A CLASS OF LATTÈS MAPS WITH CELLULAR STRUCTURES

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Abstract. We show that a class of quasiregular Lattès maps, called orthotopic Lattès maps, are cellular Markov maps. This provides examples of expanding Thurston-type maps that are also uniformly quasiregular, and whose visual metrics are quasisymmetrically equivalent to the Riemannian distance.

Contents

1. Introduction

As a higher-dimensional analog of the dynamics of rational maps on the Riemann sphere $\widehat{\mathbb{C}}$, the theory of uniformly quasiregular (abbreviated as UQR) maps has been developed in the past decades. Heuristically, a quasiregular map is a map $\mathcal{M}^n \to \mathcal{M}^n$ with bounded distortion on an oriented, connected, and closed Riemannian n-manifold $\mathcal{M}^n$, and a uniformly quasiregular map is a quasiregular map whose iterates have a uniform bound of distortion. We recall these notions in Section 2 and refer the reader to [IM01, Ka21, Ri93, Vu88] for the general quasiregular theory.

An important subclass of uniformly quasiregular maps are the quasiregular Lattès maps, which include the well-known Lattès maps on $\widehat{\mathbb{C}}$. A *quasiregular Lattès map* is a uniformly quasiregular map $f\colon \mathcal{M}^n \to \mathcal{M}^n$ that is semi-conjugate to a conformal affine map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ via a quasiregular map $h\colon \mathbb{R}^n \to \mathcal{M}^n$ strongly automorphic with respect to some discrete subgroup $\Gamma < \operatorname{Isom}(\mathbb{R}^n)$, as in the following commutative diagram.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "A"] \arrow[d, "h"'] & \mathbb{R}^n \arrow[d, "h"] \\
\mathcal{M}^n \arrow[r, "f"] & \mathcal{M}^n
\end{tikzcd}$$

We give a precise definition in Section 2, and refer to [IM01, May97, Ka22] for more information.

In dimension two, Lattès maps on $\widehat{\mathbb{C}}$ belong to the class of Thurston maps, i.e., branched covering maps with finite postcritical points and degree at least 2; cf. [BM17] for an exposition of the theory

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of Thurston maps. In the higher-dimensional case, it is natural to investigate whether (quasiregular) Lattès maps have properties analogous to Thurston maps.

A class of higher-dimensional analogs of Thurston maps called *Thurston-type maps* were introduced in [LPZ25]. The definition of Thurston-type maps is based on topological and combinatorial conditions (see Definition 2.12), which naturally generalize the postcritically-finite condition of Thurston maps, and lead to a higher-dimensional theory having potential for fruitful results. In particular, [LPZ25, Theorem A] states the following rigidity phenomenon reminiscent of the No Invariant Line Fields conjecture (see e.g. [McS98]): *In dimension* $n \geqslant 3$*, if an expanding Thurston-type map is uniformly quasiregular, then it is a Lattès map.*

In view of the discussion above, one may ask, in higher dimensions, which (quasiregular) Lattès maps are Thurston-type maps. In this article, we show that this holds for a class of Lattès maps, which we call orthotopic Lattès maps.

Theorem A. *An orthotopic Lattès map* $\mathcal{M}^n \to \mathcal{M}^n$*,* $n \geqslant 3$*, on a connected closed Riemannian* n*-manifold* $\mathcal{M}^n$ *is a cellular Markov map. Moreover, if* f *has topological degree at least two, then* f *is a Thurston-type map.*

We give a formal definition of orthotopic Lattès maps in Section 3. Heuristically, an orthotopic Lattès map is a Lattès map where the group Γ has a fundamental domain in the shape of an orthotope. Although this class of Lattès maps has not been given a name in the UQR literature, many known examples of UQR maps on closed Riemannian manifolds are orthotopic; see e.g. [AKP10] for a discussion on such examples.

We recall the definitions of cellular Markov maps and Thurston-type maps in Section 2, as well as briefly discuss their relation; see also [BM17, Chapter 5] and [LPZ25, Section 1]. Since the theory of Thurston-type maps is beyond the scope of this article, we will not include a detailed discussion.

If an orthotopic Lattès map $f\colon \mathcal{M}^n \to \mathcal{M}^n$, $n \geqslant 3$, is *chaotic* (see Definition 2.1), then it is an expanding Thurston-type map, which naturally induces a class of metrics called *visual metrics of* f. By [LPZ25, Theorem B], these visual metrics are quasisymmetrically equivalent to the Riemannian distance on $\mathcal{M}^n$. See [LPZ25, Sections 1–3 and Appendix A.2] (cf. [BM17, HP09]) for a discussion on these notions.

Corollary. *Let* $f\colon (\mathcal{M}^n, d_g) \to (\mathcal{M}^n, d_g)$*,* $n \geqslant 3$*, be a chaotic orthotopic Lattès map on a connected closed Riemannian* n*-manifold* $(\mathcal{M}^n, d_g)$*. Then each visual metric of* f *is quasisymmetrically equivalent to* d_g*.*

Finally, we note that it is not known, to our knowledge, whether all Lattès maps on $\mathcal{M}^n$ with dimension $n \geqslant 3$ are Thurston-type maps, or even have cellular branch sets. For general UQR maps, this is not the case. Indeed, by a result of Martin and Peltonen [MarPe10], the branch set of a quasiregular map on $\mathbb{S}^n$ is realizable as the branch set of a uniformly quasiregular map on $\mathbb{S}^n$, and by the construction of Heinonen and Rickman [HR98] there exist quasiregular maps $\mathbb{S}^3 \to \mathbb{S}^3$ without cellular branch sets.

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2. PRELIMINARIES

2.1. **Quasiregular maps and Lattès maps.** Recall that a continuous map $f\colon \mathcal{M}^n \to \mathcal{N}^n$ between oriented Riemannian n-manifolds is *quasiregular* if f is in the Sobolev space $W^{1,n}_{\mathrm{loc}}(\mathcal{M}^n, \mathcal{N}^n)$ and there exists $K \geqslant 1$ for which the distortion inequality

$$\|Df\|^n \leqslant K J_f \quad \text{a.e. } \mathcal{M}^n \tag{2.1}$$

holds, where $\|Df\|$ and J_f are the operator norm and the Jacobian, respectively, of the differential Df of f. In this case, we say that f is *K-quasiregular*. In particular, a holomorphic map of one complex variable is a 1-quasiregular map. In this terminology, a map is *quasiconformal* if it is a quasiregular homeomorphism. We refer to the monographs of Rickman [Ri93], Reshetnyak [Re89], and Iwaniec & Martin [IM01] for detailed expositions on quasiregular theory.

Following Iwaniec & Martin [IM96] (see also [IM01]), a quasiregular map $f\colon \mathcal{M}^n \to \mathcal{M}^n$ of an oriented Riemannian n-manifold is *uniformly quasiregular* (abbreviated as *UQR*) if there exists $K \geqslant 1$ for which each iterate f^k, $k \geqslant 1$, of f is K-quasiregular.[1] See a survey of Martin [Mar14] for a detailed discussion.

An important subclass of uniformly quasiregular maps are the quasiregular Lattès maps originating from the work of Mayer [May97, May98]. For the convenience of the reader, we recall the notion of Lattès maps (cf. [IM01, Ka22, May97] for more information).

Definition 2.1. Let $\mathcal{M}^n$, $n \geqslant 3$, be an oriented, connected, and closed Riemannian n-manifold. A triple (Γ, h, A) is a *Lattès triple* on $\mathcal{M}^n$ if the following conditions are satisfied:

(i) Γ is a discrete subgroup of the isometry group $\operatorname{Isom}(\mathbb{R}^n)$.

(ii) $h\colon \mathbb{R}^n \to \mathcal{M}^n$ is a quasiregular map which is strongly automorphic with respect to Γ, i.e., for all x, $y \in \mathbb{R}^n$, $h(x) = h(y)$ if and only if $y = \gamma(x)$ for some $\gamma \in \Gamma$.

(iii) A is a conformal affine map (i.e., $A = \lambda U + v$ for some $\lambda > 0$, $U \in \mathrm{O}(n)$, and $v \in \mathbb{R}^n$) with $A\Gamma A^{-1} \subseteq \Gamma$.

A *Lattès map* is a uniformly quasiregular map $f\colon \mathcal{M}^n \to \mathcal{M}^n$ for which there exists a Lattès triple (Γ, h, A) with $f \circ h = h \circ A$. In this case, we call f a *Lattès map with respect to* (Γ, h, A). We call a Lattès map with respect to a Lattès triple (Γ, h, A) *chaotic* if Γ is cocompact and A is expanding, i.e., $A = \lambda U + v$ for some $\lambda > 1$, $U \in \mathrm{O}(n)$, and $v \in \mathbb{R}^n$.

2.2. **Cellular Markov maps.** Here we recall the notion of cell decompositions. Our definitions of cells and cell decompositions follow [BM17, Chapter 5]. Throughout the remaining part of this section, $\mathfrak{X}$ will always be a locally-compact Hausdorff space.

Recall that, for each $n \in \mathbb{N}$, a subset c of $\mathfrak{X}$ homeomorphic to $[0,1]^n$ is called an *n-dimensional cell*, and $\dim(c) := n$ is called the dimension of c. We denote by $\partial_\circ c$ the set of points corresponding to $[0,1]^n \smallsetminus (0,1)^n$ under a homeomorphism between c and $[0,1]^n$. We call $\partial_\circ c$ the *cell-boundary* and $\operatorname{int}_\circ c := c \smallsetminus \partial_\circ c$ the *cell-interior* of c. The sets $\partial_\circ c$ and $\operatorname{int}_\circ c$ are independent of the choice of the homeomorphism. Note that the cell-boundary and cell-interior generally do not agree with the boundary ∂c and interior $\operatorname{int} c$ of c regarded as a subset of the topological space $\mathfrak{X}$. A 0-dimensional cell is a subset consisting of a single point in $\mathfrak{X}$. For a 0-dimensional cell c, we set $\partial_\circ c := \emptyset$ and $\operatorname{int}_\circ c := c$.

We recall some basic properties of cells for further discussion, omitting the standard proofs.

Lemma 2.2. *The following statements are true:*

(i) *Let $\mathfrak{X}$ be a locally-compact Hausdorff space. Then each cell $c \subseteq \mathfrak{X}$ is closed. Moreover, $c = \overline{\operatorname{int}_\circ c}$.*

(ii) *Let $\mathfrak{X}$ be a topological n-manifold. Then for each n-dimensional cell X in $\mathfrak{X}$, $\operatorname{int}_\circ X$ and $\partial_\circ X$ agree with the interior and boundary of X regarded as a subset of the topological manifold $\mathfrak{X}$, respectively.*

Definition 2.3 (Cell decompositions). A collection $\mathcal{D}$ of cells in a locally-compact Hausdorff space $\mathfrak{X}$ is a *cell decomposition of* $\mathfrak{X}$ if the following conditions are satisfied:

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[1] Note that the uniform quasiregularity of a map is independent of the choice of the Riemannian metric of $\mathcal{M}^n$, since, for Riemannian metrics g and g' on $\mathcal{M}^n$, the identity map $\mathrm{id}\colon (\mathcal{M}^n, g) \to (\mathcal{M}^n, g')$ of a closed Riemannian n-manifold is quasiconformal.

(i) The union of all cells in $\mathcal{D}$ is equal to $\mathfrak{X}$.
(ii) $\operatorname{int}_\circ(\sigma) \cap \operatorname{int}_\circ(\tau) = \emptyset$ for all distinct σ, $\tau \in \mathcal{D}$.
(iii) For each $\tau \in \mathcal{D}$, the cell-boundary $\partial_\circ \tau$ is a union of cells in $\mathcal{D}$.
(iv) Every point in $\mathfrak{X}$ has a neighborhood that meets only finitely many cells in $\mathcal{D}$.

For a collection $\mathcal{C}$ of cells in an ambient space $\mathfrak{X}$, we denote by $|\mathcal{C}| \coloneqq \bigcup \mathcal{C}$ the *space* of $\mathcal{C}$. If $\mathcal{C}$ is a cell decomposition of $|\mathcal{C}|$, then we call $\mathcal{C}$ a *cell complex*. A cell decomposition is a cell complex. A subset $\mathcal{C}' \subseteq \mathcal{C}$ is a *subcomplex* of a cell complex $\mathcal{C}$ if $\mathcal{C}'$ is a cell complex.

Let $\mathcal{D}$ be a cell decomposition of $\mathfrak{X}$. For each $S \subseteq \mathfrak{X}$, we denote

$$\mathcal{D}|_S \coloneqq \{c \in \mathcal{D} : c \subseteq S\}, \tag{2.2}$$

and call it the *restriction of* $\mathcal{D}$ *on* S. For each $k \in \mathbb{N}_0$, we denote

$$\mathcal{D}^{[k]} \coloneqq \{c \in \mathcal{D} : \dim(c) = k\},$$

and call $\left|\bigcup_{i \leqslant k} \mathcal{D}^{[i]}\right|$ the *k-skeleton of* $\mathcal{D}$.

We record some elementary properties of cell decompositions.

Lemma 2.4 ([BM17, Lemmas 5.2 and 5.3])**.** *Let $\mathcal{D}$ be a cell decomposition of $\mathfrak{X}$. Then the following statements are true:*

(i) *For each $k \in \mathbb{N}_0$, the k-skeleton of $\mathcal{D}$ is equal to $\bigcup\{\operatorname{int}_\circ c : c \in \mathcal{D}, \dim(c) \leqslant k\}$.*
(ii) *$\mathfrak{X} = \bigcup\{\operatorname{int}_\circ c : c \in \mathcal{D}\}$.*
(iii) *For each $\tau \in \mathcal{D}$, we have $\tau = \bigcup\{\operatorname{int}_\circ c : c \in \mathcal{D}, c \subseteq \tau\}$.*
(iv) *If σ and τ are two distinct cells in $\mathcal{D}$ with $\sigma \cap \tau \neq \emptyset$, then one of the following three statements is true: $\sigma \subseteq \partial_\circ \tau$, $\tau \subseteq \partial_\circ \sigma$, or $\sigma \cap \tau = \partial_\circ \sigma \cap \partial_\circ \tau$ and the intersection consists of cells in $\mathcal{D}$ of dimension strictly less than* $\min\{\dim(\sigma), \dim(\tau)\}$.
(v) *If σ, $\tau_1, \dots, \tau_k$ are cells in $\mathcal{D}$ and $\operatorname{int}_\circ(\sigma) \cap \bigcup_{j=1}^k \tau_j \neq \emptyset$, then $\sigma \subseteq \tau_i$ for some $i \in \{1, \dots, k\}$.*

Lemma 2.5 ([LPZ25, Lemma 2.10])**.** *Let $\mathcal{D}$ be a cell decomposition of a locally-compact Hausdorff space $\mathfrak{X}$ and $S \subseteq \mathfrak{X}$. Then the following statements are true:*

(i) *$\mathcal{D}|_S$ is a subcomplex of $\mathcal{D}$.*
(ii) *If S is a union of cells in $\mathcal{D}$, then $\mathcal{D}|_S$ is a cell decomposition of S. In particular, for each $c \in \mathcal{D}$, $\mathcal{D}|_c$ and $\mathcal{D}|_{\partial_\circ c}$ are cell decompositions of c and $\partial_\circ c$, respectively.*
(iii) *If $S = |\mathcal{C}|$ for a subcomplex $\mathcal{C}$ of $\mathcal{D}$, then $\mathcal{D}|_S = \mathcal{C}$.*

We also recall the notion of *refinements of cell decompositions*, and record several properties.

Definition 2.6 (Refinements). A cell decomposition $\mathcal{D}_1$ of a locally-compact Hausdorff space $\mathfrak{X}$ is a *refinement* of a cell decomposition $\mathcal{D}_0$ of $\mathfrak{X}$ if the following conditions are satisfied:

(i) For each $\sigma \in \mathcal{D}_1$, there exists $\tau \in \mathcal{D}_0$ satisfying $\sigma \subseteq \tau$.
(ii) Each cell $\tau \in \mathcal{D}_0$ is the union of cells $\sigma \in \mathcal{D}_1$ satisfying $\sigma \subseteq \tau$.

In this case, we also say that $\mathcal{D}_1$ *refines* $\mathcal{D}_0$.

Lemma 2.7 ([BM17, Lemma 5.7])**.** *Let $\mathcal{D}$ be a cell decomposition of a locally-compact Hausdorff space $\mathfrak{X}$ and let $\mathcal{D}'$ be a refinement of $\mathcal{D}$. Then for each $\sigma \in \mathcal{D}'$, there exists a minimal cell $\tau \in \mathcal{D}$ with $\sigma \subseteq \tau$, i.e., if $\sigma \subseteq \widetilde{\tau}$ for some $\widetilde{\tau} \in \mathcal{D}$, then $\tau \subseteq \widetilde{\tau}$. Moreover, τ is the unique cell in $\mathcal{D}$ with* $\operatorname{int}_\circ \sigma \subseteq \operatorname{int}_\circ \tau$.

Lemma 2.8 ([LPZ25, Lemma 2.17])**.** *Let $\mathcal{D}$ be a cell decomposition of a locally-compact Hausdorff space $\mathfrak{X}$. For each $c \in \mathcal{D}$, let $\mathcal{D}'(c)$ be a cell decomposition of c that refines $\mathcal{D}|_c$. Then $\mathcal{D}' \coloneqq \bigcup_{c \in \mathcal{D}} \mathcal{D}'(c)$ is a cell decomposition that refines $\mathcal{D}$ if and only if for all c, $\sigma \in \mathcal{D}$ with $c \subseteq \sigma$, we have $\mathcal{D}'(\sigma)|_c = \mathcal{D}'(c)$.*

Now we recall cellular maps and related notions; see [BM17, Chapter 5] or [LPZ25, Section 2] for more information. In what follows, $\mathfrak{X}$ and $\mathfrak{X}'$ stand for locally-compact Hausdorff spaces.

Definition 2.9 (Cellular maps). A continuous map $f\colon \mathfrak{X}' \to \mathfrak{X}$ is a *cellular map* if there exist cell decompositions $\mathcal{D}'$ and $\mathcal{D}$ of $\mathfrak{X}'$ and $\mathfrak{X}$, respectively, such that for each $c \in \mathcal{D}'$, $f(c)$ is a cell in $\mathcal{D}$ and the restriction $f|_c\colon c \to f(c)$ is a homeomorphism. In this case, we call f a $(\mathcal{D}', \mathcal{D})$*-cellular map*, $(\mathcal{D}', \mathcal{D})$ a *cellular pair* of f, and $\mathcal{D}'$ a *pullback* of $\mathcal{D}$ by f.

Note that the pullback $\mathcal{D}'$ of cell decomposition $\mathcal{D}$, if exists, is uniquely determined by $\mathcal{D}$ (see [LPZ25, Subsection 2.3] for a discussion), although neither each cell decomposition admits a pullback, nor cellular pairs in Definition 2.9 are unique.

Homeomorphisms are cellular maps: under a homeomorphism, each cell decomposition admits a unique pullback. For a homeomorphism $\phi\colon \mathfrak{X}' \to \mathfrak{X}$ and a cell decomposition $\mathcal{D}$ of $\mathfrak{X}$, denote

$$\phi^*(\mathcal{D}) \coloneqq \bigl\{\phi^{-1}(c) : c \in \mathcal{D}\bigr\}. \tag{2.3}$$

Then it is easy to check that $\phi^*(\mathcal{D})$ is a cell decomposition.

Lemma 2.10. *If $\phi\colon \mathfrak{X}' \to \mathfrak{X}$ is a homeomorphism and $\mathcal{D}$ is a cell decomposition of $\mathfrak{X}$, then $\phi^*(\mathcal{D}) \coloneqq \bigl\{\phi^{-1}(c) : c \in \mathcal{D}\bigr\}$ is a cell decomposition, and the unique pullback of $\mathcal{D}$. In particular, given a $(\mathcal{D}', \mathcal{D})$-cellular map $f\colon \mathfrak{X}' \to \mathfrak{X}$, for each $c' \in \mathcal{D}'$, we have $\mathcal{D}'|_{c'} = (f|_{c'})^*\bigl(\mathcal{D}|_{f(c')}\bigr)$.*

We conclude with the definition of the aforementioned cellular Markov maps and Thurston-type maps.

Definition 2.11 (Cellular Markov partitions, cellular Markov maps). A *cellular Markov partition* of a continuous map $f\colon \mathfrak{X} \to \mathfrak{X}$ is a cellular pair $(\mathcal{D}', \mathcal{D})$ of f where $\mathcal{D}'$ is a refinement of $\mathcal{D}$. We call f a *cellular Markov map* if there exists a cellular Markov partition of f.

In this article, we adopt the formulation in [HR02] and say that a discrete, open, and continuous map $f\colon \mathfrak{X} \to \mathfrak{Y}$ between topological spaces $\mathfrak{X}$ and $\mathfrak{Y}$ is a *branched cover.* Note that quasiregular maps are orientation-preserving branched covers (see e.g. [Ri93, Chapter VI] or [Ka21, Section 11]). For a branched cover $f\colon \mathfrak{X} \to \mathfrak{Y}$, we denote the *branch set* (or *critical set*) and *postbranch set* (or *postcritical set*), respectively, by

$$B_f \coloneqq \{x \in \mathfrak{X} : f \text{ is not a local homeomorphism at } x\} \quad \text{and} \quad P_f \coloneqq \bigcup_{m \geqslant 1} f^m(B_f).$$

We denote the *local multiplicity* of f at $x \in \mathfrak{X}$ by

$$i(x, f) \coloneqq \inf\{N(f, U) : U \subseteq \mathfrak{X} \text{ is an open neighborhood of } x\},$$

where $N(f, U) \coloneqq \sup\{\operatorname{card}\bigl(f^{-1}(y) \cap U\bigr) : y \in f(U)\}$

Definition 2.12 (Thurston-type maps). A branched cover $f\colon \mathcal{M}^n \to \mathcal{M}^n$ is *cellularly-postcritically-finite (CPCF)* if the following conditions hold:

(i) There exists a cellular Markov partition $(\mathcal{P}_0, \mathcal{P})$ of $f|_{P_f}\colon P_f \to P_f$ and a cell decomposition $\mathcal{D}_0$ of $\mathcal{M}^n$ such that $\mathcal{P}_0 \subseteq \mathcal{D}_0$.

(ii) There exists a cell decomposition $\mathcal{B}$ of B_f such that $f|_{B_f}\colon B_f \to P_f$ is $(\mathcal{B}, \mathcal{P})$-cellular, and that for each $c \in \mathcal{B}$, the function $i(f, \cdot)|_{\operatorname{int}_\circ c}\colon \operatorname{int}_\circ c \to \mathbb{N}$, $x \mapsto i(x, f)$ is constant.

A *Thurston-type map* is a CPCF branched cover with topological degree at least two.

Remark 2.13. We note that cellular Markov branched covers are CPCF (see [LPZ25, Proposition 3.21]), and are of Thurston-type when having topological degrees at least two; cf. [LPZ25, Section 3] for a detailed discussion.

3. Orthotopic Lattès maps

The main purpose of this article is to show that orthotopic Lattès maps are cellular Markov.

Theorem 3.1. *An orthotopic Lattès map $\mathcal{M}^n \to \mathcal{M}^n$, $n \geqslant 3$, on a connected closed Riemannian n-manifold is a cellular Markov map.*

Consequently, since Lattès maps are branched covers and by Remark 2.13, orthotopic Lattès maps with degree at least two are Thurston-type maps. Then Theorem A follows.

Now, we proceed to the definition of orthotopic Lattès maps and discuss first the underlying notion of orthotopic crystallographic groups.

3.1. **Orthotopic crystallographic groups and Lattès triples.** In this subsection, we fix an arbitrary $n \in \mathbb{N}$ and use the convention that all discussions take place in the ambient space $\mathbb{R}^n$.

First, following [SV93], we recall some basic notions on crystallographic groups and Euclidean geometry of $\mathbb{R}^n$. A discrete subgroup Γ of $\operatorname{Isom}(\mathbb{R}^n)$ is called *crystallographic* if Γ acts cocompactly on $\mathbb{R}^n$, i.e., the quotient space $\mathbb{R}^n/\Gamma$ is compact. A *fundamental domain* of a discrete subgroup $\Gamma < \operatorname{Isom}(\mathbb{R}^n)$ is a closed domain (i.e., the closure of some domain) $D \subseteq \mathbb{R}^n$ satisfying the following properties:

(i) $\{\gamma(D) : \gamma \in \Gamma\}$ is a locally-finite cover of $\mathbb{R}^n$.
(ii) If $\gamma \in \Gamma$ and $\gamma \neq \mathrm{id}$, then $\operatorname{int}(D) \cap \operatorname{int}(\gamma(D)) = \emptyset$.

Note that for a crystallographic group Γ there always exists a *normal fundamental polyhedron*, i.e., a fundamental domain P that is a convex polyhedron, and the intersection of two adjacent polyhedra in $\{\gamma(P) : \gamma \in \Gamma\}$ is a common face of them. Examples of polyhedra include simplices, cubes, orthotopes, etc. In the current section, we only consider orthotopes and do not discuss polyhedra in detail; cf. [SV93] for a more comprehensive discussion.

In what follows, for $d \in \mathbb{N}_0$ with $d \leqslant n$, we identify $\mathbb{R}^d$ with the subspace $\mathbb{R}^d \times \{0\}^{n-d}$ of $\mathbb{R}^n$. We also use the notation 0 to denote the point $(0, \dots, 0) \in \mathbb{R}^n$ when there is no ambiguity.

Fix $d \in \mathbb{N}$ with $d \leqslant n$. We call a subset of $\mathbb{R}^d$, having the form $\prod_{i=1}^d [-a_i, a_i]$, $a_i \in (0, +\infty)$ for each $1 \leqslant i \leqslant d$, a *standard d-dimensional orthotope.* A subset $Q \subseteq \mathbb{R}^n$ is a *d-dimensional orthotope* if Q is isometric to a standard d-dimensional orthotope. A 0*-dimensional orthotope* is a set containing a single point.

A special subclass of orthotopes are the *cubes.* For $d \in \mathbb{N}$ with $d \leqslant n$, a *d-dimensional cube* is a d-dimensional orthotope isometric to a *standard d-dimensional cube*, i.e., a standard orthotope of the form $[-a, a]^d$, $a \in (0, +\infty)$. Note that $C \subseteq \mathbb{R}^n$ is a d-dimensional cube if and only if C is mapped by a conformal affine map (i.e., a map in the form $\lambda \,\mathrm{id} \circ E$, where $E \in \operatorname{Isom}(\mathbb{R}^n)$ and $\lambda > 0$) to $\mathbb{I}^d := [-1, 1]^d$. A 0*-dimensional cube* is a set containing a single point.

For a standard orthotope $Q := \prod_{i=1}^d [-a_i, a_i]$, we call the linear transformation

$$\pi \colon \mathbb{R}^n \to \mathbb{R}^n, \quad (x_1, \dots, x_d, x_{d+1}, \dots, x_n) \mapsto (x_1/a_1, \dots, x_d/a_d, x_{d+1}, \dots, x_n) \tag{3.1}$$

the *cubic-stretching* of Q. Clearly $\pi(Q) = \mathbb{I}^d$.

The *orthotopic structure* of a standard orthotope $Q = \prod_{i=1}^d [-a_i, a_i]$ is the cell decomposition

$$\operatorname{Rec}_d(Q) := \prod_{i=1}^d \{[-a_i, a_i], \{-a_i\}, \{a_i\}\}.$$

For a d-dimensional orthotope $R \subseteq \mathbb{R}^n$, the orthotopic structure is given by the pullback (cf. Lemma 2.10) $\operatorname{Rec}_d(R) := T^*\operatorname{Rec}_d(Q)$, where $T \in \operatorname{Isom}(\mathbb{R}^n)$ maps R to a standard d-dimensional orthotope Q. Note that $\operatorname{Rec}_d(R)$ does not depend on the choice of T. Furthermore, we denote $\partial\operatorname{Rec}_d(R) := \operatorname{Rec}_d(R) \setminus \{R\} = \operatorname{Rec}_d(R)|_{\partial_\circ R}$. It is easy to see that $\partial\operatorname{Rec}_n(R)$ is a cell decomposition of $\partial_\circ R$. We call an element in $\operatorname{Rec}_d(R)$ a *face of R*, and call a face of dimension $d - 1$ a *facet.*

In particular, we define the *cubic structure* of $\mathbb{I}^d$ by $\mathrm{Cube}_d := \mathrm{Rec}_d\bigl(\mathbb{I}^d\bigr) = \{[-1,1], \{-1\}, \{1\}\}^d$, and the cubic structure of a d-dimensional cube C by $\mathrm{Cube}_d(C) := L^*\mathrm{Cube}_d$, where L is a conformal affine map that maps C to $\mathbb{I}^d$. We also denote $\partial\mathrm{Cube}_d(C) := \mathrm{Cube}_d(C) \smallsetminus \{C\}$ and $\partial\mathrm{Cube}_d := \partial\mathrm{Rec}_d\bigl(\mathbb{I}^d\bigr)$. Clearly, for a standard d-dimensional orthotope Q, the orthotopic structure $\mathrm{Rec}_d(Q)$ is identical to the pullback $\pi^*\mathrm{Cube}_d$ by the cubic stretching π of Q.

In addition, for a 0-dimensional orthotope (quivalently, 0-dimensional cube) P, we define $\mathrm{Rec}_0(P) = \mathrm{Cube}_0(P) := \{P\}$ and $\partial\mathrm{Rec}_0(P) = \partial\mathrm{Cube}_0(P) := \emptyset$.

In [Yu13], a crystallographic group Γ on $\mathbb{R}^n$ is called *cubic-type* if normal fundamental domains of Γ can be realized as n-dimensional cubes.

Here we introduce a similar notion called *orthotopic crystallographic group.* We call a crystallographic group Γ *orthotopic* if there exists an n-dimensional orthotope that is a normal fundamental domain of Γ. Examples of orthotopic crystallographic groups include $\Gamma_{\mathrm{tor}} \cong \mathbb{Z}^n$ and $\Gamma_{\mathrm{sph}} \cong \mathbb{Z}^n \rtimes \mathbb{Z}_2$ whose quotient spaces are $\mathbb{T}^n$ and $\mathbb{S}^n$, respectively.

With orthotopic crystallographic groups introduced, we define a special class of Lattès maps. Recall the definition of *Lattès triples* and *Lattès maps* in Definition 2.1.

Definition 3.2 (Orthotopic Lattès maps). A Lattès triple (Γ, h, A) is *orthotopic* if Γ is an orthotopic crystallographic group with a normal fundamental domain $Q = \prod_{i=1}^n [0, a_i]$, and $A = \lambda I$ where $\lambda \in \mathbb{N}$. An *orthotopic Lattès map* is a Lattès map with respect to an orthotopic Lattès triple.

For examples of orthotopic Lattès maps, see e.g. Astola, Kangaslampi, and Peltonen [AKP10].

3.2. **Symmetry group and symmetric decomposition of cube.** As in the previous subsection, we fix an arbitrary $n \in \mathbb{N}$ and use the convention that all discussions take place in the ambient space $\mathbb{R}^n$. We also identify $\mathbb{R}^d$, $0 \leqslant d \leqslant n$, with the subspace $\mathbb{R}^d \times \{0\}^{n-d}$ of $\mathbb{R}^n$.

We study an orthotopic crystallographic group $\Gamma < \mathrm{Isom}(\mathbb{R}^n)$ by decomposing its fundamental domain into some smaller polyhedra. We call such a decomposition a *symmetric decomposition.* Then in Subsection 3.3, we use symmetric decomposition to construct cellular Markov partitions for orthotopic Lattès maps.

First, we consider fundamental domains of cubic type. Then we naturally proceed to the orthotopic case, extending the results for cubic-type cases.

In preparation, we recall the notion of (cubic) symmetry. For the d-dimensional cube $\mathbb{I}^d := [-1,1]^d \subseteq \mathbb{R}^d$ (where $\mathbb{R}^d$ is identified as the subspace $\mathbb{R}^d \times \{0\}^{n-d}$ of $\mathbb{R}^n$), a *symmetry of* $\mathbb{I}^d$ *(or* $\mathbb{I}^d$*-symmetry)* is an isometry $\gamma \in \mathrm{Isom}(\mathbb{R}^n)$ with $\gamma\bigl(\mathbb{I}^d\bigr) = \mathbb{I}^d$. All symmetries of $\mathbb{I}^d$ form a subgroup $\mathrm{Sym}\bigl(\mathbb{I}^d\bigr) < \mathrm{Isom}(\mathbb{R}^n)$ called the *symmetry group of* $\mathbb{I}^d$.

For each $i \in \{1, \dots, n\}$, denote by e_i the point in $\mathbb{R}^n$ for which the i-th coordinate is 1 and the others are 0. Usually we do not distinguish the points e_i, $i \in \{1, \dots, n\}$, and the canonical orthogonal base $e_1, \dots, e_n$.

Remark 3.3. As an elementary fact, for each $\gamma \in \mathrm{Sym}\bigl(\mathbb{I}^d\bigr)$, we have $\gamma\bigl(\mathbb{R}^d\bigr) = \mathbb{R}^d$, and the restriction $\gamma|_{\mathbb{R}^d}$ is an orthogonal linear map having the form $\gamma|_{\mathbb{R}^d} = DE$ where D, $E \in \mathrm{O}(d)$ satisfy $De_i \in \{-e_i, e_i\}$ for $1 \leqslant i \leqslant d$, and the restriction of E on the base $e_1, \dots, e_d$ is a permutation, that is, there exists $\sigma \in S_d$ such that $Ee_i = e_{\sigma(i)}$, $1 \leqslant i \leqslant d$.

In what follows, we introduce symmetric decompositions. Briefly speaking, we subdivide a cube using half-spaces; cf. [SV93, Section 5.3] for a similar discussion.

Fix arbitrary $d \in \mathbb{N}$ with $d \leqslant n$. For all i, $j \in \{1, \dots, d\}$ and a, $b \in \{-1, 1\}$, we denote

$$H_{ij}^d(a,b) := \bigl\{(x_1, \dots, x_d) \in \mathbb{I}^d : ax_i \geqslant bx_j\bigr\} \subseteq \mathbb{R}^d. \tag{3.2}$$

Note that $H_{ii}^d(a,b) = \bigl\{(x_1, \dots, x_d) \in \mathbb{I}^d : (a-b)x_i \geqslant 0\bigr\}$ and $H_{ij}^d(a,b) = H_{ji}^d(-b,-a)$. Set

$$\mathcal{H}^d := \bigl\{H_{ij}^d(a,b) : a,\, b \in \{-1,1\},\, 1 \leqslant i,\, j \leqslant d\bigr\}. \tag{3.3}$$

We call $\mathcal{H}_* \subseteq \mathcal{H}^d$ a *fundamental subset* of $\mathcal{H}^d$ if, for all $1 \leqslant i \leqslant j \leqslant d$,

$$\mathcal{H}_* \cap \bigl\{H^d_{ij}(1,-1),\, H^d_{ij}(-1,1)\bigr\} \neq \emptyset \quad \text{and} \quad \mathcal{H}_* \cap \bigl\{H^d_{ij}(1,1),\, H^d_{ij}(-1,-1)\bigr\} \neq \emptyset. \tag{3.4}$$

Denote

$$\mathcal{K}^\circ_d \coloneqq \bigl\{\bigcap \mathcal{H}_* : \mathcal{H}_* \text{ is a fundamental subset of } \mathcal{H}^d\bigr\}. \tag{3.5}$$

For each d-dimensional cube $C \subseteq \mathbb{R}^n$ we denote $\mathcal{K}^\circ_d(C) \coloneqq L^*\mathcal{K}^\circ_d \coloneqq \bigl\{L^{-1}(H) : H \in \mathcal{K}^\circ_d\bigr\}$, where L is a conformal affine map with $L(C) = \mathbb{I}^d$. In addition, we denote $\mathcal{K}^\circ_0 \coloneqq \{\{0\}\}$.

Remark. Note that $\mathcal{K}^\circ_d(C)$ is well defined, i.e., independent of the choice of L (see Lemma 3.4 (i)).

Now we define symmetric decompositions of cubes. For a d-dimensional cube C with $d \in \mathbb{N}$, we call

$$\mathcal{K}_d(C) \coloneqq \bigcup_{c \in \mathrm{Cube}_d(C)} \mathcal{K}^\circ_{\dim(c)}(c) \quad \text{and} \quad \partial\mathcal{K}_d(C) \coloneqq \bigcup_{c \in \partial\mathrm{Cube}_d(C)} \mathcal{K}^\circ_{\dim(c)}(c) \tag{3.6}$$

the *symmetric decomposition* of C and $\partial_\circ C$, respectively. In particular, we denote $\mathcal{K}_d \coloneqq \mathcal{K}_d\bigl(\mathbb{I}^d\bigr)$ and $\partial\mathcal{K}_d \coloneqq \partial\mathcal{K}_d\bigl(\mathbb{I}^d\bigr)$ for convenience. Moreover, define $\mathcal{K}_0 \coloneqq \{\{0\}\}$ and $\partial\mathcal{K}_0 = \emptyset$.

Remark. Given a conformal affine map L mapping C to $\mathbb{I}^d$, we have $\mathcal{K}_d(C) = L^*\mathcal{K}_d \coloneqq \{L^{-1}(H) : H \in \mathcal{K}_d\}$ (see Lemma 3.4 (ii)).

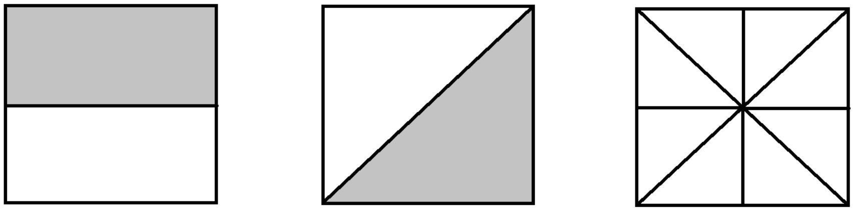

FIGURE 3.1. $H^2_{22}(1,-1)$, $H^2_{12}(1,1)$, and $\mathcal{K}_2$.

The following lemma guarantees that symmetric decompositions are well defined.

Lemma 3.4. *Let $d \in \mathbb{N}_0$ with $d \leqslant n$ be arbitrary. Then the following statements are true:*

(i) *For each $T \in \mathrm{Sym}\bigl(\mathbb{I}^d\bigr)$, $\mathcal{K}^\circ_d = T^*\mathcal{K}^\circ_d$. Moreover, for each d-dimensional cube C and each pair of conformal affine maps L, L' mapping C onto $\mathbb{I}^d$, we have $L^*\mathcal{K}^\circ_d = (L')^*\mathcal{K}^\circ_d = \mathcal{K}^\circ_d(C)$.*

(ii) *For each pair C, C' of d-dimensional cubes and each conformal affine map L mapping C to C', we have $\mathcal{K}^\circ_d(C) = L^*\mathcal{K}^\circ_d(C')$, $\mathcal{K}_d(C) = L^*\mathcal{K}_d(C')$, and $\partial\mathcal{K}_d(C) = L^*\partial\mathcal{K}_d(C')$.*

Proof. Clearly, the statements are true if $d = 0$. Now suppose $d \in \mathbb{N}$.

(i) Since T, $T^{-1} \in \mathrm{Sym}\bigl(\mathbb{I}^d\bigr)$, by Remark 3.3, we may assume $T^{-1}e_i = \delta_i e_{\sigma(i)}$, $\delta_i \in \{-1, 1\}$, for each $1 \leqslant i \leqslant d$, where $\sigma \in S_d$ is a permutation. Then for all i, $j \in \{1, \dots, d\}$ and a, $b \in \{-1, 1\}$,

$$T^{-1}\bigl(H^d_{ij}(a,b)\bigr) = \bigl\{(x_1, \dots, x_d) \in \mathbb{I}^d : a x_{\sigma(i)}/\delta_i \geqslant b x_{\sigma(j)}/\delta_j\bigr\} = H^d_{\sigma(i)\sigma(j)}(a/\delta_i, b/\delta_j),$$

which implies that for each fundamental subset $\mathcal{H}_* \subseteq \mathcal{H}^d$ (cf. (3.4) for definition), $T^*\mathcal{H}_* \coloneqq \bigl\{T^{-1}(H_*) : H_* \in \mathcal{H}_*\bigr\}$ is a fundamental subset. Thus, by the definition of $\mathcal{K}^\circ_d$ (cf. (3.5)), for each $H \in \mathcal{K}^\circ_d$, $T^{-1}(H) \in \mathcal{K}^\circ_d$. It follows that $T^*\mathcal{K}^\circ_d \subseteq \mathcal{K}^\circ_d$. Likewise, $\mathcal{K}^\circ_d \subseteq T^*\mathcal{K}^\circ_d$.

Let C be a d-dimensional cube and L, L' be conformal affine maps that map C to $\mathbb{I}^d$. It is easy to see that $L' \circ L^{-1} \in \mathrm{Sym}\bigl(\mathbb{I}^d\bigr)$, and thus, as shown above, $\bigl(L' \circ L^{-1}\bigr)^*\mathcal{K}^\circ_d = \mathcal{K}^\circ_d$. So $L^*\mathcal{K}^\circ_d = (L')^*\mathcal{K}^\circ_d = \mathcal{K}^\circ_d(C)$.

(ii) Let C, C' be d-dimensional cubes and L a conformal affine map with $L(C) = C'$. Let P and P' be conformal affine maps mapping C and C', respectively, to $\mathbb{I}^d$. It is clear that $P' \circ L \circ P^{-1} \in \operatorname{Sym}(\mathbb{I}^d)$. Thus, by (i), $\mathcal{K}_d^\circ(C) = P^*\mathcal{K}_d^\circ = P^*\bigl(P' \circ L \circ P^{-1}\bigr)^*\mathcal{K}_d^\circ = L^*(P')^*\mathcal{K}_d^\circ = L^*\mathcal{K}_d^\circ(C')$.

Now consider $\partial\mathcal{K}_d(C)$ and $\mathcal{K}_d(C)$. It is easy to verify $\operatorname{Cube}_d(C) = L^*\operatorname{Cube}_d(C')$ and $\partial\operatorname{Cube}_d(C) = L^*\partial\operatorname{Cube}_d(C')$. Combining this with the fact that $\mathcal{K}_d^\circ(C) = L^*\mathcal{K}_d^\circ(C')$ for arbitrary $d \in \mathbb{N}$ with $d \leqslant n$ (as shown above) and the definition of symmetric decomposition in (3.6), we obtain $\mathcal{K}_d(C) = L^*\mathcal{K}_d(C')$ and $\partial\mathcal{K}_d(C) = L^*\partial\mathcal{K}_d(C')$. $\square$

We further discuss the structure of $\mathcal{K}_n^\circ$, giving some elementary properties.

Lemma 3.5. *The following statements are true:*

(i) $\mathbb{I}^n = \bigcup\mathcal{K}_n^\circ$ *and* $H_0 = \bigcup\{\bigcap\mathcal{H}_0 : \mathcal{H}_0 \subseteq \mathcal{H}^n$ *is a fundamental subset with* $H_0 \in \mathcal{H}_0\}$ *for each* $H_0 \in \mathcal{H}^n$.

(ii) $H_1 \cap H_2 \in \mathcal{K}_n^\circ$ *for all* H_1, $H_2 \in \mathcal{K}_n^\circ$.

Proof. (i) Let $x := (x_1, \dots, x_n) \in \mathbb{I}^n$ be arbitrary. By (3.2), for all $1 \leqslant i \leqslant j \leqslant n$ and $\delta \in \{-1, 1\}$, we can choose a, $b \in \{-1, 1\}$ such that $ab = \delta$ and $x \in H_{ij}^n(a, b)$. This gives a fundamental subset (cf. (3.4) for definition) $\mathcal{H}_* \subseteq \mathcal{H}^n$ for which $x \in \bigcap\mathcal{H}_* \in \mathcal{K}_n^\circ$. Thus, $\mathbb{I}^n \subseteq \bigcup\mathcal{K}_n^\circ$. Clearly $\bigcup\mathcal{K}_n^\circ \subseteq \mathbb{I}^n$.

Furthermore, fix arbitrary $H_0 \in \mathcal{H}^n$ and $x_0 \in H_0$. As discussed above, there exists a fundamental subset $\mathcal{H}'_* \subseteq \mathcal{H}^n$ for which $x_0 \in \bigcap\mathcal{H}'_*$. Then $\mathcal{H}'_0 := \mathcal{H}'_* \cup \{H_0\}$ is a fundamental subset for which $x_0 \in \bigcap\mathcal{H}'_0$. It follows that $H_0 \subseteq \bigcup\{\bigcap\mathcal{H}_0 : \mathcal{H}_0 \subseteq \mathcal{H}^n$ is a fundamental subset with $H_0 \in \mathcal{H}'_*\}$. The converse inclusion is clear.

(ii) Consider H_1, $H_2 \in \mathcal{K}_n^\circ$. Suppose $H_1 = \bigcap\mathcal{H}_1$ and $H_2 = \bigcap\mathcal{H}_2$, where $\mathcal{H}_1$, $\mathcal{H}_2 \subseteq \mathcal{H}^n$ are fundamental subsets. By the definition of fundamental subsets (cf. (3.4)), $\mathcal{H}_1 \cup \mathcal{H}_2$ is a fundamental subset, and thus $H_1 \cap H_2 = \bigcap(\mathcal{H}_1 \cup \mathcal{H}_2) \in \mathcal{K}_n^\circ$. $\square$

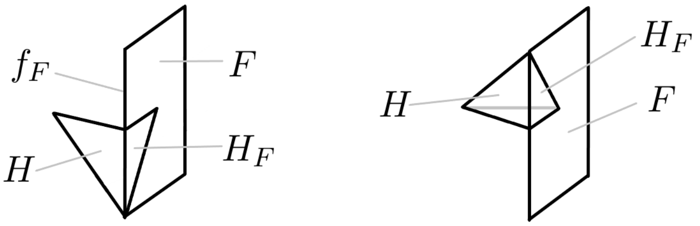

Figure 3.2. Illustration of Lemmas 3.6 (left) and 3.7 (right).

Lemma 3.6. *Fix* $H \in \mathcal{K}_n^\circ$. *Then the following statements are true:*

(i) *There exists* $T \in \operatorname{Sym}(\mathbb{I}^n)$ *such that* $T(H) \subseteq [0,1]^n$.

(ii) *If* $H \cap \partial_\circ\mathbb{I}^n \neq \emptyset$, *then there exists a facet* $F \in \operatorname{Cube}_n^{[n-1]}$ *such that* $H \cap \partial_\circ\mathbb{I}^n \subseteq F$.

(iii) *If* $F \in \operatorname{Cube}_n^{[n-1]}$ *satisfies* $H \cap F \neq \emptyset$, *then there exists* $H_F \in \mathcal{K}_{n-1}^\circ(F)$ *and* $f_F \in \operatorname{Cube}_{n-1}(F)$ *such that* $H \cap F = H_F \cap f_F$.

Proof. By the definition of $\mathcal{K}_n^\circ$ (cf. (3.5)), suppose $H = \bigcap\mathcal{H}_*$, where $\mathcal{H}_* \subseteq \mathcal{H}^n$ is a fundamental subset (cf. (3.4) for definition).

(i) By the definition of fundamental subsets, for each $i \in \{1, \dots, n\}$, pick $H_{ii}^n(a_i, -a_i) \in \mathcal{H}_*$, where $a_i \in \{-1, 1\}$. Let $T \in \mathrm{O}(n)$ be such that $Te_i = a_i e_i$, $i \in \{1, \dots, n\}$. It can be checked that $T \in \operatorname{Sym}(\mathbb{I}^n)$ and $T(H) \subseteq \bigcap_{i=1}^n T(H_{ii}^n(a_i, -a_i)) \subseteq [0,1]^n$.

(ii) Since $\mathcal{K}_n^\circ$ is invariant under $\mathbb{I}^n$-symmetries (Lemma 3.4 (i)), we may assume by (i) that $H \subseteq [0,1]^n$.

By the definition of fundamental subsets, for all $1 \leqslant i < j \leqslant n$, we can choose $H^n_{ij}(a_{ij}, a_{ij}) \in \mathcal{H}_*$, where $a_{ij} \in \{-1,\, 1\}$.

We construct a directed graph $G = (V, E)$ with n vertices $V = \{v_1,\, \dots,\, v_n\}$ by defining the set of edges $E = \{e_{ij} : 1 \leqslant i < j \leqslant n\}$ as follows: for $1 \leqslant i < j \leqslant n$, if $a_{ij} = 1$, then set $e_{ij} \coloneqq (v_i, v_j)$; otherwise, when $a_{ij} = -1$, set $e_{ij} \coloneqq (v_j, v_i)$. Using simple induction arguments, we can find an integer $1 \leqslant m \leqslant n$ such that for each $v_i \in V$, there exists a path from v_m to v_i. By the construction of G and (3.2), each $(x_1,\, \dots,\, x_n) \in \bigcap_{1\leqslant i<j\leqslant n} H^n_{ij}(a_{ij}, a_{ij})$ satisfies $x_m \geqslant x_i$, $1 \leqslant i \leqslant n$.

Let $x = (x_1,\, \dots,\, x_n) \in H \cap \partial_\circ \mathbb{I}^n$ be arbitrary. By the assumption $H \subseteq [0,1]^n$, there exists an integer $1 \leqslant k \leqslant n$ for which $x_k = 1$. Since $x \in H = \bigcap \mathcal{H}_* \subseteq \bigcap_{1\leqslant i<j\leqslant n} H^n_{ij}(a_{ij}, a_{ij})$, we have, as shown above, $x_m \geqslant x_k = 1$. Hence, $H \cap \partial_\circ \mathbb{I}^n$ is contained in the facet $\{(x_1,\, \dots,\, x_n) \in \mathbb{I}^n : x_m = 1\}$.

(iii) Since $\mathcal{K}^\circ_n$ is invariant under $\mathbb{I}^n$-symmetries (see Lemma 3.4 (i)), we may assume that H intersects the facet $F \coloneqq \{(x_1,\, \dots,\, x_n) \in \mathbb{I}^n : x_n = 1\}$. Consider the translation $L\colon (x_1,\, \dots,\, x_{n-1},\, 1) \mapsto (x_1,\, \dots,\, x_{n-1},\, 0)$ mapping F to $\mathbb{I}^{n-1}$.

Set $\mathcal{H}'_* \coloneqq \bigl\{H^{n-1}_{ij}(a,b) \in \mathcal{H}^{n-1} : H^n_{ij}(a,b) \in \mathcal{H}_*,\, a,\, b \in \{-1,\, 1\},\, 1 \leqslant i \leqslant j \leqslant n-1\bigr\}$ and $\mathcal{F} \coloneqq \bigl\{H^n_{ij}(a,b) \cap F : H^n_{ij}(a,b) \in \mathcal{H}_*,\, a,\, b \in \{-1,\, 1\},\, n \in \{i,\, j\}\bigr\}$. In what follows, we show $H_F \coloneqq L^{-1}(\bigcap \mathcal{H}'_*)$ and $f_F \coloneqq \bigcap \mathcal{F}$ satisfy the desired property.

Suppose $H^n_{ij}(a,b) \in \mathcal{H}_*$, where $1 \leqslant i \leqslant j \leqslant n$ and $a,\, b \in \{-1,\, 1\}$. Since $H \cap F = F \cap \bigcap \mathcal{H}_* \neq \emptyset$, we have $H^n_{ij}(a,b) \cap F \neq \emptyset$. If $i \neq n$ and $j \neq n$, then

$$H^n_{ij}(a,b) \cap F = \{(x_1,\, \dots,\, x_n) \in \mathbb{I}^n : ax_i \geqslant bx_j,\, x_n = 1\} = L^{-1}\bigl(H^{n-1}_{ij}(a,b)\bigr).$$

If $i = n$, then the non-empty set $H^n_{ij}(a,b) \cap F = \{(x_1,\, \dots,\, x_n) \in \mathbb{I}^n : a \geqslant bx_j,\, x_n = 1\}$ is either an $(n-2)$-dimensional face of F, or equal to F; the same holds for the case $j = n$.

The above discussion implies that $\mathcal{F}$ is a collection of faces of F and $\{H_* \cap F : H_* \in \mathcal{H}_*\} = \bigl\{L^{-1}(H'_*) : H'_* \in \mathcal{H}'_*\bigr\} \cup \mathcal{F}$. Thus, $H \cap F = \bigcap \mathcal{H}_* \cap F = L^{-1}(\bigcap \mathcal{H}'_*) \cap \bigcap \mathcal{F} = H_f \cap f_F$.

The construction of $\mathcal{H}'_*$ implies that $\mathcal{H}'_*$ is a fundamental subset of $\mathcal{H}^{n-1}$. Thus, $\bigcap \mathcal{H}'_* \in \mathcal{K}^\circ_{n-1}$ and $H_F = L^{-1}(\bigcap \mathcal{H}'_*) \in L^* \mathcal{K}^\circ_{n-1} = \mathcal{K}^\circ_{n-1}(F)$. On the other hand, the assumption $H \cap F \neq \emptyset$ implies that $f_F = \bigcap \mathcal{F}$ is non-empty, and hence, being the intersection of a collection of faces, is itself a face of F. □

Lemma 3.7. *For each facet $F \in \mathrm{Cube}^{[n-1]}_n$ and each $H_F \in \mathcal{K}^\circ_{n-1}(F)$, there exists $H \in \mathcal{K}^\circ_n$ for which $H_F = H \cap F = H \cap \partial_\circ \mathbb{I}^n$.*

Proof. Since $\mathcal{K}^\circ_n$ is invariant under $\mathbb{I}^n$-symmetries (see Lemma 3.4 (i)), we may assume $F \coloneqq \{(x_1,\, \dots,\, x_n) \in \mathbb{I}^n : x_n = 1\}$. Consider the translation $L\colon (x_1, \dots, x_{n-1}, 1) \mapsto (x_1, \dots, x_{n-1},\, 0)$.

Fix $H_F \in \mathcal{K}^\circ_{n-1}(F) = L^* \mathcal{K}^\circ_{n-1}$. By the definition of $\mathcal{K}^\circ_{n-1}$ (cf. (3.5)), suppose $H_F = L^{-1}(\bigcap \mathcal{H}_F)$, where $\mathcal{H}_F \subseteq \mathcal{H}^{n-1}$ is a fundamental subset of $\mathcal{H}^{n-1}$. It can be directly checked that

$$\mathcal{H}_* \coloneqq \{A \times [-1,1] : A \in \mathcal{H}_F\} \cup \{H^n_{in}(-1,-1),\, H^n_{in}(1,-1) : 1 \leqslant i \leqslant n\}$$

is a fundamental subset of $\mathcal{H}^n$ such that $H \coloneqq \bigcap \mathcal{H}_* \in \mathcal{K}^\circ_n$ satisfies $H_F = H \cap F = H \cap \partial_\circ \mathbb{I}^n$. □

The above two lemmas yield a description of the structure of $\partial \mathcal{K}_n$.

Lemma 3.8. *For each d-dimensional cube C, $d \in \mathbb{N}$ with $d \leqslant n$, we have*

$$\begin{aligned}\partial \mathcal{K}_d(C) =& \{H \cap \partial_\circ C : H \in \mathcal{K}^\circ_d(C),\, H \cap \partial_\circ C \neq \emptyset\} \\ =& \bigl\{H \cap F : H \in \mathcal{K}^\circ_d(C),\, F \in \mathrm{Cube}^{[d-1]}_d(C),\, H \cap F \neq \emptyset\bigr\} \\ =& \{H \cap f : H \in \mathcal{K}^\circ_d(C),\, f \in \partial \mathrm{Cube}_d(C),\, H \cap f \neq \emptyset\}.\end{aligned} \tag{3.7}$$

Proof. We argue by induction on the dimension n of the ambient space $\mathbb{R}^n$. Clearly, the lemma is true when $n = 1$. Now suppose $n \geqslant 2$ and make the induction hypothesis that the lemma holds when the dimension of the ambient space is in $\{1,\, \dots,\, n-1\}$, or equivalently, (3.7) holds for all cubes of dimension $1 \leqslant d \leqslant n-1$.

Now consider an arbitrary n-dimensional cube C. By mapping C to the standard n-dimensional cube $\mathbb{I}^n$ using a conformal affine map, it suffices to assume $C = \mathbb{I}^n$ and consider $\partial\mathcal{K}_n$.

First, by Lemma 3.6 (ii), we have

$$\begin{aligned}\{H \cap \partial_\circ \mathbb{I}^n : H \in \mathcal{K}_n^\circ,\, H \cap \partial_\circ \mathbb{I}^n \neq \emptyset\} \subseteq & \{H \cap F : H \in \mathcal{K}_n^\circ,\, F \in \mathrm{Cube}_n^{[n-1]},\, H \cap F \neq \emptyset\} \\ \subseteq & \{H \cap f : H \in \mathcal{K}_n^\circ,\, f \in \partial\mathrm{Cube}_n,\, H \cap f \neq \emptyset\}.\end{aligned}$$

Then we prove $\{H \cap f : H \in \mathcal{K}_n^\circ,\, f \in \partial\mathrm{Cube}_n,\, H \cap f \neq \emptyset\} \subseteq \partial\mathcal{K}_n$. Let $H \in \mathcal{K}_n^\circ$ and $f \in \partial\mathrm{Cube}_n$ satisfy $H \cap f \neq \emptyset$. Let F be a facet for which $f \subseteq F$. By Lemma 3.6 (iii), there exist $H_F \in \mathcal{K}_{n-1}^\circ(F)$ and $f_F \in \mathrm{Cube}_{n-1}(F)$ for which $H \cap F = H_F \cap f_F$. Thus, since $f \subseteq F$, we have $H \cap f = H_F \cap (f \cap f_F)$. Clearly $f \cap f_F \in \mathrm{Cube}_{n-1}(F)$. If $f \cap f_F = F$, then $H \cap f = H_F \in \mathcal{K}_{n-1}^\circ(F)$. Otherwise, when $f \cap f_F \neq F$, by the induction hypothesis, $H \cap f = H_F \cap (f \cap f_F) \in \partial\mathcal{K}_{n-1}(F)$. In both cases, $H \cap f \in \mathcal{K}_{n-1}(F) \subseteq \partial\mathcal{K}_n$.

Finally, we prove $\partial\mathcal{K}_n \subseteq \{H \cap \partial_\circ \mathbb{I}^n : H \in \mathcal{K}_n^\circ,\, H \cap \partial_\circ \mathbb{I}^n \neq \emptyset\}$.

Fix $H' \in \partial\mathcal{K}_n$. By the definition of $\partial\mathcal{K}_n$ (cf. (3.6)), suppose $H' \in \mathcal{K}_d^\circ(f)$ for some $0 \leqslant d \leqslant n-1$ and $f \in \mathrm{Cube}_n^{[d]}$. If $d = n-1$, then by Lemma 3.7, $H' \in \{H \cap \partial_\circ \mathbb{I}^n : H \in \mathcal{K}_n^\circ,\, H \cap \partial_\circ \mathbb{I}^n \neq \emptyset\}$.

Now suppose $d < n-1$. Let $F_i \in \mathrm{Cube}_n^{[n-1]}$, $i \in \mathcal{I}$, be all the facets of $\mathbb{I}^n$ containing f.

Fix $j \in \mathcal{I}$. Clearly $f \in \partial\mathrm{Cube}_{n-1}(F_j)$ and $H' \in \partial\mathcal{K}_{n-1}(F_j)$. By the induction hypothesis, we may choose $H_j' \in \mathcal{K}_{n-1}^\circ(F_j)$ with $H' = H_j' \cap \partial_\circ F_j$. By Lemma 3.7, there is $H_j \in \mathcal{K}_n^\circ$ such that $H_j' = H_j \cap F_j = H_j \cap \partial_\circ \mathbb{I}^n$. Thus, $H' = H_j \cap \partial_\circ F_j$.

It is easy to check $f = \bigcap_{i\in\mathcal{I}} F_i = \bigcap_{i\in\mathcal{I}} \partial_\circ F_i$. This, combined with the fact that $H_i \cap F_i = H_i \cap \partial_\circ \mathbb{I}^n$ and $H' = H_i \cap \partial_\circ F_i$ for each $i \in \mathcal{I}$, implies $H' = \bigcap_{i\in\mathcal{I}} H_i \cap \partial_\circ F_i = \bigcap_{i\in\mathcal{I}} H_i \cap F_i = \bigcap_{i\in\mathcal{I}} H_i \cap \partial_\circ \mathbb{I}^n$. Thus $H' \in \{H \cap \partial_\circ \mathbb{I}^n : H \in \mathcal{K}_n^\circ,\, H \cap \partial_\circ \mathbb{I}^n \neq \emptyset\}$ since $\bigcap_{i\in\mathcal{I}} H_i \in \mathcal{K}_n^\circ$ (see Lemma 3.5 (ii)). ☐

As a consequence of Lemma 3.8, $\mathcal{K}_n^\circ$ is the collection of cones whose bases are in $\partial\mathcal{K}_n$. Recall that for a point $x \in \mathbb{R}^n$ and a subset $Y \subseteq \mathbb{R}^n$, the cone with tip x and base Y is

$$\mathrm{Cone}(x, Y) \coloneqq \{x + \lambda(y - x) : y \in Y,\, \lambda \in [0,1]\}.$$

Corollary 3.9. *For each $d \in \mathbb{N}$ with $d \leqslant n$, we have $\mathcal{K}_d^\circ = \{\{0\}\} \cup \{\mathrm{Cone}(0, H') : H' \in \partial\mathcal{K}_d\}$. Moreover, each element in $\mathcal{K}_d$ is a simplex.*

Recall that a d-simplex ($d \leqslant n$) in $\mathbb{R}^n$ is the convex hull of $d+1$ affinely independent points.

Proof. It is easy to see from the definition of $\mathcal{K}_d^\circ$ (see (3.2) and (3.5)) that $\{0\} \in \mathcal{K}_n^\circ$ and for each $H \in \mathcal{K}_d^\circ \smallsetminus \{\{0\}\}$, $H = \mathrm{Cone}\big(0, H \cap \partial_\circ \mathbb{I}^d\big)$, and thus, by Lemma 3.8, $H = \mathrm{Cone}(0, H')$ for some $H' \in \partial\mathcal{K}_d$. Conversely, for each $H' \in \partial\mathcal{K}_d$, by Lemma 3.8, there exists $H \in \mathcal{K}_d^\circ$ such that $H' = H \cap \partial_\circ \mathbb{I}^d$, and thus, as shown above, $\mathrm{Cone}(0, H') = \mathrm{Cone}\big(0, H \cap \partial_\circ \mathbb{I}^d\big) = H$. So $\mathcal{K}_d^\circ = \{\mathrm{Cone}(0, H') : H' \in \partial\mathcal{K}_d\} \cup \{\{0\}\}$.

It follows immediately that for each d-dimensional cube C, elements in $\mathcal{K}_d^\circ(C)$ are either singletons, or cones with bases in $\partial\mathcal{K}_d(C)$. Thus, by (3.6) and since $\mathcal{K}_0 = \{\{0\}\}$, we can inductively conclude that each $H \in \mathcal{K}_d$ is a simplex. ☐

Since an element of $\mathcal{K}_n$ is a simplex, it is a topological cell. Moreover, we show that $\mathcal{K}_n$ is a cell decomposition of $\mathbb{I}^n$.

Proposition 3.10. *$\mathcal{K}_n$ is a cell decomposition of $\mathbb{I}^n$ that refines* Cube_n.

Proof. We prove the claim by induction. Clearly $\mathcal{K}_0 = \{\{0\}\}$ and $\mathcal{K}_1 = \{\{-1\}, \{0\}, \{1\}, [-1,0], [0,1]\}$ are cell decompositions refining Cube_0 and Cube_1, respectively. Fix $n \in \mathbb{N}$, $n \geqslant 2$, and make the induction hypothesis that for each $0 \leqslant d \leqslant n-1$, $\mathcal{K}_d$ is a cell decomposition that refines Cube_d (equivalently, for each d-dimensional cube C, $\mathcal{K}_d(C)$ is a cell decomposition of C that refines $\mathrm{Cube}_d(C)$).

First, by the induction hypothesis and the construction in (3.6), we can use Lemma 2.8 to conclude that $\partial\mathcal{K}_n$ is a cell decomposition that refines $\partial\mathrm{Cube}_n$.

Now, we verify conditions (i)–(iv) in Definition 2.3 for $\mathcal{K}_n$. Conditions (i) and (iv) hold since $\mathcal{K}_n$ is a finite cover of $\mathbb{I}^n$ (cf. Lemma 3.5 (i)). It remains to verify (ii) and (iii).

(ii) Let σ, $\tau \in \mathcal{K}_n$ be distinct. If σ, $\tau \in \partial\mathcal{K}_n$, then since $\partial\mathcal{K}_n$ is a cell decomposition, $\mathrm{int}_\circ(\sigma) \cap \mathrm{int}_\circ(\tau) = \emptyset$. In what follows, we consider the case where σ, $\tau \in \mathcal{K}_n^\circ$. Clearly, $\mathrm{int}_\circ \sigma \cap \mathrm{int}_\circ \tau = \emptyset$ if $\sigma = \{0\}$ or $\tau = \{0\}$. Now suppose $\sigma \neq \{0\}$ and $\tau \neq \{0\}$. Then by Corollary 3.9, there exist σ_1, $\tau_1 \in \partial\mathcal{K}_n$ such that $\sigma = \mathrm{Cone}(0, \sigma_1)$ and $\tau = \mathrm{Cone}(0, \tau_1)$. Since $\sigma \neq \tau$, we have $\sigma_1 \neq \tau_1$, and thus (since $\partial\mathcal{K}_n$ is a cell decomposition) $\mathrm{int}_\circ(\sigma_1) \cap \mathrm{int}_\circ(\tau_1) = \emptyset$. Then $\mathrm{int}_\circ \sigma = \{\lambda x : x \in \mathrm{int}_\circ \sigma_1,\ \lambda \in (0, 1)\}$ is disjoint from $\mathrm{int}_\circ \tau$. Finally, if $\sigma \in \mathcal{K}_n^\circ$ and $\tau \in \partial\mathcal{K}_n$, it is clear that $\mathrm{int}_\circ \sigma \subseteq \mathrm{int}_\circ \mathbb{I}^n$, and thus $\mathrm{int}_\circ(\sigma) \cap \mathrm{int}_\circ(\tau) = \emptyset$.

(iii) Fix $H \in \mathcal{K}_n$. If $H \in \partial\mathcal{K}_n$, then since $\partial\mathcal{K}_n$ is a cell decomposition, $\partial_\circ H$ is a union of cells in $\partial\mathcal{K}_n$. If $H = \{0\}$, then $\partial_\circ H = \emptyset$. Now suppose $H \in \mathcal{K}_n^\circ$, $H \neq \{0\}$. Then by Corollary 3.9, there exists $H_1 \in \partial\mathcal{K}_n$, for which $H = \mathrm{Cone}(0, H_1) = \{\lambda x : \lambda \in [0, 1],\ x \in H_1\}$. Since $\partial\mathcal{K}_n$ is a cell decomposition, $\partial_\circ H_1$ is a union of cells in $\partial\mathcal{K}_n$, and thus (again by Corollary 3.9) $\mathrm{Cone}(0, \partial_\circ H_1)$ is a union of cells in $\mathcal{K}_n^\circ$. It follows that $\partial_\circ H = H_1 \cup \mathrm{Cone}(0, \partial_\circ H_1)$ is a union of cells in $\mathcal{K}_n$.

Thus $\mathcal{K}_n$ is a cell decomposition. By definition $\mathcal{K}_n$ is a refinement of Cube_n. □

Next, we consider a self-similar subdivision of $\mathcal{K}_n$. Fix $l \in \mathbb{N}$. For each $\alpha := (\alpha_1, \dots, \alpha_n) \in \{0, \dots, l-1\}^n$, denote the n-dimensional cube centered at $x_\alpha := \big(\frac{2\alpha_1+1-l}{l}, \cdots, \frac{2\alpha_n+1-l}{l}\big)$ by

$$C_\alpha^n := [(2\alpha_1 - l)/l, (2\alpha_1 - l + 2)/l] \times \cdots \times [(2\alpha_n - l)/l, (2\alpha_n - l + 2)/l].$$

Using Lemma 2.8 and Proposition 3.10, we construct a cell decomposition as follows:

$$\mathcal{K}_{n,l} := \bigcup_{\alpha \in \{0, \dots, l-1\}^n} \mathcal{K}_n(C_\alpha^n). \tag{3.8}$$

For an n-dimensional cube C, we denote $\mathcal{K}_{n,l}(C) := L^*\mathcal{K}_{n,l}$, where L is a conformal affine map with $L(C) = \mathbb{I}^n$.

Remark 3.11. Note that $\mathcal{K}_{n,l}(C)$ does not depend on the choice of L since $\mathcal{K}_{n,l}$ is invariant under each $T \in \mathrm{Sym}(\mathbb{I}^n)$, which follows from Lemma 3.4 (ii) and the observation that for each $\alpha \in \{0, \dots, l-1\}^n$, there exists $\beta \in \{0, \dots, l-1\}^n$ for which $TC_\alpha^n = C_\beta^n$.

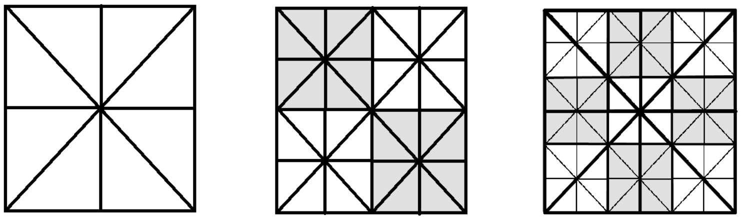

FIGURE 3.3. $\mathcal{K}_2$, $\mathcal{K}_{2,2}$, and $\mathcal{K}_{2,3}$.

Lemma 3.12. *For each $l \in \mathbb{N}$, the cell decomposition $\mathcal{K}_{n,l}$ is a refinement of $\mathcal{K}_n$.*

Proof. For each $\alpha := (\alpha_1, \dots, \alpha_n) \in \{0, \cdots, l-1\}^n$, denote $x_\alpha := \big(\frac{2\alpha_1+1-l}{l}, \cdots, \frac{2\alpha_n+1-l}{l}\big)$ and consider the conformal affine map $T_\alpha \colon x \mapsto T_\alpha x := l(x - x_\alpha)$ that maps C_α^n to $\mathbb{I}^n$ and x_α to 0.

We verify conditions (i) and (ii) in Definition 2.6.

(i) Fix $\alpha := (\alpha_1, \dots, \alpha_n) \in \{0, \cdots, l-1\}^n$. Then for all $1 \leqslant i, j \leqslant n$ and $a, b \in \{-1, 1\}$,

$$T_\alpha\big(H_{ij}^n(a, b)\big) = \{(y_1, \dots, y_n) \in T_\alpha(\mathbb{I}^n) : ay_i - by_j \geqslant 2(b\alpha_j - a\alpha_i) + (b - a)(1 - l)\}.$$

If $2(b\alpha_j - a\alpha_i) + (b - a)(1 - l) \leqslant 0$, then $H_{ij}^n(a, b) \subseteq T_\alpha\big(H_{ij}^n(a, b)\big)$; see Figure 3.4 for an illustration. Otherwise, since $2(b\alpha_j - a\alpha_i) + (b - a)(1 - l) \in 2\mathbb{Z}$, we have $H_{ij}^n(a, b) \subseteq [-1, 1]^n \subseteq T_\alpha\big(H_{ij}^n(-a, -b)\big)$.

By the above discussion, either $T_\alpha^{-1}\big(H_{ij}^n(a,b)\big) \subseteq H_{ij}^n(a,b)$ or $T_\alpha^{-1}\big(H_{ij}^n(a,b)\big) \subseteq H_{ij}^n(-a,-b)$. Thus, for each fundamental subset $\mathcal{H}_* \subseteq \mathcal{H}^n$, we can construct a fundamental subset $\mathcal{H}'_* \subseteq \mathcal{H}^n$ for which $T_\alpha^{-1}(\bigcap \mathcal{H}_*) \subseteq \bigcap \mathcal{H}'_*$. It follows that each element of $\mathcal{K}_n(C_\alpha^n) = T_\alpha^* \mathcal{K}_n$ is contained in some element of $\mathcal{K}_n$. Now condition (i) is verified.

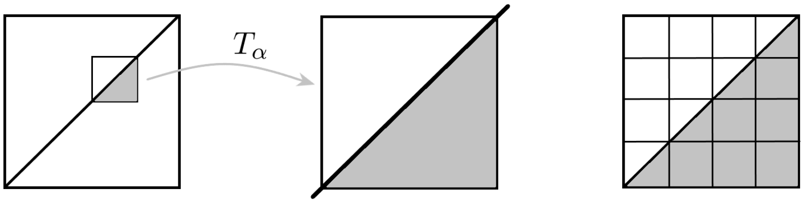

Figure 3.4

(ii) First, we show that each element in $\mathcal{H}^n$ is a union of cells in $\mathcal{K}_{n,l}$. Fix $H_0 \in \mathcal{H}^n$. It can be directly verified (see also Figure 3.4) that H_0 is covered by $\{C_\alpha^n : \alpha \in \{0, \dots, l-1\}^n,\ x_\alpha \in H_0\}$, and that for each $\alpha \in \{0, \dots, l-1\}^n$ with $x_\alpha \in H_0$, $H_0 \cap C_\alpha^n$ is either C_α^n or $T_\alpha^{-1}(H_0)$. Since it follows from Lemma 3.5 (i) that C_α^n and $T_\alpha^{-1}(H_0)$ are unions of cells in $\mathcal{K}_n(C_\alpha^n) = T_\alpha^*\mathcal{K}_n \subseteq \mathcal{K}_{n,l}$, H_0 is a union of cells in $\mathcal{K}_{n,l}$. Combining this with the definition of $\mathcal{K}_n^\circ$ (cf. (3.5)) and Lemma 2.4 (iv), we get that each element in $\mathcal{K}_n^\circ$ is a union of cells in $\mathcal{K}_{n,l}$.

It remains to show that each element in $\mathcal{K}_n$ is a union of cells in $\mathcal{K}_{n,l}$. Consider $H' \in \partial\mathcal{K}_n$. By Lemma 3.8, suppose $H' = H \cap F$ for some $H \in \mathcal{K}_n^\circ$ and some facet F of $\mathbb{I}^n$. Clearly F is a union of cells in $\mathcal{K}_{n,l}$ (which follows immediately from the definition of $\mathcal{K}_{n,l}$). Thus, since H is a union of cells in $\mathcal{K}_{n,l}$ as shown above, by Lemma 2.4 (iv), H' is a union of cells in $\mathcal{K}_{n,l}$. In conclusion, each element in $\mathcal{K}_n$ is a union of cells in $\mathcal{K}_{n,l}$. $\square$

Now we extend the discussion on symmetric decompositions to orthotopes.

For each standard n-dimensional orthotope Q, let π be the cubic-stretching (cf. (3.1)) of Q and define

$$\mathcal{K}_n(Q) \coloneqq \pi^*\mathcal{K}_n \quad \text{and} \quad \mathcal{K}_{n,l}(Q) \coloneqq \pi^*\mathcal{K}_{n,l}$$

for each $l \in \mathbb{N}$. Then $\mathcal{K}_n(Q)$ and $\mathcal{K}_{n,l}(Q)$ are cell decompositions which refine $\mathrm{Rec}_n(Q)$, and $\mathcal{K}_{n,l}(Q)$ refines $\mathcal{K}_n(Q)$. For each n-dimensional orthotope R, we define $\mathcal{K}_n(R) \coloneqq L^*\mathcal{K}_n(L(R))$ and $\mathcal{K}_{n,k}(R) \coloneqq L^*\mathcal{K}_{n,l}(L(R))$, where $L \in \mathrm{Isom}(\mathbb{R}^n)$ maps R to a standard orthotope $L(R)$.

Note that the definitions of $\mathcal{K}_n(R)$ and $\mathcal{K}_{n,l}(R)$ do not depend on the choice of L, and this follows from the observation that for each standard orthotope Q, $\mathcal{K}_n(Q)$ is invariant under Q-*symmetries*, i.e., $\gamma \in \mathrm{Isom}(\mathbb{R}^n)$ for which $\gamma(Q) = Q$.

Remark 3.13. Indeed, consider $Q \coloneqq \prod_{i=1}^n [-a_i, a_i]$ and a Q-symmetry $\gamma \in \mathrm{Isom}(\mathbb{R}^n)$. It can be checked that γ is also an $\mathbb{I}^n$-symmetry, and thus, by Remark 3.3, there exists a permutation $\sigma \in S_n$ such that $\gamma(e_i) \in \big\{-e_{\sigma(i)}, e_{\sigma(i)}\big\}$ for all $1 \leqslant i \leqslant n$. Moreover, $\gamma(Q) = Q$ guarantees that $a_i = a_{\sigma(i)}$ for all $1 \leqslant i \leqslant n$, which yields that $\pi \circ \gamma \circ \pi^{-1} \in \mathrm{Sym}(\mathbb{I}^n)$. Then since $\mathcal{K}_n$ is invariant under $\mathbb{I}^n$-symmetries (see Lemma 3.4 (i)), we have

$$\gamma^*\mathcal{K}_n(Q) = \gamma^*\pi^*\mathcal{K}_n = \pi^*\big(\pi \circ \gamma \circ \pi^{-1}\big)^*\mathcal{K}_n = \pi^*\mathcal{K}_n = \mathcal{K}_n(Q). \tag{3.9}$$

Likewise, $\gamma^*\mathcal{K}_{n,l}(Q) = \mathcal{K}_{n,l}(Q)$ since $\mathcal{K}_{n,l}$ is invariant under $\mathbb{I}^n$-symmetries (see Remark 3.11).

3.3. **Cellular Markov partitions of orthotopic Lattès maps.** Here we prove Theorem 3.1. We begin by recalling some basic results on crystallographic groups. For more details we refer the reader to [SV93, Chapter 2]. Fix a dimension $n \in \mathbb{N}$. Let $\Gamma < \mathrm{Isom}(\mathbb{R}^n)$ be a crystallographic group and P be a normal fundamental polyhedron of Γ. For each facet F of P, there exists a unique $\gamma_F \in \Gamma$ for which $\gamma_F(P) \cap P = F$.[2]

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[2] Such an element γ_F is called an *adjacency transformation* (cf. [SV93, Chapter 2]).

Let Γ be an orthotopic crystallographic group, and $Q \coloneqq \prod_{i=1}^{n}[0,a_i]$ be a normal fundamental domain. The above discussion implies that the cover $\{\gamma(Q) : \gamma \in \Gamma\}$ is given by a lattice, i.e.,

$$\{\gamma(Q) : \gamma \in \Gamma\} = \{\Sigma v + Q : v \in \mathbb{Z}^n\}, \text{ where } \Sigma \coloneqq \operatorname{diag}(a_1, \dots, a_n). \tag{3.10}$$

In what follows, we show that each $H \in \mathcal{K}_n(Q)$ is mapped by the projection $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ homeomorphically onto its image $\pi(H)$. For this, we need the following lemma.

Lemma 3.14. *For each $H \in \mathcal{K}_n$, each $x \in H$, and each $\gamma \in \operatorname{Sym}(\mathbb{I}^n)$, if $\gamma(x) \in H$, then $\gamma(x) = x$.*

Proof. Let $H \in \mathcal{K}_n$, $x \coloneqq (x_1, \dots, x_n) \in H$, and $\gamma \in \operatorname{Sym}(\mathbb{I}^n)$ satisfy $\gamma(x) \in H$.

Since, by Lemma 3.8, each element in $\mathcal{K}_n$ is contained in some element of $\mathcal{K}_n^\circ$, we may assume $H \in \mathcal{K}_n^\circ$. By the definition of $\mathcal{K}_n^\circ$ (cf. (3.5)), suppose $H = \bigcap \mathcal{H}_*$ for a fundamental subset $\mathcal{H}_* \subseteq \mathcal{H}^n$ (cf. (3.4)). By Lemma 3.6 (i), we may assume $H \subseteq [0,1]^n$.

By Remark 3.3, suppose $\gamma(e_i) = \delta_i e_{\tau(i)}$, $\delta_i \in \{-1, 1\}$, for each $1 \leqslant i \leqslant n$, where $\tau \in S_n$ is a permutation. Set $\tau = \sigma_1 \cdots \sigma_k$, where σ_i, $1 \leqslant i \leqslant k$, are disjoint cycles.

Since $x = \sum_{i=1}^{n} x_i e_i$ and $\gamma(x) = \sum_{i=1}^{n} \delta_i x_i e_{\tau(i)}$ are in $H \subseteq [0,1]^n$, we have $\gamma(x) = \sum_{i=1}^{n} x_i e_{\tau(i)}$. Thus, to show $\gamma(x) = x$, it suffices to show, for each $1 \leqslant j \leqslant k$, that $\sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i e_{\sigma_j(i)}$.

Let $\sigma \coloneqq (i_1 i_2 \dots i_l)$ be an arbitrary cycle among σ_i, $1 \leqslant i \leqslant k$. We enumerate $\bigl\{i'_1, \dots, i'_{n-l}\bigr\} \coloneqq \{1, \dots, n\} \smallsetminus \{i_1, \dots, i_l\}$. Then $x = \sum_{i=1}^{n} x_i e_i = \sum_{s=1}^{l} x_{i_s} e_{i_s} + \sum_{t=1}^{n-l} x_{i'_t} e_{i'_t}$. Since σ_i, $1 \leqslant i \leqslant k$, are disjoint cycles, we have $\bigl\{\tau(i'_1), \dots, \tau\bigl(i'_{n-l}\bigr)\bigr\} = \bigl\{i'_1, \dots, i'_{n-l}\bigr\}$ and

$$\gamma(x) = \sum_{i=1}^{n} x_i e_{\tau(i)} = \sum_{s=1}^{l} x_{i_s} e_{\sigma(i_s)} + \sum_{t=1}^{n-l} x_{i'_t} e_{\tau(i'_t)} = \sum_{s=1}^{l} x_{i_s} e_{i_{s+1}} + \sum_{t=1}^{n-l} x_{i'_t} e_{\tau(i'_t)}, \tag{3.11}$$

where we follow the convention that $i_{l+1} \coloneqq i_1$.

We show $x_{i_1} = x_{i_2} = \cdots = x_{i_l}$. We argue by contradiction and assume (without loss of generality) that $x_{i_1} \neq x_{i_2}$. First, assume $x_{i_1} > x_{i_2}$. Under such an assumption, suppose $x_{i_1} > x_{i_j}$ for some $2 \leqslant j \leqslant l$. By (3.11) and (3.2), $\gamma(x) \notin H^n_{i_2 i_{j+1}}(-1,-1)$. Then since $\gamma(x) \in H = \bigcap \mathcal{H}_*$, we have $H^n_{i_2 i_{j+1}}(-1,-1) \notin \mathcal{H}_*$, and thus, by the definition of fundamental subsets (cf. (3.4)), $H^n_{i_2 i_{j+1}}(1,1) \in \mathcal{H}_*$, which implies $x_{i_1} > x_{i_2} \geqslant x_{i_{j+1}}$. This, inductively, yields $x_{i_1} > x_{i_{l+1}} = x_{i_1}$, which is a contradiction. Similarly, $x_{i_1} < x_{i_2}$ leads to a contradiction.

Hence, $x_{i_1} = x_{i_2} = \cdots = x_{i_l}$ and $\sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i e_{\sigma(i)}$. Since σ is an arbitrary cycle among $\sigma_1, \dots, \sigma_k$, we have $\gamma(x) = x$. ☐

Corollary 3.15. *Let R be an orthotope of dimension $n \in \mathbb{N}$, and γ be a R-symmetry. If $H \in \mathcal{K}_n(R)$ and $x \in H$ satisfy $\gamma(x) \in H$, then $\gamma(x) = x$.*

Proof. Let $L \in \operatorname{Isom}(\mathbb{R}^n)$ maps R to a standard orthotope $Q \coloneqq L(R)$. Let π be the cubic stretching of Q. Then $H_1 \coloneqq \pi(L(H)) \in \mathcal{K}_n$ and $x_1 \coloneqq \pi(L(x)) \in H_1$ satisfy $\gamma_1(x_1) = \bigl(\pi \circ \bigl(L \circ \gamma \circ L^{-1}\bigr) \circ \pi^{-1}\bigr)(\pi(L(x))) = \pi(L(\gamma(x))) \in H_1$. Clearly $L \circ \gamma \circ L^{-1}$ is a Q-symmetry, and $\gamma_1 \coloneqq \pi \circ L \circ \gamma \circ L^{-1} \circ \pi^{-1} \in \operatorname{Sym}(\mathbb{I}^n)$ (cf. Remark 3.13). Then by Lemma 3.14, $\gamma_1(x_1) = x_1$, and thus $\gamma(x) = x$. ☐

Lemma 3.16. *Let Γ be an orthotopic crystallographic group and Q be an orthotope that is a normal fundamental domain of Γ. Then for each $H \in \mathcal{K}_n(Q)$ and each point $x \in H$, we have $H \cap \Gamma x = \{x\}$.*

Proof. Fix $H \in \mathcal{K}_n(Q)$ and $x \in H$. Let $\gamma \in \Gamma$ satisfy $\gamma(x) \in H \cap \Gamma x$. By the definition of normal fundamental domains, $\gamma(Q) \cap Q$ is a common face $c \in \operatorname{Rec}_n(Q) \cap \operatorname{Rec}_n(\gamma(Q))$. Let $c^* \in \operatorname{Rec}_n(Q)$ be such that $x \in c^*$ and $\gamma(c^*) = c$. Since γ is an isometry, we can extend $\gamma|_{c^*}$ to a Q-symmetry $\widetilde{\gamma}$. Then $\widetilde{\gamma}(x) = \gamma(x) \in H$. By Corollary 3.15, we have $\gamma(x) = \widetilde{\gamma}(x) = x$. ☐

Let (Γ, h, A) be an orthotopic Lattès triple on an oriented, connected, and closed Riemannian n-manifold $\mathcal{M}^n$, and $Q \coloneqq \prod_{i=1}^{n}[0,a_i]$ be a fundamental domain of Γ. Suppose $A = \lambda I$ for some $\lambda \in \mathbb{N}$. Now we show that $\mathcal{K}_n(Q)$ and $\mathcal{K}_{n,\lambda}(Q)$ induce, via h, cell decompositions of $\mathcal{M}^n$.

Proposition 3.17. *Let* (Γ, h, A) *be an orthotopic Lattès triple on an oriented, connected, and closed Riemannian* n*-manifold* $\mathcal{M}^n$*, where* Γ *has a fundamental domain* $Q \coloneqq \prod_{i=1}^n [0, a_i]$*, and* $A = \lambda I$ *for some* $\lambda \in \mathbb{N}$*. Then the restriction of* h *to each cell* c *in* $\mathcal{K}_n(Q) \cup \mathcal{K}_{n,\lambda}(Q)$ *is homeomorphic. Moreover,*

$$\mathcal{D}_0 \coloneqq \{h(c) : c \in \mathcal{K}_n(Q)\} \quad \text{and} \quad \mathcal{D}_1 \coloneqq \{h(c) : c \in \mathcal{K}_{n,\lambda}(Q)\}$$

are cell decompositions of $\mathcal{M}^n$*, and* $\mathcal{D}_1$ *is a refinement of* $\mathcal{D}_0$*.*

Proof. By Lemma 3.16, each $c \in \mathcal{K}_n(Q)$ contains at most one element in each orbit of Γ. Then $h|_c$ is injective. Since c is compact, $h|_c$ is also closed (since a continuous map between compact Hausdorff spaces is closed; cf. [Mu00, Section 26]), and thus a homeomorphism. Hence, $\mathcal{D}_0$ is indeed a collection of cells.

Since Γ is cocompact, we have $h(\mathbb{R}^n) = \mathcal{M}^n$ (which is a direct consequence of [Ka22, Theorems 1.5 and 1.7]). Since Q is a fundamental domain of Γ, we have $h(Q) = h(\mathbb{R}^n) = \mathcal{M}^n$. Then $\bigcup \mathcal{D}_0 = \mathcal{M}^n$, which verifies condition (i) in Definition 2.3. Conditions (iii) and (iv) in Definition 2.3 follow directly from the fact that $\mathcal{K}_n(Q)$ is a cell decomposition of finite cardinality.

It remains to verify condition (ii). Let $\sigma, \tau \in \mathcal{D}_0$ satisfy $\operatorname{int}_\circ(\sigma) \cap \operatorname{int}_\circ(\tau) \neq \emptyset$ and suppose $\sigma = h(\sigma_0)$, $\tau = h(\tau_0)$ for some $\sigma_0, \tau_0 \in \mathcal{K}_n(Q)$. Fix $p \in \operatorname{int}_\circ(\sigma) \cap \operatorname{int}_\circ(\tau)$, and let $x \in \operatorname{int}_\circ \sigma_0$, $y \in \operatorname{int}_\circ \tau_0$ be such that $h(x) = h(y) = p$. Since $\mathcal{K}_n(Q)$ is a refinement of $\operatorname{Rec}_n(Q)$, by Lemma 2.7 there exist $X, Y \in \operatorname{Rec}_n(Q)$ for which $\operatorname{int}_\circ \sigma_0 \subseteq \operatorname{int}_\circ X$ and $\operatorname{int}_\circ \tau_0 \subseteq \operatorname{int}_\circ Y$.

Since $h(x) = h(y) = p$, by Definition 2.1 there exists $\gamma \in \Gamma$ such that $\gamma(x) = y$. The choice of X, Y yields $\gamma(x) = y \in \operatorname{int}_\circ(\gamma(X)) \cap \operatorname{int}_\circ(Y)$. By the definition of normal fundamental domains, $\gamma(Q) \cap Q$ is a common face $Z \in \operatorname{Rec}_n(Q) \cap \operatorname{Rec}_n(\gamma(Q))$, which guarantees that $\operatorname{Rec}_n(Q) \cup \operatorname{Rec}_n(\gamma(Q))$ is a cell decomposition of $Q \cup \gamma(Q)$. Then $\gamma(X) = Y$ since $\operatorname{int}_\circ(\gamma(X)) \cap \operatorname{int}_\circ(Y) \neq \emptyset$. Extend $\gamma|_X$ to a Q-symmetry $\widetilde{\gamma}$. The invariance of $\mathcal{K}_n(Q)$ under Q-symmetries (cf. (3.9)) yields $\gamma(\sigma_0) = \widetilde{\gamma}(\sigma_0) \in \mathcal{K}_n(Q)$. Thus, since $y = \gamma(x) \in \operatorname{int}_\circ(\gamma(\sigma_0)) \cap \operatorname{int}_\circ(\tau_0) \neq \emptyset$, we have $\gamma(\sigma_0) = \tau_0$. Then since $h \circ \gamma = h$ (cf. Definition 2.1), $\sigma = h(\sigma_0) = h(\gamma(\sigma_0)) = h(\tau_0) = \tau$.

Therefore, $\mathcal{D}_0$ is a cell decomposition of $\mathcal{M}^n$. By the same arguments, $\mathcal{D}_1$ is also a cell decomposition. Since $\mathcal{K}_{n,\lambda}(Q)$ refines $\mathcal{K}_n(Q)$, $\mathcal{D}_1$ refines $\mathcal{D}_0$. ☐

Now we obtain a cellular Markov partition (cf. Definition 2.11) of the Lattès map induced by an orthotopic Lattès triple.

Proof of Theorem 3.1. Consider an orthotopic Lattès map $f \colon \mathcal{M}^n \to \mathcal{M}^n$ with respect to an orthotopic Lattès triple (Γ, h, A), where Γ has a fundamental domain $Q \coloneqq \prod_{i=1}^n [0, a_i]$, and $A = \lambda I$ for some $\lambda \in \mathbb{N}$. Set $\mathcal{D}_0 \coloneqq \{h(c) : c \in \mathcal{K}_n(Q)\}$ and $\mathcal{D}_1 \coloneqq \{h(c) : c \in \mathcal{K}_{n,\lambda}(Q)\}$.

By Proposition 3.17, D_0 and D_1 are cell decompositions of $\mathcal{M}^n$ for which D_1 refines D_0. It remains to show that f is $(\mathcal{D}_1, \mathcal{D}_0)$-cellular.

Let $c \in \mathcal{D}_1$ be arbitrary. Suppose $c = h(\sigma)$ for some $\sigma \in \mathcal{K}_{n,\lambda}$. By (3.10) and the construction of $\mathcal{K}_{n,\lambda}(Q)$ (cf. (3.8)), it is easy to see that $\tau \coloneqq A(\sigma) \in \mathcal{K}_n(\gamma(Q))$ for some $\gamma \in \Gamma$. Then since $h = h \circ \gamma^{-1}$ (cf. Definition 2.1) and by Proposition 3.17, $h|_\tau = \big(h|_{\gamma^{-1}(\tau)}\big) \circ \big(\gamma^{-1}|_\tau\big)$ is a homeomorphism for which $h(\tau) = h\big(\gamma^{-1}(\tau)\big) \in \mathcal{D}_0$. Hence, since $h|_\sigma : \sigma \to c$ is homeomorphic (see Proposition 3.17) and $f \circ h = h \circ A$ (cf. Definition 2.1), $f|_c = h|_\tau \circ A|_\sigma \circ (h|_c)^{-1}$ is a homeomorphism between $c \in \mathcal{D}_1$ and $f(c) = h(\tau) \in \mathcal{D}_0$. ☐

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