi\}$. We call $B_Q(\mathcal{S}, \Phi) := \varphi^*(B(\varphi_* \mathcal{S}, \varphi_* \Phi))$ a bubble decomposition of Q.

Note that, given a face $s \in \mathcal{S}$, the restriction $B_Q(\mathcal{S}, \Phi)|_s$ is determined by ϕ_s. Indeed, by the construction in Definition 8.3, we have that

$$B(\phi_* \mathcal{S}, \phi_* \Phi)|_{\varphi(s)} = (\phi_s \circ \varphi^{-1}|_{\varphi(s)})^*(\mathcal{D}),$$

where $\mathcal{D}$ is the restriction of C_n on $\bigcup \{r \in C_n^{[n-1]} : r \neq s_0\}$. Then, we have

$$B_Q(\mathcal{S}, \Phi)|_s = (\varphi|_s)^*(B(\varphi_* \mathcal{S}, \varphi_* \Phi)|_{\varphi(s)})$$
$$= (\varphi|_s)^*(\phi_s \circ \varphi^{-1}|_{\varphi(s)})^*(\mathcal{D}) = \phi_s^*(\mathcal{D}).$$

Now, we introduce a class of decompositions called "generators", which play a key role in the definition of cubical maps.

Definition 8.4 (Generator)**.** Fix $k \in \mathbb{N}$. Let A_k be a k-decomposition of $[0,1]^n$ and $\mathcal{S} \subseteq A_k^{[n-1]} \smallsetminus \partial A_k$ a subset of $(n-1)$-cubes. Let $s_0 := [0,1]^{n-1} \times \{0\}$ and $\phi_0 \colon s_0 \to \partial_\circ [0,1]^n \smallsetminus \mathrm{int}_\circ(s_0)$ be a homeomorphism. For each $s \in \mathcal{S}$, choose a homeomorphism $\tau_s \colon s \to s_0$ for which $A_k|_s = \tau_s^*(C_n|_{s_0})$, and let $\phi_s := \sigma_s \circ \tau_s$. Denote $\Phi := \{\phi_s : s \in \mathcal{S}\}$, and $\mathcal{S}_q := \{s \in \mathcal{S} : s \in A_k|_q\}$, $\Phi_q := \{\phi_s \in \Phi : s \in \mathcal{S}_q\}$ for each $q \in A_k^{[n]}$. Set

$$\mathcal{G}(\mathcal{S}, \Phi) := \bigcup_{q \in A_k^{[n]}} B_q(\mathcal{S}_q, \Phi_q).$$

Here we set $B_q(\mathcal{S}_q, \Phi_q) := A_k|_q$ if $\mathcal{S}_q = \emptyset$. It can be checked that $\mathcal{G}(\mathcal{S}, \Phi)$ is a cubical decomposition of $[0,1]^n$. We call $\mathcal{G}(\mathcal{S}, \Phi)$ the k-generator determined by $\mathcal{S}$ and Φ.

We give several technical lemmas regarding generators. These lemmas are prepared for Section 8.3. The reader may skip this part for now.

Recall that we denote $C_n := \{\{0\}, \{1\}, [0,1]\}^n$.

Lemma 8.5. *Let* $\mathcal{G} := \mathcal{G}(\mathcal{S}, \Phi)$ *be a* k-generator, and $c \in C_n$ *be a cube of dimension* $\dim(c) = d < n$. *Let* $Q \in \mathcal{G}$, $\dim(Q) = n$ *be arbitrary.*
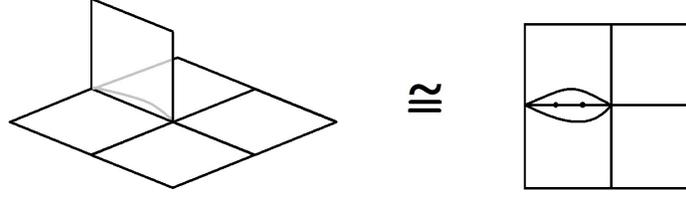
FIGURE 8.2. A 2-generator.

Suppose that $Q \in B_q(\mathcal{S}_q, \Phi_q)$ for some $q \in A_k^{[n]}$. If Q is a bubble on some face $s \in \mathcal{S}_q$, then $Q \cap c = s \cap c$. If Q is not a bubble, then $Q \cap c = q \cap c$.

Proof. First, suppose that Q is a bubble on a face $s \in \mathcal{S}_q$. Then by the construction of $B_q(\mathcal{S}_q, \Phi_q)$, we have $Q \smallsetminus s \subseteq \mathrm{int}_{\circ}(q)$, and thus $Q \smallsetminus s$ does not intersect $\partial_{\circ}[0,1]^n$. So $Q \cap c = s \cap c$.

Consider the other situation where Q is not a bubble. It is clear that each $r \in A_k|_q$ with $\dim(r) \leqslant n-2$ is contained in Q. Also if a face s of q is contained in $\partial_{\circ}[0,1]^n$, then $s \notin \mathcal{S}$, and thus $s \subseteq Q$. Since $q \cap c \subseteq \partial_{\circ}[0,1]^n$, we have $q \cap c \subseteq Q \cap c$, and thus $q \cap c = Q \cap c$. Now the proof is complete. $\qquad\square$

Lemma 8.6. *Let $\mathcal{G} := \mathcal{G}(\mathcal{S}, \Phi)$ be a k-generator, and $c \in C_n$ be of dimension $\dim(c) > 0$. Then for distinct $Q_1, Q_2 \in \mathcal{G}^{[n]}$, we have that $\max\{\dim(\sigma) : \sigma \in \mathcal{G},\ \sigma \subseteq c \cap Q_1 \cap Q_2\} < \dim(c)$. In particular, if $\{p\} \in C_n^{[0]}$, then there exists a unique $Q \in \mathcal{G}$ containing p.*

Proof. First, consider $\{p\} \in C_n^{[0]}$. Set $p = (x_1, \ldots, x_n)$, $x_i \in \{0, 1\}$ for each $i \in \{1, \ldots, n\}$. For each $1 \leqslant i \leqslant n$ if $i = 0$, let $J_i = [0, \frac{1}{k}]$, else if $x_i = 1$, let $J_i = [1 - \frac{1}{k}, 1]$. Then $J_1 \times \cdots \times J_n$ is the unique n-cube in A_k that contains p. By Lemma 8.5, p is not contained in any bubble. By the construction of generators, each n-cube in A_k contains a unique n-cube in $\mathcal{G}$ that is not a bubble, thus p is contained in a unique n-cube in $\mathcal{G}$.

Assume now that $c \in C_n$, $d := \dim(c) > 0$. We may assume $c = [0,1]^d \times \{0\}^{n-d}$. Consider distinct $Q_1, Q_2 \in \mathcal{G}^{[n]}$ with $Q_1 \cap Q_2 \neq \emptyset$. We have the following cases:

Case 1: Suppose that $Q \in \{Q_1, Q_2\}$ is a bubble on a face $s \in A_k^{[n-1]}$. Set

$$s = [i_1/k, (i_1+1)/k] \times \cdots \times \{i_t/k\} \times \cdots \times [i_n/k, (i_n+1)/k],$$

where i_j, $j \neq t$ are integers in $\{0, \ldots, k-1\}$, and $i_t \in \{0, \ldots, k\}$. By the construction of generators, s is not contained in $\partial_{\circ}[0,1]^n$, hence

$i_t \neq 0$. If $t > d$, then $s \cap c = \emptyset$. So we have $t \leqslant d$. In this case, $Q \cap c = s \cap c = \left[\frac{i_1}{k}, \frac{i_1+1}{k}\right] \times \cdots \times \left\{\frac{i_t}{k}\right\} \times \cdots \times \left[\frac{i_d}{k}, \frac{i_d+1}{k}\right] \times \{0\}^{n-d}$, which is a cube in $\mathcal{G}$ of dimension strictly less than d.

Case 2: Suppose that Q_i, $i \in \{1, 2\}$ are not any bubble. Let $Q_i \subseteq q_i$ for some $q_i \in A_k^{[n]}$. Set

$$q_i = \left[\frac{j_1^{(i)}}{k}, \frac{j_1^{(i)}+1}{k}\right] \times \cdots \times \left[\frac{j_n^{(i)}}{k}, \frac{j_n^{(i)}+1}{k}\right],$$

then we have $j_{d+1}^{(i)} = \cdots = j_n^{(i)} = 0$, and

$$Q_i \cap c = q_i \cap c = \left[\frac{j_1^{(i)}}{k}, \frac{j_1^{(i)}+1}{k}\right] \times \cdots \times \left[\frac{j_d^{(i)}}{k}, \frac{j_d^{(i)}+1}{k}\right] \times \{0\}^d.$$

Clearly $\mathrm{int}_\circ(q_1 \cap c)$ and $\mathrm{int}_\circ(q_2 \cap c)$ do not intersect. Thus, if $\sigma \in \mathcal{G}$, $\sigma \subseteq Q_1 \cap Q_2 \cap c$, then $\dim(\sigma) < d$. $\qquad\square$

Lemma 8.7. *Let p be a vertex in C_n, and $c \in C_n$ be an arbitrary cell. If $p \notin c$, then there exists a face $s \in C_n^{[n-1]}$, for which $c \subseteq s$, and $p \notin s$.*

Proof. We may assume that $p = (0, 0, \ldots, 0)$, $p \notin c$. Suppose that $c = I_1 \times \cdots \times I_n$, where $I_j \in \{\{0\}, \{1\}, [0, 1]\}$ for each $1 \leqslant j \leqslant n$. Since $p \notin c$, there exists $i_0 \in \{1, \ldots, n\}$ for which $I_{i_0} = \{1\}$. Set $s := J_1 \times \cdots \times J_n$, where $J_{i_0} = \{1\}$, and $J_i = [0, 1]$ for each $i \neq i_0$. Then s is a face of C_n as desired. $\qquad\square$

Lemma 8.8. *Let $\mathcal{G}$ be a k-generator and $q \in \mathcal{G}^{[n]}$ be an n-cube in $\mathcal{G}$. Then $\bigcap\{c \in C_n : q \cap c \neq \emptyset\} \neq \emptyset$.*

Proof. First, we prove that $\bigcap\{c \in C_n^{[n-1]} : q \cap c \neq \emptyset\} \neq \emptyset$ Suppose that $q \subseteq Q$ for some $Q \in A_k^{[n]}$. Then,

$$\bigcap\{c \in C_n^{[n-1]} : Q \cap c \neq \emptyset\} \subseteq \bigcap\{c \in C_n^{[n-1]} : q \cap c \neq \emptyset\}.$$

Suppose that

$$Q = \left[h_1/k, (h_1+1)/k\right] \times \cdots \times \left[h_n/k, (h_n+1)/k\right],$$

where $h_1, \ldots, h_n \in \{0, \ldots, k-1\}$. Since $q \cap \partial_\circ[0, 1]^n \neq \emptyset$, we have $Q \cap \partial_\circ[0, 1]^n \neq \emptyset$. Thus, we may assume that $h_{j_1}, h_{j_2}, \cdots, h_{j_l} \in \{0, k-1\}$, where $1 \leqslant j_1 < \cdots < j_l \leqslant n$, and that $h_j \notin \{0, k-1\}$ for each $j \notin \{j_1, \ldots, j_l\}$.

Let c be an $(n-1)$-cube in C_n. Suppose that $c = I_1 \times \cdots \times I_n$, where $I_1, \ldots, I_n \in \{\{0\}, \{1\}, [0, 1]\}$. We may assume that $I_j \in \{\{0\}, \{1\}\}$ for some $j \in \{1, \ldots, n\}$, and that $I_t = [0, 1]$ for each $t \neq j$. Then, $c \cap Q \neq \emptyset$ if and only if the following statements are true:

    (1) $j \in \{j_1, \ldots, j_l\}$.

(2) $I_j = \{0\}$ if $h_j = 0$, and $I_j = \{1\}$ if $h_j = k - 1$.

Thus, we have $\bigcap\{c \in C_n^{[n-1]} : Q \cap c \neq \emptyset\} = J_1 \times \cdots \times J_n$, for which the following statements are true:

(1) $J_t = [0, 1]$ for each $t \notin \{j_1, \ldots, j_l\}$.
(2) For each $t \in \{j_1, \ldots, j_l\}$, $J_t = \{0\}$ if $h_t = 0$, and $J_t = \{1\}$ if $h_t = k - 1$.

Since $\bigcap\{c \in C_n^{[n-1]} : Q \cap c \neq \emptyset\} \neq \emptyset$, we have that $\bigcap\{c \in C_n^{[n-1]} : q \cap c \neq \emptyset\} \neq \emptyset$.

Since $\bigcap\{c \in C_n^{[n-1]} : q \cap c \neq \emptyset\} \neq \emptyset$ is a union of cells in C_n, there exists a vertex $p \in \bigcap\{c \in C_n^{[n-1]} : q \cap c \neq \emptyset\}$. Consider arbitrary $c \in C_n$ with $c \cap q \neq \emptyset$, and assume that $p \notin c$. Then, by Lemma 8.7, there exists $s \in C_n^{[n-1]}$ such that $c \subseteq s$, $p \notin s$. Thus $s \cap q \neq \emptyset$, and $p \in s$, which yields a contradiction. Therefore, we have $p \in \bigcap\{c \in C_n : q \cap c \neq \emptyset\} \neq \emptyset$. □

8.2. Cubical spheres with colorings.

Given a topological sphere $\mathcal{S}^n$ and a cubical decomposition $\mathcal{D}$ of $\mathcal{S}^n$, we call the pair $(\mathcal{S}^n, \mathcal{D})$ a *cubical sphere*. Now we introduce a class of cubical decompositions of topological spheres and furthermore introduce cubical maps on topological spheres. See [BM17] for a similar discussion.

Fix $n \in \mathbb{N}$. Recall that $\mathbb{S}^n := \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$ and $\mathbb{B}^n := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leqslant 1\}$. Let $\mathbb{S}_+^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n : x_{n+1} \geqslant 0\}$, $\mathbb{S}_-^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n : x_{n+1} \leqslant 0\}$, and $\mathbb{S}_0^n := \mathbb{S}_+^n \cap \mathbb{S}_-^n$. It is clear that $\mathbb{S}_+^n$, $\mathbb{S}_-^n$ are homeomorphic to $\overline{\mathbb{B}^n}$, and that $\mathbb{S}_0^n$, the boundary of $\mathbb{S}_+^n$ (and $\mathbb{S}_-^n$), is homeomorphic to $\mathbb{S}^{n-1}$.

Let B^n be an n-cell. Given a homeomorphism $\rho \colon B^n \to [0, 1]^n$, we obtain a cubical structure $C_\rho(B^n)$. We denote

$$K_\rho(B^n) := \{\rho^{-1}(s) : s \in C_n^{(n-1)}\}.$$

Then $K_\rho(B^n)$ is a cubical decomposition of the topological $(n-1)$-sphere $\partial_\circ B^n$.

Consider two cubical structures $C_{\rho_+}(\mathbb{S}_+^n)$ and $C_{\rho_-}(\mathbb{S}_-^n)$, where $\varrho_+ \colon \mathbb{S}_+^n \to [0, 1]^n$ and $\varrho_- \colon \mathbb{S}_-^n \to [0, 1]^n$ are homeomorphisms that coincide on $\mathbb{S}_0^n$. Then we obtain a cubical decomposition K of $\mathbb{S}_0^n$ by defining $K := K_{\rho_+}(\mathbb{S}_+^n) = K_{\rho_-}(\mathbb{S}_-^n)$. Let $D_0(\mathbb{S}^n) := \{\mathbb{S}_+^n\} \cup \{\mathbb{S}_-^n\} \cup K$, then $D_0(\mathbb{S}^n)$ is a cubical decomposition of $\mathbb{S}^n$.

Let $\mathcal{S}^n$ be a topological n-sphere and $\phi \colon \mathcal{S}^n \to \mathbb{S}^n$ be a homeomorphism. Then $\phi^*(D_0(\mathbb{S}^n))$ is a cubical decomposition of $\mathcal{S}^n$ containing two n-cubes $\mathcal{S}_+^n := \phi^{-1}(\mathbb{S}_+^n)$ and $\mathcal{S}_-^n := \phi^{-1}(\mathbb{S}_-^n)$. A cubical

decomposition $\mathcal{D}_0(\mathcal{S}^n)$ of $\mathcal{S}^n$ is called a *standard two-cube decomposition* of $\mathcal{S}^n$ if there exists a homeomorphism $\psi\colon \mathcal{S}^n \to \mathbb{S}^n$ for which $\mathcal{D}_0(\mathcal{S}^n) = \psi^*(D_0(\mathbb{S}^n))$.

Let $\mathcal{D}_0$ be a standard two-cube decomposition of $\mathcal{S}^n$. Denote by X^b, X^w the n-cubes in $\mathcal{D}_0$. Suppose that $\mathcal{D}_0|_{X^b} = C_\phi(X^b)$ and $\mathcal{D}_0|_{X^w} = C_\psi(X^w)$ where $\phi\colon X^b \to [0,1]^n$ and $\psi\colon X^w \to [0,1]^n$ are homeomorphisms that coincide on $X^b \cap X^w$. Let $\mathcal{G}^b$, $\mathcal{G}^w$ be k-generators for some $k \geqslant 2$, and denote

$$\mathcal{D}_1 = \mathcal{D}_1(\mathcal{S}^n) \coloneqq \phi^*\big(\mathcal{G}^b\big) \cup \psi^*(\mathcal{G}^w).$$

We call $\mathcal{D}_1$ a refinement of $\mathcal{D}_0$ *compatible with* $\mathcal{G}^b$, $\mathcal{G}^w$.
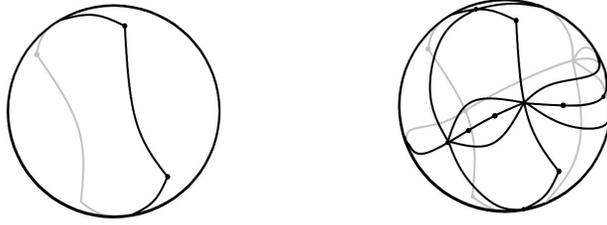

FIGURE 8.3. $\mathcal{D}_0$ and $\mathcal{D}_1$ compatible with generators in the configuration of Figure 8.2

Note that, for a standard two-cube decomposition, "joining opposite sides" actually means joining a pair of opposite faces.

Lemma 8.9. *A connected set $A \subseteq \mathcal{S}^n$ cannot be contained in a single 0-flower if and only if $A \cap q \neq \emptyset$ and $A \cap q^{\mathrm{op}} \neq \emptyset$ for some $q \in \mathcal{D}_0^{[n-1]}$.*

Proof. First, suppose that A joins faces q and q^{op}. If $A \subseteq \mathrm{Fl}_0(p)$ for some level-0 vertex p, then both q and q^{op} meet $\mathrm{Fl}_0(p)$. Thus by Lemma 2.10 (i), $p \in q$ and $p \in q^{\mathrm{op}}$, which leads to contradiction. So A cannot be contained in a single level-0 flower.

Conversely, suppose that A cannot be contained in a single level-0 flower, that is, $A \cap \mathcal{S}^n \smallsetminus \mathrm{Fl}_0(p) \neq \emptyset$ for each level-0 vertex p. Let X be a level-0 room, and suppose that $\mathcal{D}_0|_X = C_\phi(X)$, where $\phi\colon X \to [0,1]^n$ is a homeomorphism. By the construction of $\mathcal{D}_0$, we have that $\mathcal{D}_0^{[n-1]} = (\mathcal{D}_0|_X)^{[n-1]}$. Then it follows from Lemmas 2.10 and 8.7 that

$$A \cap \bigcup\{q^{\mathrm{op}} : q \in (\mathcal{D}_0|_X)^{[n-1]}, \ p \in q\} \neq \emptyset$$

for each level-0 vertex p. Consider a set $S \coloneqq \{1, 2, \ldots, n\} \times \{0, 1\}$, and the bijection $\pi\colon \mathcal{D}_0^{[n-1]} \to S$ given by

$$\pi(\phi^{-1}([0,1]^{i-1} \times \{\delta\} \times [0,1]^{n-i})) = (i, \delta).$$

It can be checked that for each $q \in \mathcal{D}_0^{[n-1]}$, if $\pi(q) = (i, \delta)$, then $\pi(q^{\mathrm{op}}) = (i, 1 - \delta)$. Let $T := \pi\big(\{q \in \mathcal{D}_0^{[n-1]} : A \cap q \neq \emptyset\}\big) = \{(i_1, \sigma_1), \ldots, (i_k, \sigma_k)\}$, where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ and $\sigma_j \in \{0, 1\}$, $j \in \{1, \ldots, k\}$. Now we show that $\mathrm{card}(T) > n$. Assume on the contrary that $k \leqslant n$, then $T' := \{(1, \delta_1), \ldots, (n, \delta_n)\}$ with $\delta_{i_j} = 1 - \sigma_{i_j}$ for each $j \in \{1, \ldots, k\}$ satisfies $T' \cap T = \emptyset$. Consider a vertex $p := \phi^{-1}(x_1, \ldots, x_n)$ where $x_{i_j} = \delta_{i_j}$ for each $j \in \{1, \ldots, k\}$, then $T' = \big\{\pi(q^{\mathrm{op}}) : q \in \mathcal{D}_0^{[n-1]}, \ p \in q\big\}$. Since $T \cap T' = \emptyset$, we have that $A \cap \bigcup \big\{q^{\mathrm{op}} : q \in (\mathcal{D}_0|_X)^{[n-1]}, \ p \in q\big\} = \emptyset$. This yields a contradiction, so A meets at least $(n + 1)$ many level-0 faces. It is clear that among such $(n + 1)$ many faces that meet A, there exists a face q such that q^{op} also meets A. $\qquad\square$

Then, the Lemma below is a direct consequence of Lemmas 2.13 and 8.9.

Lemma 8.10. *Let $\mathcal{D}_0(\mathcal{S}^n)$ be a standard two-cube decomposition. A set $A \subseteq \mathcal{S}^n$ joins opposite sides of $\mathcal{D}_0$ if and only if $A \cap q \neq \emptyset$ and $A \cap q^{\mathrm{op}} \neq \emptyset$ for some $q \in \mathcal{D}_0^{[n-1]}$.*

For a cubical sphere $(\mathcal{S}^n, \mathcal{D})$, we say a map $L: \mathcal{D}^{[n]} \to \{b, w\}$ is a *black-white coloring* of $(\mathcal{S}^n, \mathcal{D})$. Given a cubical sphere $(\mathcal{S}^n, \mathcal{D})$ and a coloring L, we call the triple $(\mathcal{S}^n, \mathcal{D}, L)$ a *colored cubical sphere*. If two distinct n-cubes Q, Q' in $\mathcal{D}$ are colored differently whenever $Q \cap Q'$ contains a cube of dimension $(n - 1)$, then we say that L satisfies *the chessboard property*. It is clear that each k-decomposition of $[0, 1]^n$ has a coloring satisfying the chessboard property. By the construction of generators, we obtain the following facts.

Lemma 8.11. *Let $G = G(S, \Phi)$ be a k-generator, the following statements hold:*

(i) *There exists a black-white coloring L on G satisfying the chessboard property.*

(ii) *If two colorings L_+ and L_- satisfy the checkerboard property, then either $L_+ = L_-$, or $L_+(q) \neq L_-(q)$ for each $q \in G^{[n]}$.*

Proof. Suppose that $G := \bigcup_{q \in A_k^{[n]}} B_q(\mathcal{S}_q, \Phi_q)$. Let $\widetilde{L}$ be a black-white coloring on A_k with chessboard property. For each $q \in A_k^{[n]}$, let $L(q_s) \neq \widetilde{L}(q)$ for each bubble q_s on the face $s \in \mathcal{S}_q$, and let $L\big(q \setminus \bigcup_{s \in \mathcal{S}_q} q_s\big) = \widetilde{L}(q)$. Since bubbles appear in pairs, we can check that L satisfies the chessboard property. Since a black-white coloring on G satisfying (i) is determined by the color of an arbitrary cube, statement (ii) holds. $\qquad\square$

We say the coloring L constructed in the proof is *induced by $\widetilde{L}$*.
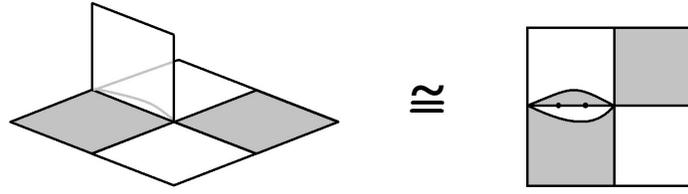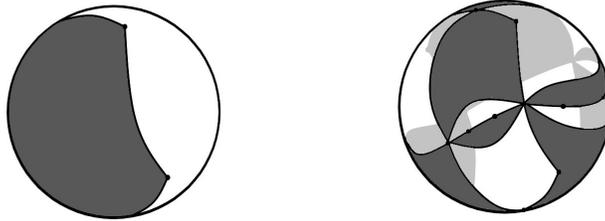
FIGURE 8.4. A generator with chessboard coloring.

Lemma 8.12. *Let $\mathcal{S}^n$ be a topological n-sphere, and $\mathcal{D}_0$ be a standard two-cube decomposition of $\mathcal{S}^n$. Let $\mathcal{G}^b$, $\mathcal{G}^w$ be two k-generators for some $k \geqslant 2$, and $\mathcal{D}_1$ be a refinement of $\mathcal{D}_0$ compatible with $\mathcal{G}^b$, $\mathcal{G}^w$. Then there exists a coloring $L \colon \mathcal{D}_1^{[n]} \to \{b, w\}$ satisfying the chessboard property*

Proof. Let L^b be a coloring on $\mathcal{G}^b$ satisfying the chessboard property. Suppose that L^b is induced by a coloring $\widetilde{L}$ on A_k. Let $\widetilde{L}'$ be the coloring on A_k where $\widetilde{L}'(a) \neq \widetilde{L}(a)$ for each $a \in A_k^{[n]}$, and L^w be the coloring on $\mathcal{G}^w$ such that L^w satisfies the chessboard property and is induced by $\widetilde{L}'$.

We define $L \colon \mathcal{D}_1^{[n]} \to \{b, w\}$ by $L(\phi^{-1}(q)) = L^b(q)$ for each $q \in (\mathcal{G}^b)^{[n]}$ and $L(\psi^{-1}(q)) = L^w(q)$ for each $q \in (\mathcal{G}^w)^{[n]}$. Then L satisfies the chessboard property as desired. $\qquad\square$

FIGURE 8.5. $\mathcal{D}_0$ and $\mathcal{D}_1$ with chessboard coloring

8.3. **Cubical maps with Markov partitions.** Now we introduce cubical maps between cubical spheres.

Definition 8.13 (Cubical maps). Let $(\mathcal{S}^n, \mathcal{D}_0, L_0)$ and $(\mathcal{S}^n, \mathcal{D}_1, L_1)$ be colored cubical spheres, where $\mathcal{D}_0 = \mathcal{D}_0(\mathcal{S}^n)$ is a standard two-cube decomposition, $\mathcal{D}_1$ is a refinement of $\mathcal{D}_0$ compatible with generators $\mathcal{G}^b$, $\mathcal{G}^w$, and L_0, L_1 are colorings with the chessboard property on $\mathcal{D}_0$, $\mathcal{D}_1$, respectively. A map $f \colon (\mathcal{S}^n, \mathcal{D}_1, L_1) \to (\mathcal{S}^n, \mathcal{D}_0, L_0)$ is a *cubical map* if the following conditions are satisfied:

(i) f is continuous, and sense-preserving.

(ii) f is cellular for $(\mathcal{D}_1, \mathcal{D}_0)$.

(iii) f preserves color, that is, for each $q \in \mathcal{D}_1^{[n]}$, $L_1(q) = L_0(f(q))$.

Remark. Note that a cubical map is a cellular Markov map.

We show some elementary properties of cubical maps.

Lemma 8.14. *A cubical map $f\colon (\mathcal{S}^n, \mathcal{D}_1, L_1) \to (\mathcal{S}^n, \mathcal{D}_0, L_0)$ is discrete and open.*

Proof. It is clear that the inverse image $f^{-1}(p)$ of a point $p \in \mathcal{S}^n$ is finite. So f is discrete.

Let $U \subseteq \mathcal{S}^n$ be an open set and $x \in U$ be arbitrary. We have the following two cases.

Case 1: Suppose first that $x \in \mathrm{int}_\circ(Q)$ for some n-cube $Q \in \mathcal{D}_1$. Since $\dim(Q) = n$, by Lemma 2.2, we have that $\mathrm{int}_\circ(Q) = \mathrm{int}(Q)$. Then we have $f(x) \in f(U \cap \mathrm{int}(Q))$. Since $f|_Q\colon Q \to f(Q)$ is a homeomorphism, $f(U \cap \mathrm{int}(Q))$ is an open subset of $\mathrm{int}(f(Q))$, and hence an open subset of $\mathcal{S}^n$. Thus $f(x) \in \mathrm{int}(f(U))$. We conclude that $f(U)$ is open.

Case 2: Suppose now that $x \notin \mathrm{int}_\circ(Q)$ for each n-cube $Q \in \mathcal{D}_1$. By Lemma 2.8, $x \in \partial_\circ Q'$ for some $Q' \in \mathcal{D}_1^{[n]}$. Then $x \in s$ for some $s \in \mathcal{D}_1^{[n-1]}$ with $s \subseteq \partial_\circ Q'$. The construction of $\mathcal{D}_1$ implies that s is contained in a black n-cube and a white n-cube. Let $\{Q_1, \ldots, Q_k\}$ be the set of all n-cubes containing x. Then there exist at least one black cube and one white cube in $\{Q_1, \ldots, Q_k\}$. Suppose that $f(x) \notin \mathrm{int}(f(U))$. Then there exists a sequence $\{y_k'\}_{k \in \mathbb{N}}$ of points in $\mathcal{S}^n \smallsetminus f(U)$, such that $y_i' \neq y_j'$ whenever $i \neq j$, and that $y_k' \to f(x)$ as $k \to +\infty$. Let $\{y_k\}_{k \in \mathbb{N}}$ be a subsequence of $\{y_k'\}$ contained in a single n-cube in $\mathcal{D}_0$. Denote the black and white n-cubes in $\mathcal{D}_0$ by X^b and X^w, respectively. Without loss of generality, assume that $y_k \in X^b$, $k \in \mathbb{N}$. Choose a black n-cube $Q \in \{Q_1, \ldots, Q_k\}$, and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of points in Q such that $f(x_i) = y_i$, $i \in \mathbb{N}$. Since $f|_Q\colon Q \to X^b$ is a homeomorphism, we get that $x_k \to x$ as $k \to +\infty$. Then, when k is large enough, we have $x_k \in U \cap Q$, and thus $y_k \in f(U)$. This is a contradiction. So $x \in \mathrm{int}(f(U))$. Thus $f(U)$ is open. $\square$

Given a cubical map $f\colon (\mathcal{S}^n, \mathcal{D}_1, L_1) \to (\mathcal{S}^n, \mathcal{D}_0, L_0)$, we obtain an essential sequence $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ generated by $\mathcal{D}_0$. It is clear that $\mathcal{D}_m$ is a cubical decomposition for each $m \in \mathbb{N}_0$.

Lemma 8.15. *Let $f\colon (\mathcal{S}^n, \mathcal{D}_1, L_1) \to (\mathcal{S}^n, \mathcal{D}_0, L_0)$ be a cubical map. Then, for each $m \in \mathbb{N}_0$, $(f^*)^m(\mathcal{D}_0)$ is a cubical decomposition of $\mathcal{S}^n$.*

Proof. To be concise, we denote $\mathcal{D}_m := (f^*)^m(\mathcal{D}_0)$ for each $m \in \mathbb{N}_0$.

We prove the lemma by induction. Clearly, $\mathcal{D}_m$ is a cubical decomposition for $m \in \{0, 1\}$.

Given an arbitrary $m \in \mathbb{N}$, $m > 1$. Make the incution hypothesis that $\mathcal{D}_l$ is a cubical decomposition for each $l \in \{0, \ldots, m-1\}$. Let $\sigma \in \mathcal{D}_m$ be arbitrary. Then $\mathcal{D}_{m-1}|_{f(\sigma)}$ is a cubical structure on $f(\sigma)$. Then, by Corollary 3.3, $\mathcal{D}_m|_\sigma = (f|_\sigma)^*(\mathcal{D}_{m-1}|_{f(\sigma)})$ is a cubical structure on σ. Therefore, $\mathcal{D}_m$ is a cubical decomposition.

By induction, $\mathcal{D}_m$ is a cubical decomposition for all $m \in \mathbb{N}_0$. $\square$

We finish this section by showing that cubical maps have combinatorially bounded essential sequences. A cubical map $f \colon (\mathcal{S}^n, \mathcal{D}_1, L_1) \to (\mathcal{S}^n, \mathcal{D}_0, L_0)$ is a cellular Markov branched cover by Lemma 8.14. It follows from Lemma 8.8 that no cell in $\mathcal{D}_1$ joins opposite sides of $\mathcal{D}_0$. By Proposition 7.18, it suffices to show that $\mathcal{D}_1$ is a transversal subdivision of $\mathcal{D}_0$; see Definition 7.15.

Proposition 8.16. *Let* $f \colon (\mathcal{S}^n, \mathcal{D}_1, L_1) \to (\mathcal{S}^n, \mathcal{D}_0, L_0)$ *be a cubical map. Then* $\mathcal{D}_1$ *is a transversal subdivision of* $\mathcal{D}_0$.

Proof. Let $Q \in \mathcal{D}_0^{[n]}$ and $c \in \mathcal{D}_0|_Q$ be arbitrary. Suppose that $\mathcal{D}_1|_Q := \phi^*(\mathcal{G})$, $\mathcal{D}_0|_Q = \phi^*(C_n)$, where $\mathcal{G}$ is a k-generator and ϕ is a homeomorphism. Suppose on the contrary that there is $\sigma \in \mathcal{D}_1|_c$ with $\dim(\sigma) = \dim(c)$, such that $\sigma \subseteq Q_1 \cap Q_2$, where $Q_1, Q_2 \in (\mathcal{D}_1|_Q)^{[n]}$ are distinct. Then $\phi(\sigma) \subseteq \phi(c) \cap \phi(Q_1) \cap \phi(Q_2)$, and $\dim(\phi(\sigma)) = \dim(\phi(c))$, which yields a contradiction by Lemma 8.6. $\square$

Corollary 8.17. *For an expanding cubical map* $f \colon (\mathbb{S}^n, \mathcal{D}_1, L_1) \to (\mathbb{S}^n, \mathcal{D}_0, L_0)$, *the sequence* $\{\mathcal{D}_m\}_{m \in \mathbb{N}_0}$ *given by* $\mathcal{D}_m := (f^*)^m(\mathcal{D}_0)$ *is a combinatorially bounded essential sequence.*

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