LOWER SEMICONTINUITY OF MULTIPLE INTEGRALS AND CONVERGENT INTEGRANDS

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Abstract. Lower semicontinuity of multiple integrals $\int_\Omega f(x, u_\alpha, P_\alpha)\, d\mu$ and $\int_\Omega f_\alpha(x, u_\alpha, P_\alpha)\, d\mu$ are studied. It is proved that the two can derive from each other under certain general hypotheses such as uniform lower compactness property and locally uniform convergence of f_α to f. The result is applied to obtain some general lower semicontinuity theorems on multiple integrals with quasiconvex integrand f, while f_α are not required to be quasiconvex.

Keywords: Lower semicontinuity, convergent integrands, lower precompactness, weakly precompact, quasiconvex.

1. Introduction and preliminaries

Let Ω be a measurable space with finite positive nonatomic complete measure μ, let $f, f_\beta : \Omega \times R^m \times R^N \to R \cup \{+\infty\}$ be extended real-valued functions satisfying certain hypotheses, and let $u : \Omega \to R^m$, $P : \Omega \to R^N$ be measurable functions in two linear topological spaces U and V respectively.

We consider integral functionals of the form

$$I(u, P) = \int_\Omega f(x, u(x), P(x))\, d\mu, \tag{1.1}$$

and

$$I_\alpha(u, P) = \int_\Omega f_\alpha(x, u(x), P(x))\, d\mu. \tag{1.2}$$

The main purpose of this paper is to study, under certain general hypotheses on U, V, f and f_β, lower semicontinuity theorems of the form

$$I(u, P) \le \underline{\lim}_{\alpha\to\infty} I(u_\alpha, P_\alpha), \tag{1.3}$$

$$I(u, P) \le \underline{\lim}_{\alpha\to\infty} I_\alpha(u_\alpha, P_\alpha), \tag{1.4}$$

and the relationship between them. The study of the relationship between (1.3) and (1.4) was motivated mainly by the needs of convergence analysis for numerical solutions to some problems in calculus of variations. For example, in many cases, we have lower semicontinuity theorems of the form (1.3), however, for one reason or another, the sequence (u_α, P_α) obtained by numerical methods is often a minimizing sequence with respect to $I_\alpha(\cdot)$ rather than with respect to $I(\cdot)$, and a lower semicontinuity theorem of the form (1.4) is needed for convergence analysis (see Li [1]). In some applications it is equally important to know under what conditions (1.3) can be derived from (1.4) (see §3).

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Some useful results are established in this paper on the relationship between (1.3) and (1.4). These results are then applied to establish some general lower semicontinuity theorems on multiple integrals with quasiconvex integrands. It is worth noticing that in these theorems f_α are not required to be quasiconvex which allows us to choose f_α more freely in numerical computations.

For better understanding of the background and relevant known results, and for the later use of this paper, we first introduce some definitions and hypotheses.

Definition 1.1. A function $f : \Omega \times R^m \times R^N \to R \cup \{+\infty\}$ is called $\mathbf{L} \otimes \mathbf{B}-$ measurable, if it is measurable respect to the σ-algebra generated by products of measurable subsets of Ω and Borel subsets of $R^m \times R^N$.

Definition 1.2. A function $f : \Omega \times R^m \times R^N \to R$ is a Carathéodory function if

(1): $f(\cdot, u, P)$ is measurable for every $u \in R^m$ and $P \in R^N$,

(2): $f(x, \cdot, \cdot)$ is continuous for almost every $x \in \Omega$.

Definition 1.3. A function $f : \Omega \times R^m \times R^N \to R \cup \{+\infty\}$ has the lower compactness property on $U \times V$ if any sequence of $f^-(x, u_\alpha(x), P_\alpha(x))$ is weakly precompact in $L^1(\Omega)$ whenever (1): u_α converge in U, P_α converge in V; and (2): $I(u_\alpha, P_\alpha) \le C < \infty$ for all $\alpha = 1, 2, \cdots$. f has the strong lower compactness property on $U \times V$ if any sequence of $f^-(x, u_\alpha(x), P_\alpha(x))$ is weakly precompact in $L^1(\Omega)$ whenever (1) holds. Here $f^- = \min\{f, 0\}$.

The assumption of lower compactness property provide us more freedom in applications than the nonnegative assumptions. For example, let U be $L^q(\Omega; R^m)$, $1 \le q \le +\infty$, with strong topology and V be $L^p(\Omega; R^N)$, $1 \le p \le +\infty$, with weak topology, let $g : \Omega \times R^m \to R^N$ be such that

$$|g(x, u)|^{p'} \le c|u|^q + b(x)$$

for some constant $c > 0$ and function $b(\cdot) \in L^1(\Omega)$, where $p' = \frac{p}{p-1}$, and let $f : \Omega \times R^m \times R^N \to R$ satisfy

$$f(x, u, P) \ge \langle g(x, u), P\rangle - c_1|u|^q + b_1(x)$$

for some constant $c_1 > 0$ and function $b_1(\cdot) \in L^1(\Omega)$, where $\langle Q, P\rangle = \sum_{i=1}^N Q_i P_i$, then it is easy to verify that f has the strong compactness property on $U \times V$.

Definition 1.4. A sequence of functions $f_\beta : \Omega \times R^m \times R^N \to R \cup \{+\infty\}$ is said to have the uniform lower compactness property, if $f_\beta^-(x, u_\alpha(x),\ P_\alpha(x))$ are uniformly weakly precompact in $L^1(\Omega)$, in other words (see [2, 3]), $f_\beta^-(x, u_\alpha(x), P_\alpha(x))$ are equi-uniformly integral continuous on Ω, *i.e.* for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\int_{\Omega'} f_\beta^-(x, u_\alpha(x), P_\alpha(x))\, d\mu| < \epsilon$$

for all α, β and any measurable subset $\Omega' \subset \Omega$ satisfying $\mu(\Omega') < \delta$, whenever $u_\alpha(\cdot)$ converge in U, $P_\alpha(\cdot)$ converge in V and $I_\beta(u_\alpha, P_\alpha) \le C < \infty$ for all α and β.

Definition 1.5. A sequence of functions $f_\alpha : \Omega \times R^m \times R^N \to R \cup \{+\infty\}$ is said to converge to $f : \Omega \times R^m \times R^N \to R \cup \{+\infty\}$ locally uniformly in $\Omega \times R^m \times R^N$, if there exists a sequence of measurable subsets $\Omega_l \subset \Omega$ with $\mu(\Omega \setminus \Omega_l) \to 0$ as $l \to \infty$ such that for each l and any compact subset $G \subset R^m \times R^N$

$$f_\alpha(x, u, P) \longrightarrow f(x, u, P), \quad \text{unifomly on } \Omega_l \times G, \text{ as } \alpha \to \infty.$$

Definition 1.6. A sequence of functions $f_\alpha : \Omega \times R^m \times R^N \to R \cup \{+\infty\}$ is said to converge to $f : \Omega \times R^m \times R^N \to R \cup \{+\infty\}$ locally uniformly in the sense of integration on $U \times V$, if there exists a sequence of measurable subsets $\Omega_l \subset \Omega$ with $\mu(\Omega \setminus \Omega_l) \to 0$ as $l \to \infty$ such that

$$\int_{\Omega_l \setminus E(u,P,K)} f_\alpha(x, u(x), P(x))\, d\mu \to \int_{\Omega_l \setminus E(u,P,K)} f(x, u(x), P(x))\, d\mu$$

uniformly in $U \times V$ for each l and any fixed $K > 0$, where

$$E(u, P, K) = \{x \in \Omega : |u(x)| > K \quad \text{or} \quad |P(x)| > K\}.$$

Remark 1.1. In numerical computations of singular minimizers, truncation methods turned out to be successful (see [1]). In such applications, f_α can be as simple as

$$f_\alpha(x, u, P) = \min\{-\alpha,\ \max\{\alpha,\ f(x, u, P)\}\}$$

which converge to f locally uniformly in $\Omega \times R^m \times R^N$ if f is a Carathéo- dory function, and which have the uniform lower compactness property on $U \times V$ if f has the lower compactness property on $U \times V$.

Remark 1.2. When Ω is a locally compact metric space, Ω_l in definition 1.5 and definition 1.6 can be taken to be compact subsets of Ω. In such a case, to say that f_α converge to f locally uniformly in $\Omega \times R^m \times R^N$ is equivalent to saying that f_α converge to f uniformly on every compact subset in $\Omega \times R^m \times R^N$. Especially, if f_α converge to f uniformly on every bounded subset in $\Omega \times R^m \times R^N$ then f_α converge to f locally uniformly in $\Omega \times R^m \times R^N$, if $f_\alpha,\ f$ are Carathéodory functions and f_α converge to f in measure on every bounded subset in $\Omega \times R^m \times R^N$ then f_α converge to f locally uniformly in the sense of integration on $U \times V$.

Remark 1.3. To cover more applications, we may consider $f_\alpha,\ f$ as mappings from $U \times V$ to $\{$measurale functions $g : \Omega \to R \cup \{+\infty\}\}$. This allows us to consider, for example, interpolations of $f(x, u(x), P(x))$ in finite element spaces. In fact, definitions and results in this paper can be extended parallely to such mappings without difficulty, as we will see that in the proofs of the theorems f_α are related to f only as such mappings.

Definition 1.7. (See [4, 5, 6]) A function $f : R^{m\times n} \to R$ is quasiconvex if

$$\int_{\Omega'} f(P + D\,\phi(x))\, dx \geq f(P)\, \mu(\Omega')$$

for every $P \in R^{m\times n}$, $\phi \in C_0^1(\Omega'; R^m)$, and every open bounded subset $\Omega' \subset R^n$, where μ is the Lebesgue measure on R^n. Let $\Omega \subset R^n$ be open and bounded. A Carathéodory function $f : \Omega \times R^m \times R^{m\times n} \to R$ is quasiconvex in P if there exists a subset $E \subset \Omega$ with $\mu(E) = 0$ such that $f(x, u, \cdot)$ is quasiconvex for all $x \in \Omega \setminus E$ and $u \in R^m$.

Example 1. Let $\Omega \subset R^2$, $n = m = 2$. Then the function

$$f(Du) = |Du|^2 - (\operatorname{tr} Du)^2$$

is quasiconvex (in fact it is polyconvex, see [6, 9, 10]) and is unbounded below. In addition, f has the strong lower compactness property on $L^p(\Omega; R^{2\times 2})$ with weak topology for all $p > 2$. If

V is taken to be $L^2(\Omega; R^{2\times 2})$ with weak topology then f has a slightly weaker lower compactness property which can be described by Chacon's biting lemma [5, 7, 8, 3] (see lemma 3.4)

Let

$$D(\Omega; R^k) = \{v : \Omega \to R^k \mid v \text{ is measurable } \}.$$

We assume that $U \subset D(\Omega; R^m)$ and $V \subset D(\Omega; R^N)$ are decomposable, *i.e.* if $v(\cdot)$ belongs to one of them, then $\chi_T(\cdot)v(\cdot)$ belongs to the same space whenever T is a measurable subset of Ω, where $\chi_T(\cdot)$ is the characteristic function of T:

$$\chi_T(x) = \begin{cases} 1, & \text{if } \ x \in T, \\ 0, & \text{if } \ x \notin T. \end{cases}$$

We assume that U and V satisfy the following hypotheses on their topologies:

- **(H1):** If $v_\alpha(\cdot), \alpha = 1, 2, \cdots$ belong to one of the spaces and converge there to zero and if $\mu T_\alpha \to 0$, then $\chi_{T_\alpha}(\cdot)v_\alpha(\cdot)$ also converge to zero.
- **(H2):** The topology in U is not weaker than the topology of convergence in measure; the topology of V is not weaker than the topology induced in V by the weak topology of $L^1(\Omega; R^N)$.

It is easy to see that if U is taken to be $L^q(\Omega; R^m)$, $1 \le q \le +\infty$, with strong topology and V is taken to be $L^p(\Omega; R^m)$ with weak topology for $1 \le p < +\infty$ or weak* topology for $p = +\infty$, then (H1) and (H2) are satisfied. This covers most applications in Sobolev spaces. For applications concerning Orlicz spaces see for example [2].

Remark 1.4. Here and throughout this paper, assumptions and statements are referred to sets with measure-negligible projections on Ω, *i.e.* they hold on a subset $\Omega' \subset \Omega$ with $\mu\,\Omega' = \mu\,\Omega$.

In Reshetnyak's result (see theorem 1.2 in [11]), Ω is taken to be a local compact metric space, $f, f_\alpha : \Omega \times R^N \to R$ are nonnegative functions such that for any $\epsilon > 0$ there is a compact set $A \subset \Omega$ with $\mu(\Omega \setminus A) < \epsilon$ and $f(x, u), f_\alpha(x, u)$ being continuous on $A \times R^N$, $f(x, \cdot), f_\alpha(x, \cdot)$ are convex for almost all $x \in \Omega$, and $f_\alpha \to f$ locally uniformly in $\Omega \times R^N$ as $\alpha \to \infty$, and V is taken to be $L^1(\Omega; R^N)$ with weak topology.

In the case when $f(x, u, \cdot)$ is convex, (1.3) was proved (see Ioffe [2]) under the hypotheses that f satisfies lower compactness property and U , V satisfy (H1) and (H2), and (1.4) was proved (see Li [12]), by using (1.3) , under the hypotheses that f has the lower compactness property, f_α have the uniform compactness property and $f_\alpha \to f$ locally uniformly in $\Omega \times R^m \times R^N$.

The results of the type (1.3) concerning quasiconvexity of $f(x, u, \cdot)$ and $P = Du$ can be found in [5, 6]. The following theorem, which will be used in §3, was established by Acerbi and Fusco [5, 6].

Theorem 1.1. *Let $\Omega \subset R^n$ be bounded and open. Let*

$$F(u) = \int_\Omega f(x, u, Du)\, dx, \quad u \in W^{1,p}(\Omega; R^m),$$

where $1 \le p < \infty$, and where $f : \Omega \times R^m \times R^{m\times n} \to R$ satisfies

- **(1):** *$f(\cdot, \cdot, \cdot)$ is a Carathéodory function;*
- **(2):** *$f(x, u, \cdot)$ is quasiconvex;*

(3): $0 \le f(x,u,P) \le a(x) + C(|u|^p + |P|^p)$ *for every* $x \in \Omega, u \in R^m$ *and* $P \in R^{m\times n}$, *where* $C > 0$ *and* $a(\cdot) \in L^1(\Omega)$.

Then, the functional $u \to F(u)$ *is sequentially weakly lower semicontinuous on* $W^{1,p}(\Omega; R^m)$, i.e. *(1.3) holds for* $P = Du$ *and* $P_\alpha = Du_\alpha$ *with* $u_\alpha \rightharpoonup u$ *in* $W^{1,p}(\Omega; R^m)$, *where and in what follows "$\rightharpoonup$" means "converges weakly to".*

Ball and Zhang [6] generalized the above result to cover the case when

$$|f(x,u,P)| \le a(x) + C(|u|^p + |P|^p),$$

and proved that (1.3) holds on each $\Omega \backslash E_k$, where $\{E_k\}$ is a nonincreasing sequence of measurable subsets of Ω with $\lim_{k\to\infty} \mu E_k = 0$.

In §2, It is shown, under certain general hypotheses, such as uniform lower compactness property and locally uniform convergence of f_α to f (see theorem 2.1 for details), that (1.3) and (1.4) can somehow be derived from each other, and the results are further developed into a theorem concerning Γ-limit (see theorem 2.3, for Γ-limit, Γ-convergence and their applications in calculus of variations see for example [13, 14]). These results generalized the result of Li [12] which was proved for the case when $f(x,u,\cdot)$ is convex. In §3, the results established in §2 are applied to prove some lower semicontinuity theorems of the form (1.3) and (1.4) for quasiconvex integrands, which generalize the results of Acerbi and Fusco [5], Ball and Zhang [6] and Li [12].

2. Lower semicontinuity and convergent integrands

The following theorem establishes the relationship between the lower semicontinuity theorems of the form (1.3) and those of the form (1.4).

Theorem 2.1. *Let* Ω *be a measurable space with finite positive non- atomic complete measure* μ. *Let* U *and* V *satisfy (H1) and (H2). Let* $\{u_\alpha\}, u \in U$ *and* $\{P_\alpha\}, P \in V$ *be such that*

$$u_\alpha \longrightarrow u, \quad in \quad U, \tag{2.5}$$

and

$$P_\alpha \longrightarrow P, \quad in \quad V. \tag{2.6}$$

Let $f, \{f_\beta\} : \Omega \times R^m \times R^N \to R \cup \{+\infty\}$ *satisfy*

(i): $f, \{f_\beta\}$ *are* $\mathbf{L} \otimes \mathbf{B}$*- measurable;*

(ii): $f^-(x,u(x),P(x)), f^-(x,u_\alpha(x),P_\alpha(x))$, *and* $f_\alpha^-(x,u_\alpha(x),P_\alpha(x))$ *are weakly precompact in* $L^1(\Omega)$*;*

(iii): $f_\alpha \to f$ *locally uniformly in* $\Omega \times R^m \times R^N$.

Then, we have (a):

$$\int_\Omega f(x,u,P)\,d\mu \le \underline{\lim}_{\alpha\to\infty} \int_\Omega f_\alpha(x,u_\alpha,P_\alpha)\,d\mu, \tag{2.7}$$

provided that

$$\int_{\Omega'} f(x,u,P)\,d\mu \le \underline{\lim}_{\alpha\to\infty} \int_{\Omega'} f(x,u_\alpha,P_\alpha)\,d\mu, \tag{2.8}$$

for all measurable subset $\Omega' \subset \Omega$*;*

and (b):

$$\int_\Omega f(x,u,P)\,d\mu \le \underline{\lim}_{\alpha\to\infty} \int_\Omega f(x,u_\alpha,P_\alpha)\,d\mu, \tag{2.9}$$

provided that $f, \{f_\beta\}$ satisfy a further hypothesis

(iv): $f^+_\alpha(x, u_\alpha(x), P_\alpha(x)) \le a(x) + f^+(x, u_\alpha(x), P_\alpha(x))$, *where* $a(\cdot) \in L^1(\Omega)$ *is nonnegative; and*

$$\int_\Omega f(x, u, P)\, d\mu \le \underline{\lim}_{\alpha\to\infty} \int_\Omega f_\alpha(x, u_\alpha, P_\alpha)\, d\mu. \tag{2.10}$$

To prove the theorem, we need the following lemmas.

Lemma 2.1. *Let* $\{u_\alpha\}, u \in U$ *and* $\{P_\alpha\}, P \in V$ *satisfy (2.5) and (2.6) respectively. Let*

$$\begin{aligned} E^1_\alpha(K) &= \{x \in \Omega : |u_\alpha(x)| > K\}, \\ E^2_\alpha(K) &= \{x \in \Omega : |P_\alpha(x)| > K\}, \end{aligned}$$

and

$$E_\alpha(K) = E^1_\alpha(K) \cup E^2_\alpha(K). \tag{2.11}$$

Then

$$\mu\, E_\alpha(K) \longrightarrow 0, \quad \textit{uniformly for } \alpha \textit{ as } \quad K \to \infty.$$

Proof. For any $\epsilon > 0$, since $u \in U$, there exists $K_1(\epsilon) > 1$ such that

$$\mu\, \{x \in \Omega : |u(x)| > K\} < \epsilon/2, \quad \forall K > K_1(\epsilon).$$

Thus, by (2.5) and (H2), there exists $\alpha(\epsilon) > 1$ such that

$$\mu\, E^1_\alpha(K) < \epsilon/2, \quad \forall \alpha > \alpha(\epsilon) \quad \text{and} \quad K > K_1(\epsilon) + 1. \tag{2.12}$$

Since $u_\alpha \in U$ for each α, we have

$$\lim_{K\to\infty} \mu E^1_\alpha(K) = 0, \qquad \text{for each } \alpha.$$

Thus, for $\alpha \in \{1, 2, \cdots, \alpha(\epsilon)\}$, there exists $K_2(\epsilon) > 1$ such that

$$\mu\, E^1_\alpha(K) < \epsilon/2, \quad \forall \alpha \in \{1, 2, \cdots, \alpha(\epsilon)\} \quad \text{and} \quad K > K_2(\epsilon). \tag{2.13}$$

Let $K(\epsilon) = \max\{K_1(\epsilon) + 1, K_2(\epsilon)\}$, then (2.12) and (2.13) give

$$\mu\, E^1_\alpha(K) < \epsilon/2, \quad \forall \alpha \ge 1 \quad \text{and} \quad K > K(\epsilon). \tag{2.14}$$

On the other hand, it follows from (2.6) and (H2) that

$$\int_\Omega |P_\alpha(x)|\, d\mu \le C,$$

for some constant $C > 0$. Thus, for any $\epsilon > 0$ there exists $K(\epsilon) > 1$ such that

$$\mu\, E^2_\alpha(K) < \epsilon/2, \quad \forall \alpha \ge 1 \quad \text{and} \quad K > K(\epsilon). \tag{2.15}$$

Hence the lemma follows from (2.14) and (2.15). □

Lemma 2.2. *Let $f, \{f_\alpha\}$ satisfy the hypotheses in theorem 2.1. Let $\bar{f}_A : \Omega \times R^m \times R^N \to R$ be defined by*

$$\bar{f}_A(x, u, P) = \min\{A, f(x, u, P)\}. \tag{2.16}$$

Let $\{u_\alpha\}, u \in U$ and $\{P_\alpha\}, P \in V$ satisfy (2.5) and (2.6) respectively. Let $\{\Omega_l\}$ be a sequence of measurable subsets of Ω, the existence of which is guaranteed by the hypothesis (iii) for f_α, such that

$$\mu(\Omega \setminus \Omega_l) \longrightarrow 0, \quad \text{as } l \to \infty, \tag{2.17}$$

and

$$f_\alpha \longrightarrow f, \quad \text{uniformly on } \Omega_l \times G, \tag{2.18}$$

for each l and any compact set $G \subset R^m \times R^N$.

Suppose

$$\int_\Omega f_\alpha(x, u_\alpha, P_\alpha)\, d\mu \leq C,$$

for some constant $C > 0$.

Then, for any $\epsilon > 0, A > 1$, there exist $l(\epsilon) \geq 1$ and $\alpha(\epsilon, A, l) \geq 1$ such that

$$\begin{array}{ll} \int_{\Omega_l} \bar{f}_A(x, u_\alpha, P_\alpha)\, d\mu & \leq \int_\Omega f_\alpha(x, u_\alpha, P_\alpha)\, d\mu + \epsilon \\ & \forall l \geq l(\epsilon) \quad \text{and} \quad \alpha \geq \alpha(\epsilon, A, l). \end{array} \tag{2.19}$$

Proof.

$$\begin{array}{ll} & \int_{\Omega_l} \bar{f}_A(x, u_\alpha, P_\alpha)\, d\mu \\ = & \int_\Omega f_\alpha(x, u_\alpha, P_\alpha)\, d\mu + \int_{\Omega\setminus\Omega_l}(-f_\alpha(x, u_\alpha, P_\alpha))\, d\mu + \\ & + \int_{\Omega_l}(\bar{f}_A(x, u_\alpha, P_\alpha) - f_\alpha(x, u_\alpha, P_\alpha))\, d\mu \\ = & \int_\Omega f_\alpha(x, u_\alpha, P_\alpha)\, d\mu + I_1 + I_2. \end{array}$$

By (2.5), (2.6), (2.17) and (ii), there exists $l(\epsilon) > 0$ such that

$$\begin{array}{rl} I_1 = & \int_{\Omega\setminus\Omega_l}(-f_\alpha(x, u_\alpha, P_\alpha))\, d\mu \\ \leq & \int_{\Omega\setminus\Omega_l}(-f_\alpha^-(x, u_\alpha, P_\alpha))\, d\mu \\ < & \epsilon/2, \qquad \text{if} \quad l \geq l(\epsilon). \end{array} \tag{2.20}$$

By (2.16), we have

$$\begin{array}{rl} I_2 \leq & \int_{\Omega_l\setminus E_\alpha(K)}(f(x, u_\alpha, P_\alpha) - f_\alpha(x, u_\alpha, P_\alpha))\, d\mu + \\ & + \int_{E_\alpha(K)}(A - f_\alpha(x, u_\alpha, P_\alpha))\, d\mu \\ = & I_{21} + I_{22}, \end{array}$$

where $E_\alpha(K)$ is defined by (2.11).

By lemma 2.1, $E_\alpha(K) \to 0$ uniformly for α as $K \to \infty$. Thus it follows from (ii) that there exists $K(\epsilon, A) > 1$ such that

$$\begin{array}{rl} I_{22} \leq & \int_{E_\alpha(K)}(A - f_\alpha^-(x, u_\alpha, P_\alpha))\, d\mu \\ < & \epsilon/4, \qquad \text{if} \quad K \geq K(\epsilon, A). \end{array}$$

Let $\bar{K} = K(\epsilon, A)$, then

$$G(\bar{K}) = \{u \in R^m : |u| \leq \bar{K}\} \times \{P \in R^N : |P| \leq \bar{K}\}$$

is a compact set in $R^m \times R^N$. It follows from (2.18) that there exists $\alpha(\epsilon, A, l) > 0$ such that

$$|I_{21}| < \epsilon/4, \qquad \forall \alpha \geq \alpha(\epsilon, A, l).$$

Thus, we have

$$|I_2| < \epsilon/2, \qquad \forall \alpha \geq \alpha(\epsilon, A, l). \tag{2.21}$$

Thus (2.19) follows from (2.20) and (2.21). □

Lemma 2.3. *Let $f, \{f_\alpha\}$ satisfy the hypotheses in theorem 2.1. Let $\{u_\alpha\}, u \in U$ and $\{P_\alpha\}, P \in V$ satisfy (2.5) and (2.6) respectively. Let*

$$F(\alpha, A) = \{x \in \Omega : f(x, u_\alpha(x), P_\alpha(x)) > A\}.$$

Suppose

$$\int_\Omega f_\alpha(x, u_\alpha, P_\alpha)\, d\mu \leq C, \tag{2.22}$$

for some constant $C > 0$.

Then, for any $\epsilon > 0$ and $K \geq 1$, there exist $A(\epsilon) > 1$ and $\alpha(\epsilon, K) > 1$ such that

$$\begin{aligned} \mu\, F(\alpha, A) \quad &\leq \mu\, E_\alpha(K) + \epsilon, \\ &\textit{if} \;\; A \geq A(\epsilon) \quad \textit{and} \quad \alpha \geq \alpha(\epsilon, K), \end{aligned} \tag{2.23}$$

where $E_\alpha(K)$ is defined by (2.11).

Proof. By (iii), there is a sequence of measurable subsets $\{\Omega_l\}$ in Ω such that

$$\mu\,(\Omega \setminus \Omega_l) \longrightarrow 0, \qquad \text{as } l \to \infty, \tag{2.24}$$

and

$$f_\alpha \longrightarrow f, \qquad \text{uniformly on } \Omega_l \times G, \tag{2.25}$$

for each l and any compact set $G \subset R^m \times R^N$.

For any $\epsilon > 0$, by (2.24), there is $l_1(\epsilon) \geq 1$ such that

$$\mu\,(\Omega \setminus \Omega_l) < \epsilon/2, \qquad \text{if } l \geq l_1(\epsilon). \tag{2.26}$$

By (2.22),

$$\begin{aligned} & \textstyle\int_{\Omega_l \setminus E_\alpha(K)} f(x, u_\alpha, P_\alpha)\, d\mu \\ = \;& \textstyle\int_{\Omega_l \setminus E_\alpha(K)} (f(x, u_\alpha, P_\alpha) - f_\alpha(x, u_\alpha, P_\alpha))\, d\mu \\ & + \textstyle\int_{\Omega \setminus (\Omega_l \setminus E_\alpha(K))} (-f_\alpha(x, u_\alpha, P_\alpha))\, d\mu + C \\ = \;& I_1 + I_2 + C, \end{aligned}$$

It follows from (ii) that

$$\begin{aligned} I_2 \;\; &\leq \int_{\Omega \setminus (\Omega_l \setminus E_\alpha(K))} (-f_\alpha^-(x, u_\alpha, P_\alpha))\, d\mu \\ &\leq \int_\Omega (-f_\alpha^-(x, u_\alpha, P_\alpha))\, d\mu \\ &\leq C_1, \end{aligned} \tag{2.27}$$

for some constant $C_1 > 0$.

It follows from (2.25) that there exists $\alpha(l,K) > 1$ such that

$$\begin{aligned} |I_1| &\leq \int_{\Omega_l \setminus E_\alpha(K)} \quad |f(x,u_\alpha,P_\alpha) - f_\alpha(x,u_\alpha,P_\alpha)|\, d\mu \\ &\leq 1, \qquad\qquad \forall \alpha \geq \alpha(l,K). \end{aligned} \tag{2.28}$$

Thus we have

$$\int_{\Omega_l \setminus E_\alpha(K)} f(x,u_\alpha,P_\alpha)\, d\mu \leq C_2, \quad \forall \alpha \geq \alpha(l,K), \tag{2.29}$$

where $C_2 = C + C_1 + 1$ is a constant.

Denote $\Omega_\alpha^- = \{x \in \Omega : f(x,u_\alpha(x),P_\alpha(x)) < 0\}$ and $f^+ = \max\{f,0\}$; then, by (2.29)

$$\begin{aligned} & \int_{\Omega_l \setminus E_\alpha(K)} f^+(x,u_\alpha,P_\alpha)\, d\mu \\ \leq\ & \int_{\Omega_l \setminus E_\alpha(K)} (-f^-(x,u_\alpha,P_\alpha))\, d\mu + C_2 \\ =\ & \int_{(\Omega_l \setminus E_\alpha(K)) \cap \Omega_\alpha^-} (-f(x,u_\alpha,P_\alpha))\, d\mu + C_2 \\ =\ & \int_{(\Omega_l \setminus E_\alpha(K)) \cap \Omega_\alpha^-} (f_\alpha(x,u_\alpha,P_\alpha) - f(x,u_\alpha,P_\alpha))\, d\mu \\ & \quad + \int_{(\Omega_l \setminus E_\alpha(K)) \cap \Omega_\alpha^-} (-f_\alpha(x,u_\alpha,P_\alpha))\, d\mu + C_2 \\ \leq\ & \int_{\Omega_l \setminus E_\alpha(K)} |f(x,u_\alpha,P_\alpha) - f_\alpha(x,u_\alpha,P_\alpha)|\, d\mu \\ & \quad + \int_\Omega (-f_\alpha^-(x,u_\alpha,P_\alpha))\, d\mu + C_2. \end{aligned}$$

It follows from this and (2.27), (2.28) that

$$\int_{\Omega_l \setminus E_\alpha(K)} f^+(x,u_\alpha,P_\alpha)\, d\mu \leq C_3, \quad \forall \alpha \geq \alpha(l,K), \tag{2.30}$$

where $C_3 = C_1 + C_2 + 1$ is a constant.

Now (2.30) implies that there exists $A(\epsilon) > 0$ such that

$$\begin{aligned} \mu\, \{x \in \Omega_l \setminus E_\alpha(K) : \quad & f(x,u_\alpha(x),P_\alpha(x)) > A\} < \epsilon/2, \\ & \text{if } A \geq A(\epsilon) \text{ and } \alpha \geq \alpha(l,K). \end{aligned} \tag{2.31}$$

Since

$$F(\alpha,A) \subset E_\alpha(K) \cup (\Omega \setminus \Omega_l) \cup F(l,K,\alpha,A),$$

where

$$F(l,K,\alpha,A) = \{x \in \Omega_l \setminus E_\alpha(K) : f(x,u_\alpha(x),P_\alpha(x)) > A\},$$

we have

$$\mu\, F(\alpha,A) \leq \mu\, E_\alpha(K) + \mu\,(\Omega \setminus \Omega_l) + \mu\, F(l,K,\alpha,A).$$

Taking $l = l_1(\epsilon)$ and $\alpha(\epsilon,K) = \alpha(l_1(\epsilon),K)$, by (2.26) and (2.31), we conclude that (2.23) is true. □

Proof of Theorem 2.1.

Without loss of generality, we assume that

$$\int_\Omega f_\alpha(x,u_\alpha,P_\alpha)\, d\mu \leq C,$$

for some constant $C > 0$. It follows from (iii) that there exists a sequence of measurable subsets $\{\Omega_l\}$ of Ω such that

$$\Omega_l \subset \Omega_{l+1}, \ \forall l, \quad \text{and} \quad \lim_{l\to\infty} \mu\,(\Omega \setminus \Omega_l) = 0, \tag{2.32}$$

and

$$f_\alpha \longrightarrow f, \qquad \text{uniformly on} \quad \Omega_l \times G, \tag{2.33}$$

for each l and any compact set $G \subset R^m \times R^N$.

Let $E_\alpha(K)$ be defined by (2.11). It follows from lemma 2.1 that there exists an increasing sequence $\{K_i\}$ such that

$$\sum_{i=1}^{\infty} \sup_{1\le\alpha<\infty} \{\mu\, E_\alpha(K_i)\} < \infty. \tag{2.34}$$

Let $\epsilon_i > 0, i = 1, 2, \cdots$ be a decreasing sequence of numbers satisfying $\lim_{i\to\infty} \epsilon_i = 0$. Let $A_i = A(\epsilon_i/2^i)$, $l_i = l(\epsilon_i)$, $\alpha_i = \max\{\alpha(l_i, K_i),$
$\alpha(\epsilon_i, A_i, l_i)\}$ and $F_i = F(\alpha_i,\ A_i) = \{x \in \Omega : f(x, u_{\alpha_i}(x), P_{\alpha_i}(x)) > A_i\}$ where $A(\cdot)$, $\alpha(\cdot,\ \cdot)$ are defined by lemma 2.3 and $l(\cdot)$, $\alpha(\cdot,\ \dot{;}\ \cdot)$ are defined by lemma 2.2. Then, by lemma 2.2, we have

$$\int_{\Omega_{l_i}} \bar{f}_{A_i}(x, u_{\alpha_i}, P_{\alpha_i})\, d\mu \le \int_\Omega f_{\alpha_i}(x, u_{\alpha_i}, P_{\alpha_i})\, d\mu + \epsilon_i \quad \forall i, \tag{2.35}$$

and by lemma 2.3, we have

$$\sum_{i=1}^{\infty} \mu\ F_i \le \sum_{i=1}^{\infty} (\mu\ E_{\alpha_i}(K_i) + \epsilon_i/2^i) < \infty. \tag{2.36}$$

Let $H_j = ((\Omega \setminus \Omega_{l_j}) \cup (\cup_{i\ge j} F_i))$ and $G_j = \Omega \setminus H_j$. It follows from (2.28) and (2.32) that

$$G_j \subset G_{j+1}, \ \forall j, \quad \text{and} \quad \lim_{j\to\infty} (\Omega \setminus G_j) = 0. \tag{2.37}$$

Thus, by the definition of $\bar{f}_{A_i}$ and F_i, we have

$$\begin{aligned} & \textstyle\int_{G_j} f(x, u_{\alpha_i}, P_{\alpha_i})\, d\mu \\ = \ & \textstyle\int_{\Omega_{l_i}\setminus F_i} f(x, u_{\alpha_i}, P_{\alpha_i})\, d\mu \quad + \int_{(\Omega_{l_i}\setminus F_i)\setminus G_j} (-f(x, u_{\alpha_i}, P_{\alpha_i}))\, d\mu \\ \le \ & \textstyle\int_{\Omega_{l_i}\setminus F_i} f(x, u_{\alpha_i}, P_{\alpha_i})\, d\mu \quad + \int_{\Omega\setminus G_j} (-f^-(x, u_{\alpha_i}, P_{\alpha_i}))\, d\mu \\ \le \ & \textstyle\int_{\Omega_{l_i}\setminus F_i} \bar{f}_{A_i}(x, u_{\alpha_i}, P_{\alpha_i})\, d\mu \quad + \int_{H_j} (-f^-(x, u_{\alpha_i}, P_{\alpha_i}))\, d\mu, \quad \forall i \ge j. \end{aligned}$$

It follows from this and (2.35) that

$$\begin{aligned} & \textstyle\int_{G_j} f(x, u_{\alpha_i}, P_{\alpha_i})\, d\mu \\ \le \ & \textstyle\int_\Omega f_{\alpha_i}(x, u_{\alpha_i}, P_{\alpha_i})\, d\mu \\ & \textstyle + \int_{H_j} (-f^-(x, u_{\alpha_i}, P_{\alpha_i}))\, d\mu \quad + \epsilon_i, \quad \forall i \ge j. \end{aligned} \tag{2.38}$$

Let $i \to \infty$ in (2.38). By (2.8), we have

$$\begin{aligned} & \textstyle\int_{G_j} f(x, u, P)\, d\mu \\ \le \ & \underline{\lim}_{i\to\infty} \textstyle\int_\Omega f_{\alpha_i}(x, u_{\alpha_i}, P_{\alpha_i})\, d\mu \\ & + \overline{\lim}_{i\to\infty} \textstyle\int_{H_j} (-f^-(x, u_{\alpha_i}, P_{\alpha_i}))\, d\mu. \end{aligned} \tag{2.39}$$

By (ii) and (2.37), we have

$$\lim_{j\to\infty} (\sup_{i\ge1} \int_{H_j} (-f^-(x, u_{\alpha_i}, P_{\alpha_i}))\, d\mu) = 0.$$

It follows from this and (2.37), (2.39) that

$$\begin{aligned}\int_\Omega f(x,u,P)\,d\mu &= \lim_{j\to\infty}\int_{G_j} f(x,u,P)\,d\mu \\ &\le \underline{\lim}_{i\to\infty}\int_\Omega f_{\alpha_i}(x,u_{\alpha_i},P_{\alpha_i})\,d\mu.\end{aligned}$$

This completes the proof of (a).

Next, we prove (b). Without loss of generality, we assume that

$$\int_\Omega f(x,u_\alpha,P_\alpha)\,d\mu \le C < \infty. \tag{2.40}$$

It follows from (ii), (iv) and (2.40) that

$$\int_\Omega f_\alpha(x,u_\alpha,P_\alpha)\,d\mu \le C_1 < \infty.$$

By (2.10), for any $\epsilon > 0$, there exists $\hat\alpha(\epsilon) > 0$ such that

$$\int_\Omega f(x,u,P)\;d\mu \le \int_\Omega f_\alpha(x,u_\alpha,P_\alpha)\;d\mu + \epsilon \quad \forall\alpha \ge \hat\alpha(\epsilon). \tag{2.41}$$

By (ii), for any $\epsilon > 0$, there exists $\delta_1(\epsilon) > 0$ such that

$$|\int_{\Omega'} f^-(x,u_\alpha,P_\alpha)\,d\mu| < \epsilon, \quad \forall\mu\Omega' \le \delta_1(\epsilon), \tag{2.42}$$

$$|\int_{\Omega'} f_\alpha^-(x,u_\alpha,P_\alpha)\,d\mu| < \epsilon, \quad \forall\mu\Omega' \le \delta_1(\epsilon). \tag{2.43}$$

By (iv) and (ii), for any $\epsilon > 0$, there exists $\delta_2(\epsilon) > 0$ such that

$$\begin{aligned}&\int_{\Omega'} f_\alpha^+(x,u_\alpha,P_\alpha)\,d\mu \\ \le\;& \int_{\Omega'} f^+(x,u_\alpha,P_\alpha)\,d\mu + \int_{\Omega'} a(x)\,d\mu \\ \le\;& \int_{\Omega'} f(x,u_\alpha,P_\alpha)\,d\mu + \epsilon, \quad \forall\Omega' \subset \Omega \text{ and } \mu\Omega' \le \delta_2(\epsilon).\end{aligned} \tag{2.44}$$

Let

$$E_\alpha(K) = \{x \in \Omega : |u_\alpha(x)| > K \;\text{ or }\; |P_\alpha(x)| > K\}.$$

By lemma 2.1, for any $\delta > 0$ there exists $K(\delta) > 0$ such that

$$\mu E_\alpha(K) < \delta, \quad \forall K \ge K(\delta). \tag{2.45}$$

By (iii), there exists a sequence of measurable subsets $\Omega_l \subset \Omega$ such that $\mu(\Omega \setminus \Omega_l) \to 0$ as $l \to \infty$ and $f_\alpha \to f$ uniformly on $\Omega_l \times G$ for each fixed l and compact set $G \subset R^m \times R^N$. Thus, for any $\delta > 0$ there exists $l(\delta) \ge 1$ such that

$$\mu\,(\Omega \setminus \Omega_l) < \delta, \quad \forall l \ge l(\delta), \tag{2.46}$$

and for any $\epsilon > 0$, $l \ge 1$ and compact set $G \subset R^m \times R^N$ there exists $\alpha(\epsilon,l,G) > 0$ such that

$$|f_\alpha(x,v,Q) - f(x,v,Q)| < \frac{\epsilon}{\mu\,\Omega} \tag{2.47}$$

for all $\alpha \ge \alpha(\epsilon,l,G)$ and $(x,v,Q) \in \Omega_l \times G$.

Now, by taking

$$\delta = \min\{\delta_1(\epsilon), \delta_2(\epsilon)\},$$

$$K = K(\delta),$$
$$l = l(\delta),$$
$$G = \{(v, Q) \in R^m \times R^N : |v| \leq K \text{ and } |P| \leq K\}$$

and

$$\alpha(\epsilon) = \max\{\hat{\alpha}(\epsilon), \alpha(\epsilon, l, G)\},$$

we get from (2.41) – (2.47) that

$$\begin{aligned}
& \int_\Omega f(x, u, P)\, d\mu \\
\leq \ & \int_\Omega f_\alpha(x, u_\alpha, P_\alpha)\, d\mu + \epsilon \\
\leq \ & \int_{\Omega_l \setminus E_\alpha(K)} f_\alpha(x, u_\alpha, P_\alpha)\, d\mu + \int_{\Omega \setminus (\Omega_l \setminus E_\alpha(K))} f_\alpha^+(x, u_\alpha, P_\alpha)\, d\mu + \epsilon \\
\leq \ & \int_{\Omega_l \setminus E_\alpha(K)} f(x, u_\alpha, P_\alpha)\, d\mu + \int_{\Omega \setminus (\Omega_l \setminus E_\alpha(K))} f^+(x, u_\alpha, P_\alpha)\, d\mu + 3\epsilon \\
\leq \ & \int_\Omega f(x, u_\alpha, P_\alpha)\, d\mu + 4\epsilon, \quad \forall \alpha > \alpha(\epsilon).
\end{aligned}$$

This and the arbitrariness of ϵ imply (2.9). □

Corollary 2.1. *If the hypothesis (iii) is replaced by the hypothesis*

(iii)′: *$f_\alpha \to f$ locally uniformly in the sense of integration on $U \times V$,*

in theorem 2.1, the conclusions of the theorem still hold.

PROOF. Since in the proof of theorem 2.1 the hypothesis (iii) was only used to show that there exists $\alpha(\epsilon, l, K) > 0$ such that

$$|\int_{\Omega_l \setminus E_\alpha(K)} f(x, u_\alpha, P_\alpha) - f_\alpha(x, u_\alpha, P_\alpha)\, d\mu| < \epsilon/4$$

for $\alpha \geq \alpha(\epsilon, l, K)$, the result follows. □

Corollary 2.2. *If the hypothesis (ii) is replaced by the hypothesis*

(ii)′: *f has the strong lower compactness property and $\{f_\alpha\}$ have the uniform lower compactness property,*

in theorem 2.1, the conclusions of the theorem still hold.

PROOF. The proof of the part (a) of theorem 2.1 remains valid, since (ii)′ and the assumption $\int_\Omega f_\alpha(x, u_\alpha(x), P_\alpha(x))\, d\mu \leq C < \infty$ give (ii).

The proof of the part (b) of theorem 2.1 still holds, because (ii)′, (iv) and the assumption $\int_\Omega f(x, u_\alpha(x), P_\alpha(x))\, d\mu \leq C < \infty$ imply

$$\int_\Omega f_\alpha(x, u_\alpha(x), P_\alpha(x))\, d\mu \leq C_1 < \infty$$

and hence (ii). □

Theorem 2.2. *Let Ω be a measurable space with finite positive non-atomic complete measure μ. Let U and V satisfy (H1) and (H2). Let $f, \{f_\alpha\} : \Omega \times R^m \times R^N \to R \cup \{+\infty\}$ satisfy*

- **(i):** *$f, \{f_\beta\}$ are $\mathbf{L} \otimes \mathbf{B}$- measurable;*
- **(ii)$''$:** *f has the lower compactness property and f_α have the uniform lower compactness property.*
- **(iii)$'$:** *$f_\alpha \to f$ locally uniformly in the sense of integration on $U \times V$;*
- **(iv)$'$:** *$f_\alpha^+(x, v, Q) \le a(x) + f^+(x, v, Q)$ for all $(v, Q) \in R^m \times R^N$, where $a(\cdot) \in L^1(\Omega)$ is nonnegative.*

Let $\{u_\alpha\}, u \in U$ and $\{P_\alpha\}, P \in V$ be such that

$$u_\alpha \longrightarrow u, \quad \text{in} \quad U,$$

and

$$P_\alpha \longrightarrow P, \quad \text{in} \quad V.$$

Suppose

$$\int_\Omega f(x, u, P)\, d\mu \le \underline{\lim}_{\alpha\to\infty} \int_\Omega f_\alpha(x, u_\alpha, P_\alpha)\, d\mu.$$

Then

$$\int_\Omega f(x, u, P)\, d\mu \le \underline{\lim}_{\alpha\to\infty} \int_\Omega f(x, u_\alpha, P_\alpha)\, d\mu.$$

Proof. Without loss of generality, we may assume

$$\int_\Omega f(x, u_\alpha(x), P_\alpha(x))\, d\mu \le C_1 < \infty.$$

Thus (ii)$''$, (iv)$'$ give (ii). It is easily seen that (iv)$'$ imply (iv). Hence the theorem follows from the same arguments as in the proof of the part (b) of theorem 2.1. □

So far, only individual sequence $\{(u_\alpha,\ P_\alpha)\}$ satisfying (2.5) and (2.6) is considered. If all sequences $\{(u_\alpha,\ P_\alpha)\}$ satisfying (2.5) and (2.6) are considered at the same time, some Γ-convergence results can be obtained from the above theorems. First, recall that the Γ-limit of a sequence of integrals $I_\alpha(\cdot,\cdot)$ can be defined by

$$\begin{aligned}&\Gamma(U^-, V^-) \lim_{\alpha\to\infty} I_\alpha(u, P)\\ &= \inf\{\underline{\lim}_{\alpha\to\infty} I_\alpha(u_\alpha, P_\alpha) : (u_\alpha, P_\alpha) \to (u, P) \text{ in } U \times V\}\end{aligned}$$

provided it exists and the topologies of U and V are metrizable [13, 14]. We have the following theorem as a consequence of theorem 2.1 and theorem 2.2.

Theorem 2.3. *Let Ω be a measurable space with finite positive non-atomic complete measure μ. Let U and V satisfy (H1), (H2) and be metrizable. Let $f, \{f_\alpha\} : \Omega \times R^m \times R^N \to R \cup \{+\infty\}$ satisfy*

- **(i):** *$f, \{f_\alpha\}$ are $\mathbf{L} \otimes \mathbf{B}$- measurable;*
- **(ii)$''$:** *f has the lower compactness property and f_α have the uniform lower compactness property.*
- **(iii)$'$:** *$f_\alpha \to f$ locally uniformly in the sense of integration on $U \times V$.*

Then, (a):

$$\int_\Omega f(x,u,P)\,d\mu \le \Gamma(U^-,V^-)\lim_{\alpha\to\infty} I_\alpha(u,P), \tag{2.48}$$

provided that $I(u,P;\Omega') = \int_{\Omega'} f(x,u(x),P(x))\,d\mu$ *is lower semicontinuous at* $(u,P)\in U\times V$ *for all measurable subsets* $\Omega'\subset\Omega$*;*
and (b):

$$\int_\Omega f(x,u,P)\,d\mu \le \underline{\lim}_{\alpha\to\infty}\int_\Omega f(x,u_\alpha,P_\alpha)\,d\mu \tag{2.49}$$

for all $(u_\alpha,P_\alpha)\to(u,P)$ *in* $U\times V$, i.e. $\int_\Omega f(x,u,P)\,d\mu$ *is lower semicontinuous at* $(u,P)\in U\times V$*, provided that* $f,\{f_\alpha\}$ *satisfy a further hypothesis*

- **(iv)′:** $f_\alpha^+(x,v,Q)\le a(x)+f^+(x,v,Q)$ *for all* $(v,Q)\in R^m\times R^N$*, where* $a(\cdot)\in L^1(\Omega)$ *is nonnegative;*

and

$$\int_\Omega f(x,u,P)\,d\mu \le \Gamma(U^-,V^-)\lim_{\alpha\to\infty} I_\alpha(u,P).$$

Corollary 2.3. *Let* Ω *be a measurable space with finite positive non-atomic complete measure* μ*. Let* U *and* V *satisfy (H1), (H2) and be metrizable. Let* $f,\{f_\alpha\}:\Omega\times R^m\times R^N\to R\cup\{+\infty\}$ *satisfy*

- **(i):** $f,\{f_\alpha\}$ *are* $\mathbf{L}\otimes\mathbf{B}$*- measurable;*
- **(ii)″:** f *has the lower compactness property and* f_α *have the uniform lower compactness property.*
- **(iii)′:** $f_\alpha\to f$ *locally uniformly in the sense of integration on* $U\times V$*.*
- **(iv)′:** $f_\alpha^+(x,v,Q)\le a(x)+f^+(x,v,Q)$ *for all* $(v,Q)\in R^m\times R^N$*, where* $a(\cdot)\in L^1(\Omega)$ *is nonnegative.*

Then

$$\int_\Omega f(x,u,P)\,d\mu = \Gamma(U^-,V^-)\lim_{\alpha\to\infty} I_\alpha(u,P), \tag{2.50}$$

provided that $I(u,P;\Omega') = \int_{\Omega'} f(x,u(x),P(x))\,d\mu$ *is lower semicontinuous at* $(u,P)\in U\times V$ *for all measurable subsets* $\Omega'\subset\Omega$*. In addition, we have in this case*

$$\Gamma(U^-,V^-)\lim_{\alpha\to\infty} I_\alpha(u,P) = \underline{\lim}_{\alpha\to\infty}\int_\Omega f_\alpha(x,u(x),P(x))\,d\mu. \tag{2.51}$$

Proof. By theorem 2.3, we only need to show that

$$\int_\Omega f(x,u,P)\,d\mu \ge \underline{\lim}_{\alpha\to\infty}\int_\Omega f_\alpha(x,u(x),P(x))\,d\mu. \tag{2.52}$$

This can be easily verified by using the inequality

$$\begin{array}{rl} & \int_\Omega f(x,u(x),P(X))\,d\mu \\ \ge & \int_\Omega f_\alpha(x,u(x),P(X))\,d\mu \\ & +\int_{\Omega_l\setminus E(K)}(f(x,u(x),P(X))-f_\alpha(x,u(x),P(X)))\,d\mu \\ & -\int_{\Omega\setminus(\Omega_l\setminus E(K))} f_\alpha^+(x,u(X),P(X))\,d\mu \\ & +\int_{\Omega\setminus(\Omega_l\setminus E(K))} f^-(x,u(X),P(X))\,d\mu \end{array}$$

for all $l\ge 1$ and $K>0$, where

$$E(K)=\{x\in\Omega : |u(x)|>K \;\text{ or }\; |P(x)|>K\},$$

and by using lemma 2.1 and the hypotheses (ii)$''$, (iii)$'$ and (iv)$'$. □

3. Lower semicontinuity theorems for quasiconvex integrands

In this section, some lower semicontinuity theorems in the form of (1.3) and (1.4) are established for quasiconvex integrand $f(x,u,P)$. In the following, U is taken to be $L^p(\Omega; R^m)$ with strong topology and V is taken to be $L^p(\Omega; R^{m\times n})$ with weak topology, where $1 \le p < \infty$, and μ is taken to be the Lebesgue measure on R^n.

Theorem 3.1. *Let $\Omega \subset R^n$ be open and bounded. Let $f : \Omega \times R^m \times R^{m\times n} \to R$ satisfy*

- **(i):** *$f(\cdot,\cdot,\cdot)$ is a Carathéodory function;*
- **(ii):** *$f(x,u,P)$ is quasiconvex in P;*
- **(iii):** *f has the strong lower compactness property on $U \times V$;*
- **(iv):** *$f(x,u,P) \le a(x) + b(x)(|u|^p + |P|^p)$ for every $x \in \Omega$, $u \in R^m$ and $P \in R^{m\times n}$, where $a(\cdot), b(\cdot) \in L^1(\Omega)$ are nonnegative functions.*

Let $\{f_\beta\} : \Omega \times R^m \times R^{m\times n} \to R$ satisfy

- **(a):** *$f_\beta(\cdot,\cdot,\cdot)$ are $\mathbf{L}\otimes\mathbf{B}$- measurable;*
- **(b):** *f_β have the uniform lower compactness property;*
- **(c):** *$f_\beta \to f$ locally uniformly in $\Omega \times R^m \times R^{m\times n}$.*

Let

$$u_\alpha \rightharpoonup u, \qquad in \quad W^{1,p}(\Omega;\ R^m). \tag{3.53}$$

Then

$$\int_\Omega f(x,u,Du)\, d\mu \le \underline{\lim}_{\alpha\to\infty} \int_\Omega f_\alpha(x,u_\alpha,Du_\alpha)\, d\mu. \tag{3.54}$$

To prove the theorem, we begin with the following lemmas.

Lemma 3.1. *Let $\Omega \subset R^n$ be open and bounded. Let $f : \Omega \times R^m \times R^{m\times n} \to R$ satisfy (i) –(iv) in theorem 3.1. For all integers $\beta \ge 1$, let functions $\hat f_\beta : \Omega \times R^m \times R^{m\times n} \to R$ be defined by*

$$\hat f_\beta(x,u,P) = \max\{-\beta, f(x,u,P)\}.$$

Let $u_\alpha \rightharpoonup u$ in $W^{1,p}(\Omega; R^m)$. Then, for each fixed $\beta \ge 1$,

$$\int_\Omega \hat f_\beta(x,u(x),Du(x))\, d\mu \le \underline{\lim}_{\alpha\to\infty} \int_\Omega \hat f_\beta(x,u_\alpha(x),Du_\alpha(x))\, d\mu. \tag{3.55}$$

Proof. Let

$$E(b,K) = \{x \in \Omega : |b(x)| \le K\}.$$

Denote $\chi_K(x)$ the characteristic function of the set $E(b,K)$. It is easily verified that the functions

$$\tilde f_{\beta,K} = \chi_K\, \hat f_\beta + \beta, \quad \beta \ge 1,\ K \ge 1$$

satisfy the hypotheses of theorem 1.1. Hence we have

$$\begin{aligned} &\int_\Omega \chi_K(x) \hat f_\beta(x,u(x),Du(x))\, d\mu \\ \le\ &\underline{\lim}_{\alpha\to\infty} \int_\Omega \chi_K(x) \hat f_\beta(x,u_\alpha(x),Du_\alpha(x))\, d\mu. \end{aligned}$$

Since

$$\begin{array}{rl} & \int_\Omega \chi_K(x)\hat{f}_\beta(x,u(x),Du(x))\,d\mu \\ \geq & \int_\Omega \hat{f}_\beta^-(x,u(x),Du(x))\,d\mu + \int_\Omega \chi_K(x)\hat{f}_\beta^+(x,u(x),Du(x))\,d\mu \end{array}$$

and

$$\begin{array}{rl} & \int_\Omega \chi_K(x)\hat{f}_\beta(x,u_\alpha(x),Du_\alpha(x))\,d\mu \\ \leq & \int_\Omega \hat{f}_\beta(x,u_\alpha(x),Du_\alpha(x))\,d\mu \;+ \\ & \quad |\int_\Omega (1-\chi_K(x))\hat{f}_\beta^-(x,u_\alpha(x),Du_\alpha(x))\,d\mu| \\ \leq & \int_\Omega \hat{f}_\beta(x,u_\alpha(x),Du_\alpha(x))\,d\mu + I(K,\alpha,\beta), \end{array}$$

we have

$$\begin{array}{rl} & \displaystyle\int_\Omega \hat{f}_\beta^-(x,u(x),Du(x))\,d\mu + \int_\Omega \chi_K(x)\hat{f}_\beta^+(x,u(x),Du(x))\,d\mu \\ \leq & \displaystyle\underline{\lim}_{\alpha\to\infty}\int_\Omega \hat{f}_\beta(x,u_\alpha(x),Du_\alpha(x))\,d\mu + \overline{\lim}_{\alpha\to\infty} I(K,\alpha,\beta), \\ & \quad \text{for fixed } K \text{ and } \beta. \end{array} \tag{3.56}$$

By (iii) and the definition of $\hat{f}_\beta$, we know that for any $\epsilon > 0$ there exists $K(\epsilon) \geq 1$ such that

$$I(K,\alpha,\beta) < \epsilon, \quad \forall \alpha,\beta, \quad \text{and} \quad K \geq K(\epsilon),$$

since $\lim_{K\to\infty}\mu(\Omega \setminus E(b,K)) = 0$.

Thus (3.55) is obtained by letting $k \to \infty$ in (3.56) and passing to the limit. □

Lemma 3.2. *Let $\Omega \subset R^n$ be open and bounded. Let $f : \Omega \times R^m \times R^{m\times n} \to R$ satisfy (i) –(iv) in theorem 3.1. For all integers $\beta \geq 1$, let functions $\hat{f}_\beta : \Omega \times R^m \times R^{m\times n} \to R$ be defined by*

$$\hat{f}_\beta(x,u,P) = \max\{-\beta, f(x,u,P)\}.$$

Let $u_\alpha \rightharpoonup u$ in $W^{1,p}(\Omega;R^m)$. Then

$$\int_\Omega f(x,u(x),Du(x))\,d\mu \leq \underline{\lim}_{\alpha\to\infty}\int_\Omega \hat{f}_\alpha(x,u_\alpha(x),Du_\alpha(x))\,d\mu. \tag{3.57}$$

Proof. Denote

$$F(\beta) = \underline{\lim}_{\alpha\to\infty}\int_\Omega \hat{f}_\beta(x,u_\alpha(x),Du_\alpha(x))\,d\mu. \tag{3.58}$$

Since $f_\beta \geq f_{\beta+1}$ for all β, $F(\beta)$ is nonincreasing. On the other hand, it follows from (iii) that $F(\beta)$ is bounded from below. Hence $\lim_{\beta\to\infty} F(\beta)$ exists. Denote the limit by F.

Since $f \leq \hat{f}_\beta$ for all β, by lemma 3.1, we have

$$\int_\Omega f(x,u(x),Du(x))\,d\mu \leq F. \tag{3.59}$$

Given $\epsilon > 0$, by (3.58) and $F(\beta) \geq F$, there exists $\alpha(\epsilon,\beta) \geq \beta$ such that

$$\int_\Omega \hat{f}_\beta(x,u_\alpha(x),Du_\alpha(x))\,d\mu > F - \epsilon, \quad \forall \alpha \geq \alpha(\epsilon,\beta). \tag{3.60}$$

Let

$$I(\alpha,\beta) = \int_\Omega (\hat{f}_\alpha(x,u_\alpha(x),Du_\alpha(x)) - \hat{f}_\beta(x,u_\alpha(x),Du_\alpha(x)))\,d\mu.$$

We have

$$\begin{array}{rl} I(\alpha,\beta) \geq & \int_{E(\alpha,\beta)} \hat{f}_\alpha(x,u_\alpha(x),Du_\alpha(x))\,d\mu \\ \geq & \int_{E(\alpha,\beta)} f^-(x,u_\alpha(x),Du_\alpha(x))\,d\mu, \;\; \forall \alpha \geq \beta, \end{array}$$

where $E(\alpha,\beta) = \{x \in \Omega : f^-(x, u_\alpha(x), Du_\alpha(x)) < -\beta\}$. Hence, by (iii), for any $\epsilon > 0$ there exists $\beta(\epsilon) > 0$ such that

$$I(\alpha,\beta) > -\epsilon, \quad \forall \beta \geq \beta(\epsilon) \quad \text{and} \quad \alpha \geq \beta. \tag{3.61}$$

It follows from (3.60) and (3.61) that

$$\begin{aligned} & \int_\Omega \hat{f}_\alpha(x, u_\alpha(x), Du_\alpha(x))\, d\mu \\ = \; & \int_\Omega \hat{f}_{\beta(\epsilon)}(x, u_\alpha(x), Du_\alpha(x))\, d\mu + I(\alpha,\beta) \\ > \; & F - 2\epsilon, \quad \forall \alpha \geq \alpha(\epsilon, \beta(\epsilon)). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this and (3.59) imply (3.57). □

Lemma 3.3. *Let $\Omega \subset R^n$ be open and bounded. Let $f : \Omega \times R^m \times R^{m\times n} \to R$ satisfy (i) –(iv) in theorem 3.1. Let $u_\alpha \rightharpoonup u$ in $W^{1,p}(\Omega; R^m)$. Then*

$$\int_{\Omega'} f(x, u(x), Du(x))\, d\mu \leq \underline{\lim}_{\alpha\to\infty} \int_{\Omega'} f(x, u_\alpha(x), Du_\alpha(x))\, d\mu, \tag{3.62}$$

for all measurable subset $\Omega' \subset \Omega$.

Proof. Let $\chi_{\Omega'}$ be the characteristic function of Ω'. Let $\tilde{f} = \chi_{\Omega'} f$, $\tilde{f}_\beta = \max\{-\beta, \chi_{\Omega'} f\}$. It is easily verified that $\tilde{f}$ satisfies the hypotheses (i) – (iv) in theorem 3.1 as well. Hence, by lemma 3.2, we have

$$\int_{\Omega'} f(x, u(x), Du(x))\, d\mu \leq \underline{\lim}_{\alpha\to\infty} \int_{\Omega'} \hat{f}_\alpha(x, u_\alpha(x), Du_\alpha(x))\, d\mu, \tag{3.63}$$

where $\hat{f}_\alpha = \max\{-\alpha, f\}$.

By (iii) and the definition of $\hat{f}_\alpha$, we know that $\hat{f}_\alpha$ have the uniform lower compactness property and $\hat{f}_\alpha \to f$ locally uniformly in $\Omega \times R^m \times R^{m\times n}$. Thus (3.62) follows from (3.63) and the part (a) of theorem 2.1. □

Proof of theorem 3.1. The conclusion follows directly from lemma 3.3 and the part (b) of theorem 2.1. □

In theorem 3.1, the hypothesis that f has the strong compact property is not essential. Actually, we have the following stronger result.

Theorem 3.2. *Let $\Omega \subset R^n$ be open and bounded. Let $u_\alpha \rightharpoonup u$ in $W^{1,p}(\Omega;$ $R^m)$. Let $f, f_\beta : \Omega\times R^m\times R^{m\times n} \to R$ satisfy (i), (ii), (iv) and (a), (c) in theorem 3.1 respectively, and satisfy also the following hypothesis*

(h): *$f_\alpha^-(x, u_\alpha(x), Du_\alpha(x))$ are weakly precompact in $L^1(\Omega)$.*

Then

$$\int_\Omega f(x, u, Du)\, d\mu \leq \underline{\lim}_{\alpha\to\infty} \int_\Omega f_\alpha(x, u_\alpha, Du_\alpha)\, d\mu.$$

Proof. Let $f^N = \max\{f, -N\}$ and $f_\alpha^N = \max\{f_\alpha, -N\}$. Then, for each fixed $N > 0$, f^N and f_α^N satisfy (i) – (iv) and (a) – (c) in theorem 3.1 respectively.

Thus, by theorem 3.1, we have, for each fixed $N > 0$,

$$\int_\Omega f^N(x, u, Du)\, d\mu \leq \underline{\lim}_{\alpha\to\infty} \int_\Omega f_\alpha^N(x, u_\alpha, Du_\alpha)\, d\mu. \tag{3.64}$$

Let $N \to \infty$, by (h) and by passing to the limit in (3.64), we get the result. $\square$

The following lemma is a version of Chacon's biting lemma (see [5], see also [7, 8, 3]).

Lemma 3.4. *Let $\Omega \subset R^n$ be bounded and measurable, and let g_α be a bounded sequence in $L^1(\Omega)$. Then there exist a subsequence g_{α_j} of g_α and a nonincreasing sequence of measurable subsets E_k with $\lim_{k\to\infty} \mu E_k = 0$ such that g_{α_j} are weakly precompact in $L^1(\Omega \setminus E_k)$ for each fixed k,* i.e. *for any $\epsilon > 0$ and fixed k there exists $\delta(\epsilon, k) > 0$ such that*

$$\int_{\Omega'} |g_{\alpha_j}(x)| \, d\mu < \epsilon, \quad \forall j,$$

provided that $\Omega' \subset \Omega \setminus E_k$ and $\mu(\Omega') < \delta(\epsilon, k)$.

Lemma 3.5. *Let $\Omega \subset R^n$ be open and bounded. Let $f, \{f_\beta\} : \Omega \times R^m \times R^{m\times n} \to R$ satisfy*

- **(L):** *$f(x, u, P)$ and $f_\beta(x, u, P)$ are bounded below by $-(a(x) + b(x)(|u|^p + |P|^p))$, where $a(\cdot), b(\cdot) \in L^1(\Omega)$ and $a(x) \geq 0$, $b(x) \geq 0$, $\forall x \in \Omega$.*

Let $u_\alpha \rightharpoonup u$ in $W^{1,p}(\Omega; R^m)$. Then there exist a subsequence u_{α_j} of u_α and a nonincreasing sequence of measurable subsets E_k with $\lim_{k\to\infty} \mu E_k = 0$ such that $f^-(x, u_{\alpha_j}(x), Du_{\alpha_j}(x))$, $f^-_{\alpha_j}(x, u_{\alpha_j}(x), Du_{\alpha_j}(x))$ are weakly precompact in $L^1(\Omega \setminus E_k)$ for each fixed k.

Proof. By lemma 3.4, there exist a subsequence u_{α_j} of u_α and a nonincreasing sequence of measurable subsets $\tilde{E}_k$ with $\lim_{k\to\infty} \mu \tilde{E}_k = 0$ such that $h_{\alpha_j}(x) = (|u_{\alpha_j}(x)|^p + |Du_{\alpha_j}(x)|^p)$ are weakly precompact in $L^1(\Omega \setminus \tilde{E}_k)$ for each fixed k.

Let $\hat{E}_k = \{x \in \Omega : |b(x)| > k\}$ and $E_k = \hat{E}_k \cup \tilde{E}_k$.Then E_k are nonincreasing with $\lim_{k\to\infty} \mu E_k = 0$ and $g_{\alpha_j}(x) = a(x) + b(x)(|u_{\alpha_j}(x)|^p + |Du_{\alpha_j}(x)|^p)$ are weakly precompact in $L^1(\Omega \setminus E_k)$ for each fixed k. This and the hypothesis (L) give the result. $\square$

Theorem 3.3. *Let $\Omega \subset R^n$ be open and bounded. Let $f, \{f_\beta\} : \Omega \times R^m \times R^{m\times n} \to R$ satisfy*

- **(i):** *$f, \{f_\beta\}$ are Carathéodory functions;*
- **(ii):** *$f(x, u, P)$ is quasiconvex in P;*
- **(iii):** *$|f(x, u, P)| \leq a(x) + b(x)(|u|^p + |P|^p)$ and $|f_\beta(x, u, P)| \leq a(x) + b(x)(|u|^p + |P|^p)$ for every $x \in \Omega$, $u \in R^m$ and $P \in R^{m\times n}$, where $a(\cdot), b(\cdot) \in L^1(\Omega)$ are nonnegative functions;*
- **(iv):** *$f_\beta \to f$ locally uniformly in $\Omega \times R^m \times R^{m\times n}$.*

Let

$$u_\alpha \rightharpoonup u, \quad in \quad W^{1,p}(\Omega;\ R^m). \tag{3.65}$$

Then there exist a subsequence u_{α_j} of u_α and a nonincreasing sequence of measurable subsets E_k with $\lim_{k\to\infty} \mu E_k = 0$ such that

$$\begin{aligned} \int_{\Omega\setminus E_k} f(x, u, Du) \, d\mu \quad \leq \quad & \underline{\lim}_{j\to\infty} \int_{\Omega\setminus E_k} f_{\alpha_j}(x, u_{\alpha_j}, Du_{\alpha_j}) \, d\mu, \\ & \textit{for each } \ k. \end{aligned} \tag{3.66}$$

Proof. By (iii), (3.65) and lemma 3.5, there exist a subsequence u_{α_j} of u_α and a nonincreasing sequence of measurable subsets E_k with $\lim_{k\to\infty} \mu E_k = 0$ such that $f^-_{\alpha_j}(x, u_{\alpha_j}(x), Du_{\alpha_j}(x))$ are uniformly weakly precompact in $L^1(\Omega \setminus E_k)$.

Let $f^k = \chi_{\Omega\setminus E_k} f$ and $f^k_{\alpha_j} = \chi_{\Omega\setminus E_k} f_{\alpha_j}$. Applying theorem 3.2 to $f^k, f^k_{\alpha_j}$ and u_{α_j}, we obtain (3.66). $\square$

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