Numerical Methods for Minimizers and Microstructures in Nonlinear Elasticity

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Abstract

A standard finite element method and a finite element truncation method are applied to solve the boundary value problems of nonlinear elasticity with certain nonconvex stored energy functions such as those of St. Venant-Kirchhoff materials. Finite element solutions are proved to exist and to be in the form of minimizers in appropriate sets of admissible finite element functions for both methods. Convergence of the finite element solutions to a solution in the form of a minimizer or microstructure for the boundary value problem is established. it is also shown that in the presence of Lavrentiev phenomenon in the problem the finite element truncation method can overcome the difficulty and converges to the absolute minimum while the standard finite element method converges to a poseudominimum which is a minimum in a slightly small set of admissible functions.

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Key words : numerical method, minimizer, microstructure, weak convergence, strong convergence, nonlinear elasticity

1 Introduction

In this paper, we consider numerical methods for the boundary value problems of nonlinear elasticity, or in mathematical terms the problem of minimizing the functional

$$I(u) = \int_\Omega W(x, F)\, dx - \int_\Omega f \cdot u\, dx - \int_{\partial\Omega_1} g \cdot u\, ds \qquad (1.1)$$

in the set of admissible functions

$$A = \{u \in W^{1,p}(\Omega; R^3) : adj\, F \in L^q(\Omega; M^3),\ \det F \in L^r(\Omega),$$
$$\det F > 0, a.e.\ \text{in}\ \Omega,\ \text{and}\ u = u_0\ \text{on}\ \partial\Omega_0\}, \qquad (1.2)$$

where $\Omega \subset R^3$ is a connected open set with Lipschitz continuous boundary $\partial\Omega$, $\partial\Omega_0 \subset \partial\Omega$, $\partial\Omega_1 \subset \partial\Omega$ with $\partial\Omega_0 \cap \partial\Omega_1 = \emptyset$ and $area(\partial\Omega_0) \neq 0$, $F = I + Du$ with Du being the displacement gradient and I being the 3×3 identity matrix, $adj\, F$ is the transpose of the matrix of the cofactors of F and $\det F$ is the determinant of F, f is the body force and g is the traction.

There is no general existence theorem for the problem. However, if the stored energy function $W(x, F)$ satisfies polyconvex conditions and certain growth and coerceiveness conditions, the minimum of $I(\cdot)$ in A can be proved to be obtained (see [1] [2]) for any f and g satisfying the hypothesis

(c) $\Phi(u) = \int_\Omega f \cdot u\, dx + \int_{\partial\Omega_1} g \cdot u\, ds$ is continuous in $W^{1,p}(\Omega; R^3)$.

The numerical methods for the corresponding problems have also been studied [3] [4] [5] [6].

In the case when $W(x, F)$ is not polyconvex, such as the case for St. Venant-Kirchhoff materials [7] where

$$W(x, F) \equiv W^*(E) = \frac{\lambda}{2}(tr\, E)^2 + \mu tr\, E^2 \qquad (1.3)$$

with $E = \frac{1}{2}(F^T F - I)$ being the strain tensor and $\lambda > 0$, $\mu > 0$ being the Lamé parameters, the problem is more complicated. In the present paper, the following hypotheses recently introduced by Li [8] for stored energy functions $W(x, F)$ are considered.

(H1) There is a continuous function $G : \bar\Omega \times M^3 \times M^3 \times M^3 \times R \to R \cup \{+\infty\}$ such that $G(x, \cdot, \cdot, \cdot)$ is strictly convex and

$$W(x, F) = G(x, E, F, adj\, F, det\, F), \quad \text{for all}\ x \in \bar\Omega\ \text{and}\ F \in M_+^3,$$

where $M^3 = \{\text{all}\ 3 \times 3\ \text{matrices}\}$, $M_+^3 = \{F \in M^3 : det\, F > 0\}$, $E = \frac{1}{2}(F^T F - I)$.

(H2) There are constants $C_0 \in R$, $C_1 > 0$, $s > 1$, $p \geq 2\, s$, $q \geq \frac{p}{p-1}$, $r > 1$ such that

$$G(x, E, F, H, \delta) \geq C_0 + C_1(|E|^s + |F|^p + |H|^q + |\delta|^r),$$

for all $(x, E, F, H, \delta) \in \bar\Omega \times M^3 \times M^3 \times M^3 \times R_+$.

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(H3) $G(x, E, F, H, \delta) = +\infty$ if $\delta \le 0$ and is finite elesewhere,

$$G(x, E_n, F_n, H_n, \delta_n) \longrightarrow +\infty,$$

if $|E_n| + |F_n| + |H_n| + |\delta_n| \to +\infty$ or $\delta_n \to 0^+$.

Li proved the following result.

Theorem 1.1 *[8]. Let $W(x, F)$ satisfy the hypotheses (H1)-(H3). Let f, g satisfy the hypothesis (c). Let $\{u_j\}$ be a minimizing sequence of $I(\cdot)$ in A. Suppose that $\inf_{v \in A} I(v) < +\infty$.*

Then, (i): There exist a subsequence $\{u_\nu\}$ of $\{u_j\}$, function $u \in A$ and function $\hat{E} \in L^s(\Omega; R^3)$ such that

$$
\begin{align}
u_\nu \quad &\rightharpoonup \quad u \qquad in \ W^{1,p}(\Omega; R^3), \tag{1.4}\\
adj\ F_\nu \quad &\rightharpoonup \quad adj\ F \quad in \ L^q(\Omega; M^3), \tag{1.5}\\
\det F_\nu \quad &\rightharpoonup \quad \det F \quad in \ L^r(\Omega), \tag{1.6}\\
E_\nu \quad &\rightharpoonup \quad \hat{E} \qquad in \ L^s(\Omega; M^3), \tag{1.7}
\end{align}
$$

where $F_\nu = I + Du_\nu$, $E_\nu = \frac{1}{2}(F_\nu F_\nu - I)$, and

$$I(\hat{E}, u) \le \underline{\lim}_{\nu \to \infty} I(u_\nu), \tag{1.8}$$

where

$$I(\hat{E}, u) = \int_\Omega G(x, \hat{E}, F, adj\ F, \det F)\, dx - \int_\Omega f \cdot u\, dx - \int_{\partial\Omega_1} g \cdot u\, ds. \tag{1.9}$$

(ii): Either the weak convergence in (1.4) - (1.7) are in fact all strong convergence, in this case u is a minimizer of $I(\cdot)$ in A and $\hat{E} = \frac{1}{2}((I + Du)^T(I + Du) - I)$, or $\{u_\nu\}$ gives microtructure (see lemma 2.4 for the definition).

(iii): If the equality holds in (1.8) then

$$\hat{E} = \frac{1}{2}((I + Du)^T(I + Du) - I)$$

and u is a minimizer of $I(\cdot)$ in A. Otherwise, $\{u_\nu\}$ gives microtructure.

Remark 1.1. Theorem 1.1 is an extension of Ball's result. In fact, if $G(x, \cdot, \cdot, \cdot, \cdot)$ does not explicitly depend on E, then the theorem simply reduces to the existence theorem for polyconvex materials [8] [9].

Remark 1.2. The result covers a large amount of nonlinear elastic materials including many homogeneous, isotropic materials whose stored energy functions are of the form

$$W(x, F) = \frac{\lambda}{2}(tr\, E)^2 + \mu tr\, E^2 + o(|E|^2),$$

where $o(|E|^2)$ is such that (H1)–(H3) are satisfied by $W(x, F)$ [8]. The theory can also be easily adjusted, roughly speaking by removing the restriction $\det F > 0$ on the admissible functions and the hypothesis (H3) for $W(x, F)$, to cover St.Venant-Kirchhoff materials.

Our purpose is to show taht the minimizers and microstructures can be approximated by the minimizers of some finite problems obtained by applying certain finite element methods. First, a standard finite element method is applied to solve the problem. We will prove the existence of minimizers for the induced finite problem and discuss the convergence of the method in §2. Then, in §3, a finite element truncation method [11] is applied and the corresponding existence and convergence results are obtained. The results show that while the finite element truncation method can always produce a solution to the problem of minimizing $I(\cdot)$ in A, the standard finite element method may fail to do so, in fact it produces a solution to the problem of minimizing $I(\cdot)$ in $\hat{A}$, where $\hat{A}$ is in general a genuine subset of A (see §2). As a consequence, the standard finite element method typically fail both to find the absolute minimum and to produce a solution to the original problem in which Lavrentiev phenonmenon [12] [13] occurs. To actually find numerically the minimizers of the corresponding finite problem is by no means trivial and is out of the scope of this paper.

2   Existence and convergence of a standard finite element method

As in a standard application of the finite element method, we introduce regular triangulations T_h on Ω [10], where h is the mesh size. For simplicity, we assume that $\bar{\Omega} = \cup_{K \in T_h} K$, $\partial \Omega_0$ consists of the faces of T_h and $u(x) = 0$ on $\partial \Omega_0$. Define

$$\hat{A}_h = \{ \quad u \in C(\bar{\Omega}) : u \text{ is affine on each } K \in T_h;$$
$$u(x) = 0, \forall x \in \partial \Omega_0, \ \det(I + Du) > 0, \ a.e. \text{ in } \Omega\}. \quad (2.1)$$

We consider the finite problems of minimizing the functional $I(\cdot)$ defined by (1.1) with $W(x, F)$ satisfying the hypotheses (H1) - (H3) and

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f, g satisfying the hypothesis (c) (see §1) in the sets of admissible finite element functions $\hat{A}_h$.

Theorem 2.1 . *For each fixed $h > 0$, there exists a function $u_h \in \hat{A}_h$ such that*

$$I(u_h) = \inf_{v \in \hat{A}_h} I(v). \tag{2.2}$$

Proof. It is easily seen that $\hat{A}_h \neq \emptyset$ and $\inf_{v \in \hat{A}_h} I(v) < +\infty$. let $\{u_j\} \subset \hat{A}_h$ be a minimizing sequence of $I(\cdot)$ in $\hat{A}_h$. By (H2), (c) and the equivalence of the convergence in weak topology of $W^{1,p}(\Omega; R^3)$ and that in strong topology of $W^{1,\infty}(\Omega; R^3)$ in a finite dimensional function spaces [10], we conclude that there exist a subsequence of $\{u_j\}$, denoted again by $\{u_j\}$, and $u_h \in \{v \in C(\bar{\Omega}) : v|_K$ is affine $, v|_{\partial\Omega_0} = 0\}$ such that

$$
\begin{array}{rcll}
u_j & \rightarrow & u_h & \text{in } W^{1,\infty}(\Omega; R^3), & (2.3) \\
F_j & \rightarrow & F_h & \text{in } L^{\infty}(\Omega; M^3), & (2.4) \\
adj\, F_j & \rightarrow & adj\, F_h & \text{in } L^{\infty}(\Omega; M^3), & (2.5) \\
\det F_j & \rightarrow & \det F_h & \text{in } L^{\infty}(\Omega), & (2.6) \\
E_j & \rightarrow & E_h & \text{in } L^{\infty}(\Omega; M^3), & (2.7)
\end{array}
$$

where $F_j = I + Du_j$, $F_h = I + Du_h$, $E_j = \frac{1}{2}(F_j^T F_j - I)$ and $E_h = \frac{1}{2}(F_h^T F_h - I)$. By (H1) and (c), $I(E, u)$ defined by (1.9) is sequentially weakly lower semicontinuous [14]. Since $I(E_j, u_j) = I(u_j)$ for all j, we have

$$I(E_h, u_h) \leq \underline{\lim}_{j \to \infty} I(u_j) = \inf_{v \in \hat{A}_h} I(v) < +\infty.$$

This and (H3) imply that $\det F_h(x) > 0$ for almost all $x \in \Omega$ and hence $u_h \in \hat{A}_h \subset A$. Thus $I(u_h) = I(E_h, u_h)$ and (2.2) follows. $\qquad\square$

To discuss the convergence of the finite element solutions, we introduce a different set of admissible functions $\hat{A}$. Let

$$
\begin{aligned}
A(j) = \{u \quad \in \quad & W^{1,\infty}(\Omega; R^3) \cap A : |Du(x)| \leq j, \ a.e. \ \text{in } \Omega \\
& \text{and } \det(I + Du(x)) \geq \tfrac{1}{j}, \ a.e. \ \text{in } \Omega\}. \tag{2.8}
\end{aligned}
$$

Define $d : A \times A \to R$ by

$$d(u, v) = |u - v|_{1,p} + |I(u) - I(v)|. \tag{2.9}$$

Define

$$\hat{A} = \{u \in A : d(u, v_j) \to 0, \ \text{for a sequence } v_j \in A(j), j = 1, 2, \cdots\}. \tag{2.10}$$

It is known that $\hat{A}$ is in general a genuine subset of A.

Theorem 2.2 . *Let $h_j > 0, j = 1, 2, \cdots$, and $\lim_{j \to \infty} h_j = 0$. For each j, let u_j be a minimizer of $I(\cdot)$ in A_{h_j}. Then, $\{u_j\}$ is a minimizing sequence of $I(\cdot)$ in $\hat{A}$.*

Proof. By the definition of $\hat{A}$, it is sufficient to show that for any $M > 1$ and $u \in A(M)$,

$$\lim_{j \to \infty} |I(u) - I(u_{h_j})| = 0,$$

where u_{h_j} are the interpolations of u in $A_{h_j}, j = 1, 2, \cdots$. But this follows directly from the fact [10] that

$$\|u - u_{h_j}\|_{1,\infty} \to 0$$

and hence also

$$\|\det(I + Du) - \det(I + Du_{h_j})\|_\infty \to 0,$$

and the fact that the stored energy function $W(x, F)$ is continuous on $\bar{\Omega} \times M_+^3$ (see (H1)). □

Lemma 2.1 . *Let $\{u_j\}$ be a minimizing sequence of $I(\cdot)$ in $\hat{A}$. Then, there exist a subsequence of $\{u_j\}$, denoted again by $\{u_j\}$, function $u \in A$ and function $\hat{E} \in L^s(\Omega; M^3)$ such that*

$$
\begin{align}
u_j &\rightharpoonup u &&\text{in } W^{1,p}(\Omega; R^3), &&(2.11)\\
F_j &\rightharpoonup F &&\text{in } L^p(\Omega; M^3), &&(2.12)\\
adj\, F_j &\rightharpoonup adj\, F &&\text{in } L^q(\Omega; M^3), &&(2.13)\\
\det F_j &\rightharpoonup \det F &&\text{in } L^r(\Omega), &&(2.14)\\
E_j &\rightharpoonup \hat{E} &&\text{in } L^s(\Omega; M^3), &&(2.15)
\end{align}
$$

and

$$I(\hat{E}, u) \le \lim_{j \to \infty} I(u_j) = \inf_{v \in \hat{E}} I(v), \qquad (2.16)$$

where $I(\hat{E}, u)$ is defined by (1.9).

Proof. It follows from (H2), (c) and the sequentially weak continuity of Jacobians [1] [2] that (2.11) - (2.15) hold for a function $u \in W^{1,p}(\Omega; R^3)$ with $u = 0$ on $\partial\Omega_0$ and function $\hat{E} \in L^s(\Omega; M^3)$. (2.16) follows from the sequentially weakly lower semicontinuity of $I(\hat{E}, u)$ which is a consequence of (H1) and (c) and a standard lower semicontinuity theorem [14]. Now, (H3) and (2.16) imply that $\det F > 0$ for almost all $x \in \Omega$ and hence $u \in A$. □

Definition 2.1 . *A sequence of $\{u_j\} \subset \hat{A}$ is said to be a generalized solution to the problem of minimizing $I(\cdot)$ in $\hat{A}$, if it is a minimizing sequence of $I(\cdot)$ in $\hat{A}$, and (2.11) – (2.16) hold for some functions $u \in A$ and $\hat{E} \in L^s(\Omega; R^3)$.*

Thus, as a collorary of theorem 2.1, theorem 2.2 and lemma 2.1, we have

Theorem 2.3 . *By solving the finite problem of minimizing $I(\cdot)$ in $\hat{A}_h, h > 0$, we can obtain at least one generalized solution to the problem of minimizing $I(\cdot)$ in $\hat{A}$.*

Next, we are to see what can be expected from a generalized solution.

Lemma 2.2 . *Let $\{u_j\}$ be a generalized solution to the problem of minimizing $I(\cdot)$ in $\hat{A}$, and let $(u_j, E_j) \rightharpoonup (u, \hat{E})$ in $W^{1,p}(\Omega; R^3) \times L^s(\Omega; M^3)$. Suppose that*

$$I(u) = I(\hat{E}, u) = \lim_{j \to \infty} I(u_j) = \inf_{v \in \hat{A}} I(v). \qquad (2.17)$$

Then

$$
\begin{align}
\hat{E} &= \frac{1}{2}((I + Du)^T(I + Du) - I), & (2.18) \\
u_j &\to u & \text{in } W^{1,p}(\Omega; R^3), & (2.19) \\
E_j &\to \hat{E} = E \text{ in } L^s(\Omega; M^3), & (2.20)
\end{align}
$$

and u is a minimizer of $I(\cdot)$ in $\hat{A}$.

Proof. The result follows from the same arguments as in [8] (see (b) and (c) in the proof of theorem 2.1 in [8] for the arguments). □

Lemma 2.3 . *Let $\{u_j\}$ be a generalized solution to the problem of minimizing $I(\cdot)$ in $\hat{A}$, and let $(u_j, E_j) \rightharpoonup (u, \hat{E})$ in $W^{1,p}(\Omega; R^3) \times L^s(\Omega; M^3)$. Suppose that*

$$\hat{E} = \frac{1}{2}((I + Du)^T(I + Du) - I), \qquad (2.21)$$

and

$$I(u) = I(\hat{E}, u) < \lim_{j \to \infty} I(u_j) = \inf_{v \in \hat{A}} I(v). \qquad (2.22)$$

Then, Lavrentiev phenomenon [12] occurs in the problem, i.e.

$$\inf_{v \in A} I(v) < \inf_{v \in \hat{A}} I(v). \qquad (2.23)$$

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Proof. The result is straight forward. □

Lemma 2.4 . *Let $\{u_j\}$ be a generalized solution to the problem of minimizing $I(\cdot)$ in $\hat{A}$, and let $(u_j, E_j) \rightharpoonup (u, \hat{E})$ in $W^{1,p}(\Omega; R^3) \times L^s(\Omega; M^3)$. Suppose that*

$$\hat{E} \neq \frac{1}{2}((I + Du)^T(I + Du) - I), \qquad (2.24)$$

Then, $\{u_j\}$ gives microstructure, i.e. there exists a measurable subset $\hat{\Omega} \subset \Omega$ with $meas\,(\hat{\Omega}) > 0$ such that

$$\underline{\lim}_{j \to \infty} \int_{\Omega'} |Du_j - Du|\, dx \; > \; 0, \qquad (2.25)$$

$$\lim_{j \to \infty} \int_{\Omega'} (Du_j - Du)\, dx \; = \; 0, \qquad (2.26)$$

for any measurable subset $\Omega' \subset \hat{\Omega}$ with $meas\,(\Omega') > 0$.

Proof. Let

$$\hat{\Omega} = \{x \in \Omega : \hat{E} \neq \frac{1}{2}((I + Du)^T(I + Du) - I)\}.$$

By (2.24), $meas\,(\hat{\Omega}) > 0$.

(2.26) is obvious, since $u_j \rightharpoonup u$, in $W^{1,p}(\Omega; R^3)$.

Suppose that the lemma is not true. Then, there would exist a measurable subset $\Omega' \subset \hat{\Omega}$ with $meas\,(\Omega') > 0$, and a subsequence of $\{u_j\}$, denoted again by $\{u_j\}$, such that

$$\lim_{j \to \infty} \int_{\Omega'} |Du_j - Du|\, dx = 0.$$

By extracting a further subsequence, we would then be able to find a subsequence $\{u_\nu\}$ of $\{u_j\}$ satisfying

$$Du_\nu \to Du \quad a.e. \text{ in } \Omega',$$

and thus

$$E_\nu \to \frac{1}{2}((I + Du)^T(I + Du) - I)\ a.e. \text{ in } \Omega'.$$

This would imply

$$\hat{E} = \frac{1}{2}((I + Du)^T(I + Du) - I)\ a.e. \text{ in } \Omega',$$

since $E_\nu \rightharpoonup \hat{E}$ in $L^s(\Omega; M^3)$ and hence also in $L^s(\Omega'; M^3)$. But this contradicts the assumption that $\Omega' \subset \hat{\Omega}$. □

Theorem 2.4 . *Let $\{u_j\}$ be a generalized solution to the problem of minimizing $I(\cdot)$ in $\hat{A}$, and let $(u_j, E_j) \rightharpoonup (u, \hat{E})$ in $W^{1,p}(\Omega; R^3) \times L^s(\Omega; M^3)$. Then, there are three possibilities*

(i):

$$I(\hat{E}, u) = \lim_{j \to \infty} I(u_j) = \inf_{v \in \hat{A}} I(v).$$

In this case,

$$
\begin{aligned}
\hat{E} &= \tfrac{1}{2}((I + Du)^T(I + Du) - I), \\
u_j &\to u \quad \text{in } W^{1,p}(\Omega; R^3), \\
E_j &\to \hat{E} = E \text{ in } L^s(\Omega; M^3),
\end{aligned}
$$

and u is a minimizer of $I(\cdot)$ in $\hat{A}$.

(ii):

$$I(\hat{E}, u) < \lim_{j \to \infty} I(u_j) = \inf_{v \in \hat{A}} I(v),$$

and

$$\hat{E} = \frac{1}{2}((I + Du)^T(I + Du) - I).$$

In this case, Lavrentiev phenomenon occurs, i.e.

$$\inf_{v \in A} I(v) < \inf_{v \in \hat{A}} I(v).$$

(iii):

$$\hat{E} \neq \frac{1}{2}((I + Du)^T(I + Du) - I)$$

In this case, $\{u_j\}$ gives microstructure.

Proof. It is easily seen that (i) – (iii) include all the possibilities. The conclusions in (i), (ii) and (iii) follows from directly from lemma 2.2, lemma 2.3 and lemma 2.4 respectively. □

The results in this section show that a standard application of finite element methods can be successful in producing regular solutions, but it may fail in general to produce a satisfactory result when a solution is singular.

3   Existence and convergence of a finite element truncation method

In this section, we apply a finite element truncation method, which was designed to compute singular minimizers [11], to solve the problem of minimizing $I(\cdot)$ in A.

Let T_h be regular triangulations of Ω [10]. As in §2, for simplicity, assume that $\bar{\Omega} = \cup_{K \in T_h} K$, $\partial \Omega_0$ consists of faces of T_h and $u(x) = 0$ on $\partial \Omega_0$. Let

$$A_h = \{u \in C(\bar{\Omega}) \ : \ u \text{ is affine on each } K \in T_h,$$
$$u(x) = 0, \ \forall x \in \partial \Omega_0\}. \tag{3.1}$$

Let $\{\alpha_M\}$ be an increasing sequence of positive numbers satisfying

$$\lim_{M \to \infty} \alpha_M = +\infty. \tag{3.2}$$

Define

$$W_M(x, F) = \min\{W(x, F), \alpha_M(1 + |F|^p)\}, \ \forall F \in M^3. \tag{3.3}$$

We consider the finite problem of minimizing the functional

$$I_M(u) = \int_\Omega W_M(x, F)\,dx - \int_\Omega f \cdot u\,dx - \int_{\partial \Omega_1} g \cdot u\,ds, \tag{3.4}$$

with $W(x, F)$ satisfying hypotheses (H1) – (H3) and f, g satisfying hypothesis (c) (see §1), in the set of admissible finite element functions A_h. We should notice that A_h here is free from the restriction that $\det(I + Du(x)) > 0$ for almost all $x \in \Omega$. This makes the problem much easier to tackle.

Theorem 3.1 . *For each fixed $M \geq 1$ and $h > 0$, there exists a $u_h \in A_h$ such that*

$$I_M(u_h) = \inf_{v \in A_h} I_M(v). \tag{3.5}$$

Proof. It is easily seen from (H1) and (3.3) that $W_M(x, F)$ is continuous on A_h for each $M \geq 1$ and all $h > 0$. Let $\{u_j\}$ be a minimizing sequence of $I_M(\cdot)$ in A_h. By the boundedness of $\|u_j\|_{1,p}$, which follows from (3.3) and (H2), the finite dimensionality of A_h and continuity of $I_M(\cdot)$, which follows from the continuity of $W_M(x, F)$ and (c), we conclude that there exist a subsequence of $\{u_j\}$, denoted again by $\{u_j\}$, and function $u_h \in A_h$ such that

$$u_j \to u_h, \quad \text{in } W^{1,p}(\Omega; R^3), \tag{3.6}$$

$$I_M(u_h) = \lim_{j \to \infty} I_M(u_j) = \inf_{v \in A_h} I_M(v). \tag{3.7}$$

$\square$

Theorem 3.2 . *There exists a function $h(\cdot, \cdot) : R_+ \times R_+ \to R_+$,which is nondecreasing with respect to the first variable and nonincreasing with respect to the second variable, such that for any $\epsilon > 0$ and $M \geq 1$*

$$\inf_{v \in A_h} I_M(v) < \inf_{v \in A} I(v) + \epsilon,$$

provided that $0 < h < h(\epsilon, M)$.

Proof. Let $u \in A$ be such that

$$I(u) < \inf_{v \in A} I(v) + \frac{\epsilon}{2}. \tag{3.8}$$

Let $u_h \in A_h, h > 0$ be such that

$$\lim_{h \to 0} \|u_h - u\|_{1,p} = 0. \tag{3.9}$$

Consider

$$
\begin{aligned}
& I_M(u_h) - I(u) \\
= \ & \int_\Omega (W_M(x, F_h) - W_M(x, F)) \, dx + \int_\Omega (W_M(x, F) - W(x, F)) \, dx \\
& + \int_\Omega f \cdot (u - u_h) \, dx + \int_{\partial \Omega_1} g \cdot (u - u_h) \, ds \\
= \ & I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.10}
$$

By (c) and (3.9), there exists $h_1(\epsilon) > 0$ such that

$$|I_3 + I_4| < \frac{\epsilon}{4}, \quad \forall h \in (0, h_1(\epsilon)). \tag{3.11}$$

It follows from (3.3) that

$$I_2 < 0. \tag{3.12}$$

We claim that for any $\epsilon > 0$ and $M \geq 1$ there exists $h_2(\epsilon, M) > 0$ such that

$$|I_1| < \frac{\epsilon}{4}, \quad \forall h \in (0, h_2(\epsilon, M)). \tag{3.13}$$

Suppose otherwise. Then, there would be $\epsilon_0 > 0$, $M_0 > 0$ and a decreasing sequence $\{h_j\}$ with $\lim_{j \to \infty} h_j = 0$ such that

$$|\int_\Omega (W_{M_0}(x, F_j) - W_{M_0}(x, F)) \, dx| \geq \frac{\epsilon_0}{4}, \quad \forall j.$$

By (3.9), we may assume without loss of generality that

$$
\begin{aligned}
u_j & \to u, & a.e. \text{ in } \Omega, \\
Du_j & \to Du, & a.e. \text{ in } \Omega.
\end{aligned}
$$

Thus, by the continuity of $W_{M_0}(\cdot, \cdot)$,

$$W_{M_0}(x, F_j) \to W_{M_0}(x, F), \quad a.e. \text{ in } \Omega. \tag{3.14}$$

On the other hand, by (3.3),

$$|W_{M_0}(x, F_j) - W_{M_0}(x, F)| \le \alpha_{M_0}(2 + |F_j|^p + |F|^p). \tag{3.15}$$

By (3.9), the right handside of (3.15) is uniformly integral continuous. Hence by (3.14) and (3.15)

$$\lim_{j \to \infty} \int_\Omega |W_{M_0}(x, F_j) - W_{M_0}(x, F)| \, dx = 0.$$

This is a contradiction.

Now, the theorem follows from (3.8) and (3.10) – (3.13) by taking $h(\epsilon, M) = \min\{h_1(\epsilon), h_2(\epsilon, M)\}$. □

To obtain the convergence result, we first introduce the following lemmas.

Lemma 3.1 . *Let $u_{M,h} \in A_h$ be such that*

$$I_M(u_{M,h}) \le \inf_{v \in A_h} I_M(v) + 1. \tag{3.16}$$

Let $\hat\Omega_{M,h} = \hat\Omega_M(u_{M,h})$ where

$$\hat\Omega_M(v) = \{x \in \Omega : W_M(x, F(x)) \ne W(x, F(x))\}, \tag{3.17}$$

where $F(x) = I + Dv(x)$.

Then, there exist constants $M_1 \ge 1$ and $C > 0$ such that

$$\|u_{M,h}\|_{1,p}^p + \|E_{M,h}\|_s^s \le C, \quad \forall M \ge M_1 \text{ and } h > 0, \tag{3.18}$$

and

$$\|u_{M,h}\|_{1,p,\Omega_{M,h}}^p + \|adj\, F_{M,h}\|_{q,\Omega_{M,h}}^q \quad +\|\det F_{M,h}\|_{r,\Omega_{M,h}}^r + \|E_{M,h}\|_{s,\Omega_{M,h}}^s$$
$$\le C, \quad \forall M \ge M_1 \text{ and } h > 0, \tag{3.19}$$

where $E_{M,h} = \frac{1}{2}(F_{M,h}^T F_{M,h} - I)$ and $\Omega_{M,h} = \Omega \setminus \hat\Omega_{M,h}$. Furthermore, let

$$\hat A_{M,h} = \{u \in A_h : I_M(u) \le \inf_{v \in A_h} I_M(v) + 1\}$$

and let

$$\delta_{M,h} = \sup_{v \in \hat A_{M,h}} meas\, \hat\Omega_M(v).$$

Then

$$\delta_{M,h} \to 0, \quad uniformly \text{ for } h > 0 \text{ as } M \to \infty. \tag{3.20}$$

Proof. By (H1) and (3.2), there exists $M_1 \geq 1$ such that

$$\max_{x \in \bar{\Omega}} W(x, I) \leq \alpha_M, \quad \forall M \geq M_1.$$

Thus, by (3.3)

$$W_M(x, I) = W(x, I), \quad \forall x \in \bar{\Omega} \text{ and } M \geq M_1. \tag{3.21}$$

Since $u_{M,h}$ satisfies (3.16), (3.21) implies that

$$I_M(u_{M,h}) \leq I_M(0) + 1 = I(0) + 1, \quad \forall M \geq M_1 \text{ and } h > 0. \tag{3.22}$$

By (H2) and (3.3), we have

$$W_M(x, F) \geq \hat{C}_0 + \hat{C}_1 |F|^p, \quad \forall x \in \bar{\Omega} \text{ and } F \in M^3, \tag{3.23}$$

for some constants $\hat{C}_0 \in R$ and $\hat{C}_1 > 0$.

(3.22), (3.23) together with (c) and Poincaré inequality [15] give

$$\|u_{M,h}\|_{1,p}^p \leq \hat{C}, \quad \forall M \geq M_1 \text{ and } h > 0, \tag{3.24}$$

where $\hat{C}$ is a constant. Since $2s \leq p$, (3.24) implies (3.18).

It follows from (H2), (3.3), (3.22) and the positiveness of $W_M(x, F_{M,h})$ on $\hat{\Omega}_{M,h}$ that

$$\int_{\Omega_{M,h}} [C_0 + C_1(|F_{M,h}|^p + |adj\, F_{M,h}|^q + |\det F_{M,h}|^r + |E_{M,h}|^s)]\, dx$$

$$\leq \int_{\Omega_{M,h}} W(x, F_{M,h})\, dx = \int_{\Omega_{M,h}} W_M(x, F_{M,h})\, dx$$

$$\leq I_M(u_{M,h}) + \int_\Omega f \cdot u_{M,h}\, dx + \int_{\partial\Omega_1} g \cdot u_{M,h}\, ds$$

$$\leq I(0) + 1 + \int_\Omega f \cdot u_{M,h}\, dx + \int_{\partial\Omega_1} g \cdot u_{M,h}\, ds,$$
$$\forall M \geq M_1 \text{ and } h > 0.$$

Thus, by (c) and (3.24), we have (3.19).

For any $u_{M,h} \in A_h$ satisfying (3.16), it follows from (3.3), (3.17) and (3.22) that

$$\alpha_M meas\, \hat{\Omega}_{M,h} \leq \int_{\hat{\Omega}_{M,h}} W_M(x, F_{M,h})\, dx$$

$$= I_M(u_{M,h}) - \int_{\Omega_{M,h}} W_M(x, F_{M,h})\, dx +$$

$$+ \int_\Omega f \cdot u_{M,h}\, dx + \int_{\partial\Omega_1} g \cdot u_{M,h}\, ds$$

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$$\leq I(0) + 1 - \int_{\Omega_{M,h}} W(x, F_{M,h})\, dx +$$

$$+ \int_{\Omega} f \cdot u_{M,h}\, dx + \int_{\partial\Omega_1} g \cdot u_{M,h}\, ds,$$
$$\forall M \geq M_1 \text{ and } h > 0.$$

Thus, (3.20) follows from (H2), (c), (3.18), (3.19) and (3.2). $\qquad\square$

Lemma 3.2 . *Let $\{M_j\}$ be such that*

$$\lim_{j\to\infty} M_j = +\infty, \tag{3.25}$$

$$\sum_{j=1}^{\infty} \delta_{M_j} < +\infty, \tag{3.26}$$

where $\delta_M = \sup_{h>0} \sup_{v\in\hat{A}_{M,h}} meas\, \hat{\Omega}_M(v)$ with

$$\hat{A}_{M,h} = \{u \in A_h : I_M(u) \leq \inf_{v\in A_h} I_M(v) + 1\}$$

and $\hat{\Omega}_M(v)$ being defined by (3.17). Let $h_j \in R_+$ and $u_j \in A_{h_j}$ be such that

$$I_{M_j}(u_j) \leq \inf_{v\in A_{h_j}} I_{M_j}(v) + 1. \tag{3.27}$$

Then, there exist a subsequence $\{u_\nu\}$ of $\{u_j\}$, decreasing measurable subsets $\Omega_l \subset \Omega$, functions $u \in W^{1,p}(\Omega; R^3)$ with $u = 0$ on $\partial\Omega_0$ and $\hat{E} \in L^s(\Omega; M^3)$ such that

$$\lim_{l\to\infty} meas\, \Omega_l = 0, \tag{3.28}$$

$$u_\nu \;\rightharpoonup\; u \quad in\ W^{1,p}(\Omega; R^3), \tag{3.29}$$

$$adj\, F_\nu \;\rightharpoonup\; adj\, F \ in\ L^q(\Omega \setminus \Omega_l; M^3),\ for\ each\ fixed\ l, \tag{3.30}$$

$$\det F_\nu \;\rightharpoonup\; \det F \ in\ L^r(\Omega \setminus \Omega_l),\ for\ each\ fixed\ l, \tag{3.31}$$

$$E_\nu \;\rightharpoonup\; \hat{E} \quad in\ L^s(\Omega; M^3), \tag{3.32}$$

Proof. Let

$$\Omega_l = \cup_{j=l}^{\infty} \hat{\Omega}_j,$$

where $\hat{\Omega}_j = \hat{\Omega}_{M_j}(u_j)$ is defined by (3.17). It is obvious by the definition and (3.26) that Ω_l are decreasing and satisfy (3.28).

By lemma 3.1, $(F_j, adj\, F_j, \det F_j, E_j)$ are bounded in $L^p(\Omega; M^3) \times L^q(\Omega \setminus \hat{\Omega}_j; M^3) \times L^r(\Omega \setminus \hat{\Omega}_j) \times L^s(\Omega; M^3)$ for each j and the bounds can

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be chosen to be independent of j, and hence they are uniformly bounded in $L^p(\Omega; M^3) \times L^q(\Omega \setminus \hat{\Omega}_l; M^3) \times L^r(\Omega \setminus \hat{\Omega}_l) \times L^s(\Omega; M^3)$ for all $j \geq l$.

Extracting a subsequence $\{u_\nu\}$ form $\{u_j\}$ by applying the diagonal process and making use of the weak continuity of Jacobians [1] [2], we obtain $(3.29) - (3.32)$. $\qquad\square$

Lemma 3.3 . *Let $\Psi : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ satisfy*

(i) *Ψ is an extended continuous function,*

(ii) *$\Psi(x, u, \cdot)$ is convex,*

(iii) *$\Psi(x, u, P) \geq C$, for some constant $C \in R$.*

Let $\Psi_M : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ satisfy

(1) *Ψ_M is an extended continuous function,*

(2) *$\Psi_M \to \Psi$ locally uniformly in $\Omega \times R^m \times R^n$, i.e. there exists a sequence of measurable subsets $\Omega_i \subset \Omega$ with $\lim_{i \to \infty} \text{meas}\,(\Omega \setminus \Omega_i) = 0$ such that for each fixed i and any compact subset $G \subset R^m \times R^n$*

$$\Psi_M(x, u, P) \to \Psi(x, u, P), \quad \text{uniformly on } \Omega_i \times G \text{ as } M \to +\infty,$$

(3) *$\Psi_M(x, u, P) \geq C$, for some constant $C \in R$.*

Let $1 \leq p_1 \leq +\infty$, $1 \leq p_2 \leq +\infty$, and let $\{u_M\}$, $u \in L^{p_1}(\Omega; R^m)$ and $\{P_M\}$, $P \in L^{p_2}(\Omega; R^n)$ be such that

$$
\begin{aligned}
u_M &\to u & \text{in } L^{p_1}(\Omega; R^m), \\
P_M &\rightharpoonup P & \text{in } L^{p_2}(\Omega; R^n).
\end{aligned}
$$

Then
$$\int_\Omega \Psi(x, u, P)\, dx \leq \underline{\lim}_{M \to \infty} \int_\Omega \Psi_M(x, u_M, P_M)\, dx.$$

Lemma 3.3 is a special case of a general lower semicontinuity theorem given by Li [16], where the theorem is proved under weaker hypotheses for Ψ and Ψ_M.

Theorem 3.3 . *Let $\epsilon_j > 0$ satisfy $\lim_{j \to \infty} \epsilon_j = 0$. Let M_j satisfy (3.25) and (3.26). Let $0 < h_j < h(\epsilon_j, M_j)$, where $h(\cdot, \cdot)$ is determined by theorem 3.2. Let $u_j \in A_{h_j}$ be such that*

$$I_{M_j}(u_j) \leq \inf_{v \in A_{h_j}} I_{M_j}(v) + \epsilon_j. \tag{3.33}$$

Then, there exist a subsequence $\{u_\nu\}$ of $\{u_j\}$, decreasing measurable subsets $\Omega_l \subset \Omega$, functions $u \in A$ and $\hat{E} \in L^s(\Omega; M^3)$ such that (3.28) – (3.32) hold and

$$I(\hat{E}, u) \leq \lim_{j \to \infty} I_{M_j}(u_j) = \inf_{v \in A} I(v), \qquad (3.34)$$

where $I(\hat{E}, u)$ is defined by (1.9).

Proof. It follows from lemma 3.2 that (3.28) – (3.32) hold for some decreasing measurable subsets $\Omega_l \subset \Omega$, functions $u \in W^{1,p}(\Omega; R^3)$ with $u = 0$ on $\partial\Omega_0$ and $\hat{E} \in L^s(\Omega; M^3)$.

It is easily seen that $W(x, P)$ and $W_M(x, P)$ satisfy (i) – (iii) and (1) – (3) in lemma 3.3 respectively. Thus by lemma 3.3, hypothesis (c) (see §1), we have

$$\int_{\Omega \setminus \Omega_l} W(x, \hat{E}, F) \, dx - \int_\Omega f \cdot u \, dx - \int_{\partial\Omega_1} g \cdot u \, ds$$
$$\leq \underline{\lim}_{\nu \to \infty} \left(\int_{\Omega \setminus \Omega_l} W_{M_\nu}(x, F_{M_\nu}) \, dx - \int_\Omega f \cdot u_{M_\nu} \, dx - \int_{\partial\Omega_1} g \cdot u_{M_\nu} \, ds \right)$$
$$= \underline{\lim}_{\nu \to \infty} \left(I_{M_\nu}(u_\nu) - \int_{\Omega_l} W_{M_\nu}(x, F_{M_\nu}) \, dx \right), \qquad (3.35)$$

where $W(x, \hat{E}, F) = G(x, \hat{E}, F, adj\, F, \det F)$ (see (H1)).

By (H2) and (3.3), there is a constant $\hat{C}_0 \geq 0$ such that

$$-W_{M_\nu}(x, F_\nu) \leq \hat{C}_0.$$

Thus, by theorem 3.2, (3.33) and (3.35), we have

$$\int_{\Omega \setminus \Omega_l} W(x, \hat{E}, F) \, dx - \int_\Omega f \cdot u \, dx - \int_{\partial\Omega_1} g \cdot u \, ds$$
$$\leq \inf_{v \in A} I(v) + \hat{C}_0 \, meas\, \Omega_l, \quad \forall l. \qquad (3.36)$$

This and (H3) imply that

$$\det F > 0 \quad a.e. \text{ in } \Omega \setminus \Omega_l, \quad \forall l,$$

and thus, by (3.28)

$$\det F > 0 \quad a.e. \text{ in } \Omega. \qquad (3.37)$$

Hence we have $u \in A$.

Let $l \to +\infty$ in (3.36), noticing that $W(x, \hat{E}, F)$ is bounded from below by a constant and by passing to limit, we obtain (3.34). $\qquad \square$

Lemma 3.4 *(see [9]). Let* $\Psi_j, \Psi, \xi_j, \xi, \eta_j, \eta \in L^1(\Omega)$ *with* $\xi_j \geq \Psi_j \geq \eta_j$ *for all* j. *Suppose that* $\Psi_j \to \Psi$, $\xi_j \to \xi$, $\eta_j \to \eta$, *a.e. in* Ω, *and* $\int_\Omega \xi_j \, dx \to \int_\Omega \xi \, dx$, $\int_\Omega \eta_j \, dx \to \int_\Omega \eta \, dx$, *as* $j \to \infty$.

Then

$$\int_\Omega \Psi_j \, dx \to \int_\Omega \Psi \, dx, \quad as \ j \to \infty.$$

Proof. The result follows by applying Fatou's lemma to the sequences $\xi_j - \Psi_j$ and $\Psi_j - \eta_j$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Lemma 3.5 *Let* $\Omega' \subset \Omega$. *Let* $\{u_\nu\}, u$ *be such that*

$$
\begin{aligned}
u_\nu &\rightharpoonup u && in \ W^{1,p}(\Omega'; R^3), \\
adj\, F_\nu &\rightharpoonup adj\, F && in \ L^q(\Omega'; M^3) \\
\det F_\nu &\rightharpoonup \det F && in \ L^r(\Omega'), \\
E_\nu &\rightharpoonup E && in \ L^s(\Omega'; M^3),
\end{aligned}
$$

and

$$\int_{\Omega'} W(x, F)\, dx = \lim_{\nu \to +\infty} \int_{\Omega'} W(x, F_\nu)\, dx < +\infty.$$

Then

$$
\begin{aligned}
u_\nu &\to u && in \ W^{1,p}(\Omega'; R^3), \\
adj\, F_\nu &\to adj\, F && in \ L^q(\Omega'; M^3) \\
\det F_\nu &\to \det F && in \ L^r(\Omega'), \\
E_\nu &\to E && in \ L^s(\Omega'; M^3).
\end{aligned}
$$

The result follows from a standard argument (see [9]) that, roughly speaking, for a strictly convex integral functional $J(v)$ satisfying certain coerceivness hypothesis, such as (H2) in this case, $v_j \rightharpoonup v$ and $J(v_j) \to J(v)$ imply $v_j \to v$. For details of the argument see [9] or (b) in the proof of theorem 2.1 in [8].

Theorem 3.4 . *Let* ϵ_j, M_j, h_j *and* $u_j \in A_{h_j}$ *satisfy the same conditions as in theorem 3.3. Let* $\{u_\nu\} \subset \{u_j\}$, $\Omega_l = \cup_{j=l}^\infty \hat{\Omega}_{M_j}(u_j)$ *with* $\hat{\Omega}_{M_j}(u_j)$ *being defined by (3.17), $u \in A$ and $\hat{E} \in L^s(\Omega; M^3)$ be such that (3.28) – (3.32) and (3.34) hold.*

Then, either

(a):

$$\hat{E} = \frac{1}{2}((I + Du)^T(I + Du) - I). \qquad\qquad (3.38)$$

In this case, $u \in A$ is a minimizer of $I(\cdot)$ in A, and we have

$$u_\nu \quad \rightarrow \quad u \qquad in \ W^{1,p}(\Omega; R^3), \tag{3.39}$$
$$adj \ F_\nu \quad \rightarrow \quad adj \ F \quad in \ L^q(\Omega \setminus \Omega_l; M^3), \quad for \ each \ l, \tag{3.40}$$
$$\det F_\nu \quad \rightarrow \quad \det F \quad in \ L^r(\Omega \setminus \Omega_l), \ \ for \ each \ l, \tag{3.41}$$
$$E_\nu \quad \rightarrow \quad E = \hat{E} \quad in \ L^s(\Omega; M^3). \tag{3.42}$$

or
(b):

$$\hat{E} \neq \frac{1}{2}((I + Du)^T(I + Du) - I). \tag{3.43}$$

In this case, $\{u_\nu\}$ gives microstructure.

Proof. In case (a), it follows from (3.34) that

$$I(u) = \inf_{v \in A} I(v), \tag{3.44}$$

a.e. $u \in A$ is a minimizer of $I(\cdot)$ in A.

Further, let $\Omega' \subset \Omega \setminus \Omega_l$ for some l be a measurable subset of Ω. Then, it follows from lemma 3.3 and (3.38) that

$$\int_{\Omega'} W(x, F) \, dx \leq \underline{\lim}_{\nu \to \infty} \int_{\Omega'} W_{M_\nu}(x, F_\nu) \, dx. \tag{3.45}$$

This implies

$$\int_{\Omega \setminus \Omega_l} W(x, F) \, dx = \underline{\lim}_{\nu \to \infty} \int_{\Omega \setminus \Omega_l} W_{M_\nu}(x, F_\nu) \, dx, \ \ for \ each \ l. \tag{3.46}$$

Since otherwise, there would be some $l_0 \geq 1$ such that

$$\int_{\Omega \setminus \Omega_{l_0}} W(x, F) \, dx - \underline{\lim}_{\nu \to \infty} \int_{\Omega \setminus \Omega_{l_0}} W_{M_\nu}(x, F_\nu) \, dx = \delta < 0,$$

and thus we would have

$$\begin{aligned} \int_{\Omega \setminus \Omega_l} W(x, F) \, dx &= \int_{\Omega_{l_0} \setminus \Omega_l} W(x, F) \, dx + \int_{\Omega \setminus \Omega_{l_0}} W(x, F) \, dx \\ &\leq \underline{\lim}_{\nu \to \infty} \int_{\Omega \setminus \Omega_l} W_{M_\nu}(x, F_\nu) \, dx + \delta. \end{aligned}$$

Hence, by passing to limit, we would have

$$I(u) \leq \inf_{v \in A} I(v) + \delta < \inf_{v \in A} I(v).$$

This contradicts (3.44).

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By the definition of Ω_l and (3.46), we have

$$\int_{\Omega \setminus \Omega_l} W(x, F) \, dx = \underline{\lim}_{\nu \to \infty} \int_{\Omega \setminus \Omega_l} W(x, F_\nu) \, dx. \quad \text{for each } l \qquad (3.47)$$

Thus, (3.29) – (3.42), (3.38) and lemma3.5 imply (3.40), (3.41) and

$$
\begin{aligned}
u_\nu &\to u, &\text{in } W^{1,p}(\Omega \setminus \Omega_l; R^3), &\quad \text{for each } l, &(3.48) \\
E_\nu &\to E = \hat{E}, &\text{in } L^s(\Omega \setminus \Omega_l; M^3), &\quad \text{for each } l. &(3.49)
\end{aligned}
$$

By (3.40), (3.41), (3.48) and (3.49), we can extract a further subsequence of $\{u_\mu\}$, again denoted by $\{u_\mu\}$, such that

$$(u_\mu, adj\, F_\mu, \det F_\mu, E_\mu) \to (u, adj\, F, \det F, E), \quad a.e. \text{ in } \Omega. \qquad (3.50)$$

Thus, by (3.3), we also have

$$W_{M_\mu}(x, F_\mu(x)) \to W(x, F(x)), \quad a.e. \text{ in } \Omega. \qquad (3.51)$$

On the other hand, it follows from (H2) and (3.3) that there is a constant $C > 0$ such that

$$
\begin{aligned}
0 &\leq |F_\mu(x) - F(x)|^p + |E_\mu(x) - E(x)|^s \\
&\leq C \left(1 + W_{M_\mu}(x, F_\mu(x)) + W(x, F(x))\right), \forall \mu. \qquad (3.52)
\end{aligned}
$$

Now, it follows from (3.34), (3.44), (3.50) – (3.52) and lemma 3.4 that

$$\|F_\mu - F\|_p^p - \|E_\mu - E\|_s^s \to 0, \quad \text{as } \mu \to \infty.$$

This gives (3.39) and (3.42).

In case (b), $\{u_\nu\}$ gives microstructure on

$$\hat{\Omega} = \{x \in \Omega : \hat{E} \neq \frac{1}{2}((I + Du)^T(I + Du) - I)\}.$$

The argument is similar as that in lemma 2.4 (see also [8]). □

The results in this section show that, by solving the finite problem of minimizing $I_M(\cdot)$ in A_h with M sufficiently large and h sufficiently small, we can obtain good information on the solutions of the problem of minimizing $I(\cdot)$ in A. For example, let $u_{M,h} \in A_h$ be a minimizer of $I_M(\cdot)$ in A_h with M sufficiently large and h sufficiently small, if $u_{M,h}$ oscilates violently somewhere in Ω, then it suggests that microstructure probably exists as a solution, on the other hand, if $u_{M,h}$ does not oscilate anywhere in Ω, then it should give a good approximation to a minimizer of $I(\cdot)$ in A.

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