A Theorem on Lower Semicontinuity of Integral Functionals

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Abstract

A general lower semicontinuity theorem, in which not only mappings u_M and P_M but also the integrands f_M depend on M, is proved for integrands f, f_M under certain general hypotheses including that $f(x,u,P)$ is convex respect to P and f_M converge to f locally uniformly, but $f_M(x,u,P)$ are not required to be convex respect to P and $f_M(x,\cdot,\cdot)$ do not even need to be lower semicontinuous. Some more usable criteria, as corollaries of the main theorem, for lower semicontinuity of integral functionals are also given.

1 Introduction

In the present paper, we consider integral functionals of the form

$$I(u,P) = \int_\Omega f(x,u(x),P(x))\,d\mu, \tag{1.1}$$

and

$$I_M(u,P) = \int_\Omega f_M(x,u(x),P(x))\,d\mu, \tag{1.2}$$

where Ω is a measurable space with finite positive nonatomic complete measure μ, $f, f_M : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ are extended real-valued functions satisfying certain hypotheses, and $u : \Omega \to R^m$, $P : \Omega \to R^n$ are measurable functions in two linear topological spaces U and V respectively.

The purpose is to establish, under certain general hypotheses on U, V, f and f_M, a lower semicontinuity theorem of the form

$$I(u,P) \le \underline{\lim}_{M\to\infty} I_M(u_M,P_M), \tag{1.3}$$

for every sequence (u_M, P_M) converging to (u, P) in $U \times V$.

In Reshetnyak's result (see theorem 1.2 in [1]), Ω is taken to be a local compact metric space, $f, f_M : \Omega \times R^n \to R$ are nonnegative functions such that for any $\epsilon > 0$ there is a compact set $A \subset \Omega$ with $\mu(\Omega \setminus A) < \epsilon$ and $f(x,u), f_M(x,u)$ being continuous on $A \times R^n$, $f(x,\cdot), f_M(x,\cdot)$ are convex for almost all $x \in \Omega$, and $f_M \to f$ locally uniformly in $\Omega \times R^n$ as $m \to \infty$, and V is taken to be $L^1(\Omega; R^n)$ with weak topology.

In the case when $f_M \equiv f$, there is a standard lower semicontinuity theorem by Ioffe [2]. To present the theorem, we first introduce some definitions and hypotheses which will also be used throughout this paper.

Definition 1.1 *: A function $f : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ is called* $\mathbf{L} \otimes \mathbf{B}-$ *measurable, if it is measurable respect to the σ-algebra generated by products of measurable subsets of Ω and Borel subsets of $R^m \times R^n$.*

Definition 1.2 *: A function $f : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ is said to satisfy the lower compactness property on $U \times V$ if any sequence of $f^-(x, u_M(x), P_M(x))$ is weakly precompact in $L^1(\Omega)$ whenever u_M converge in U, P_M converge in V and $I(u_M, P_M) \le C < \infty$ for all $M = 1, 2, \cdots$. Here $f^- = \min\{f, 0\}$.*

Let

$$D(\Omega; R^k) = \{v : \Omega \to R^k \mid v \text{ is measurable }\}.$$

We assume that $U \subset D(\Omega; R^m)$ and $V \subset D(\Omega; R^n)$ are decomposable, *i.e.* if $v(\cdot)$ belongs to one of them, then $\chi_T(\cdot)v(\cdot)$ belongs to the same space whenever T is a measurable subset of Ω, where $\chi_T(\cdot)$ is the characteristic function of T:

$$\chi_T(x) = \begin{cases} 1, & \text{if } \ x \in T, \\ 0, & \text{if } \ x \notin T. \end{cases}$$

We assume that U and V satisfy the following hypotheses on their topologies:

(H1) If $v_M(\cdot), M = 1, 2, \cdots$ belong to one of the spaces and converge there to zero and if $\mu T_M \to 0$, then $\chi_{T_M}(\cdot)v(\cdot)$ also converge to zero.

(H2) The topology in U is not weaker than the topology of convergence in measure; the topology of V is not weaker than the topology induced in V by the weak topology of $L^1(\Omega; R^n)$

The following theorem is given by Ioffe [2] in 1977.

Theorem 1.1 (Ioffe) *: Let U and V satisfy (H1) and (H2). Assume that $f(x, u, P)$ is $\mathbf{L} \otimes \mathbf{B}-$ measurable, lower semicontinuous in (u, P) and convex in P. In order that $I(\cdot, \cdot)$ be lower semicontinuous on $U \times V$ and everywhere on $U \times V$ more than $-\infty$, it is necessary and (if $I(\cdot, \cdot)$ is finite at least at one point in $U \times V$) sufficient that f satisfy the lower compactness property.*

Remark 1.1 *: Here and throughout this paper, assumptions and statements are referred to sets with measure-negligible projections on Ω,* i.e. *they hold on a subset $\Omega' \subset \Omega$ with $\mu\,\Omega' = \mu\,\Omega$.*

A lot of developments have since been made, mainly devoted to replacing the convexity conditions by some weaker conditions and for the case when u, P are related in some way(see for example Ball [3], Acerbi and Fusco [4], Ball and Zhang [5]).

The main result of this paper, theorem 2.1, generalizes theorem 1.1 as well as the result of Reshetnyak in the form of (1.3). The proof of the theorem in fact depends on Ioffe's result, *i.e.* theorem 1.1.

The statement and the proof of the main result are given in §2. In §3, some more usable criteria, as corollaries of theorem 2.1, for (1.3) to hold are given. These criteria are natural generalizations of Ioffe's results which are the corollaries of theorem 1.1.

2 Lower Semicontinuity Theorem

Let Ω be a measurable space with finite positive nonatomic complete measure μ.

Before stating the theorem, we introduce some definitions concerning f_M.

Definition 2.1 *: A sequence of functions $f_M : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ is said to have the uniform lower compactness property, if $f_M^-(x, u_M(x), P_M(x))$ are uniformly weakly precompact in $L^1(\Omega)$, in other words (see [2]),*

$f_M^-(x, u_M(x), P_M(x))$ *are equi-uniformly integral continuous on* Ω, *i.e. for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|\int_{\Omega'} f_M^-(x, u_M(x), P_M(x))\, d\mu| < \epsilon$$

for all M *and any measurable subset* $\Omega' \subset \Omega$ *satisfying* $\mu(\Omega') < \delta$, *whenever* $u_M(\cdot)$ *converge in* U, $P_M(\cdot)$ *converge in* V *and* $I_M(u_M, P_M) \leq C < \infty$.

Definition 2.2 *: A sequence of functions* $f_M : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ *is said to converge to* $f : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ *locally uniformly in* $\Omega \times R^m \times R^n$, *if there exists a sequence of measurable subsets* $\Omega_l \subset \Omega$ *with* $\mu(\Omega \setminus \Omega_l) \to 0$ *as* $l \to \infty$ *such that for each* l *and any compact subset* $G \subset R^m \times R^n$

$$f_M(x, u, P) \longrightarrow f(x, u, P), \quad \textit{unifomly on } \Omega_l \times G, \textit{ as } M \to \infty.$$

Remark 2.1 *: When* Ω *is a locally compact metric space,* Ω_l *in definition 2.2 can be taken to be compact subsets of* Ω.

Theorem 2.1 *: Let* U *and* V *satisfy (H1) and (H2). Let* $f : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ *satisfy*

(i) $f(\cdot, \cdot, \cdot)$ *is* $\mathbf{L} \otimes \mathbf{B}$*- measurable,*

(ii) $f(x, \cdot, \cdot)$ *is lower semicontinuous,*

(iii) $f(x, u, \cdot)$ *is convex,*

(iv) $f(x, u, P)$ *has the lower compactness property.*

Let $f_M : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ *satisfy*

(a) $f_M(\cdot, \cdot, \cdot)$ *are* $\mathbf{L} \otimes \mathbf{B}$*- measurable,*

(b) $f_M(x, u, P)$ *have the uniform lower compactness property,*

(c) $f_M \to f$ *locally uniformly in* $\Omega \times R^m \times R^n$.

Let $\{u_M\}, u \in U$ *and* $\{P_M\}, P \in V$ *be such that*

$$u_M \longrightarrow u, \qquad in \quad U, \tag{2.1}$$

and

$$P_M \longrightarrow P, \qquad in \quad V, \tag{2.2}$$

Then

$$\int_\Omega f(x,u,P)\,d\mu \leq \underline{\lim}_{M\to\infty} \int_\Omega f_M(x,u_M,P_M)\,d\mu. \tag{2.3}$$

To prove the theorem, we need the following lemmas.

Lemma 2.1 *: Let* $\{u_M\}, u \in U$ *and* $\{P_M\}, P \in V$ *satisfy (2.1) and (2.2) respectively. Let*

$$E^1_M(K) \quad = \{x \in \Omega : |u_M(x)| > K\}, \tag{2.4}$$

$$E^2_M(K) \quad = \{x \in \Omega : |P_M(x)| > K\}, \tag{2.5}$$

and

$$E_M(K) = E^1_M(K) \cup E^2_M(K). \tag{2.6}$$

Then

$$\mu\,E_M(K) \longrightarrow 0, \quad \textit{uniformly for } M \textit{ as} \quad K \to \infty. \tag{2.7}$$

Proof: For any $\epsilon > 0$, since $u \in U$, there exists $K_1(\epsilon) > 1$ such that

$$\mu\,\{x \in \Omega : |u(x)| > K\} < \epsilon/2, \quad \forall K > K_1(\epsilon).$$

Thus, by (2.1) and (H2), there exists $M(\epsilon) > 1$ such that

$$\mu\,E^1_M(K) < \epsilon/2, \quad \forall M > M(\epsilon) \quad \text{and} \quad K > K_1(\epsilon) + 1. \tag{2.8}$$

Since $u_M \in U$ for each M, we have

$$\lim_{K\to\infty} \mu E^1_M(K) = 0, \qquad \text{for each } M.$$

Thus, for $M \in \{1, 2, \cdots, M(\epsilon)\}$, there exists $K_2(\epsilon) > 1$ such that

$$\mu\, E_M^1(K) < \epsilon/2, \quad \forall M \in \{1, 2, \cdots, M(\epsilon)\} \quad \text{and} \quad K > K_2(\epsilon). \tag{2.9}$$

Let $K(\epsilon) = \max\{K_1(\epsilon) + 1, K_2(\epsilon)\}$, then (2.8) and (2.9) give

$$\mu\, E_M^1(K) < \epsilon/2, \quad \forall M \geq 1 \quad \text{and} \quad K > K(\epsilon). \tag{2.10}$$

On the other hand, it follows from (2.2) and (H2) that

$$\int_\Omega |P_M(x)|\, d\mu \leq C,$$

for some constant $C > 0$. Thus, for any $\epsilon > 0$ there exists $K(\epsilon) > 1$ such that

$$\mu\, E_M^2(K) < \epsilon/2, \quad \forall M \geq 1 \quad \text{and} \quad K > K(\epsilon). \tag{2.11}$$

Hence (2.7) follows from (2.10) and (2.11). □

Lemma 2.2 *: Let $f, \{f_M\}$ satisfy the hypotheses in theorem 2.1. Let $\bar{f}_A : \Omega \times R^m \times R^n \to R$ be defined by*

$$\bar{f}_A(x, u, P) = \min\{A, f(x, u, P)\}. \tag{2.12}$$

Let $\{u_M\}, u \in U$ and $\{P_M\}, P \in V$ satisfy (2.1) and (2.2) respectively. Let $\{\Omega_l\}$ be a sequence of measurable subsets of Ω, the existence of which is guaranteed by the hypothesis (c) for f_M, such that

$$\mu\,(\Omega \setminus \Omega_l) \longrightarrow 0, \quad \textit{as} \;\; l \to \infty, \tag{2.13}$$

and

$$f_M \longrightarrow f, \quad \textit{uniformly on } \Omega_l \times G, \tag{2.14}$$

for each l and any compact set $G \subset R^m \times R^n$.

Suppose

$$\int_\Omega f_M(x, u_M, P_M)\, d\mu \leq C,$$

for some constant $C > 0$.

Then, for any $\epsilon > 0, A > 1$, there exist $l(\epsilon) \geq 1$ and $M(\epsilon, A, l) \geq 1$ such that

$$\begin{aligned} \textstyle\int_{\Omega_l} \bar{f}_A(x, u_M, P_M)\, d\mu \;\; &\leq \textstyle\int_\Omega f_M(x, u_M, P_M)\, d\mu + \epsilon \\ &\forall l \geq l(\epsilon) \quad \textit{and} \quad M \geq M(\epsilon, A, l). \end{aligned} \tag{2.15}$$

Proof:

$$
\begin{aligned}
& \textstyle\int_{\Omega_l} \bar{f}_A(x,u_M,P_M)\,d\mu \\
= \;& \textstyle\int_\Omega f_M(x,u_M,P_M)\,d\mu + \int_{\Omega\setminus\Omega_l}(-f_M(x,u_M,P_M))\,d\mu + \\
& + \textstyle\int_{\Omega_l}(\bar{f}_A(x,u_M,P_M) - f_M(x,u_M,P_M))\,d\mu \\
= \;& \textstyle\int_\Omega f_M(x,u_M,P_M)\,d\mu + I_1 + I_2.
\end{aligned}
$$

By (2.1), (2.2), (2.13) and (b), there exists $l(\epsilon) > 0$ such that

$$
\begin{aligned}
I_1 = \;& \textstyle\int_{\Omega\setminus\Omega_l}(-f_M(x,u_M,P_M))\,d\mu \\
\leq \;& \textstyle\int_{\Omega\setminus\Omega_l}(-f_M^-(x,u_M,P_M))\,d\mu \\
< \;& \epsilon/2, \qquad \text{if} \quad l \geq l(\epsilon).
\end{aligned}
\tag{2.16}
$$

By (2.12), we have

$$
\begin{aligned}
I_2 \leq \;& \textstyle\int_{\Omega_l\setminus E_M(K)}(f(x,u_M,P_M) - f_M(x,u_M,P_M))\,d\mu + \\
& + \textstyle\int_{E_M(K)}(A - f_M(x,u_M,P_M))\,d\mu \\
= \;& I_{21} + I_{22},
\end{aligned}
$$

where $E_M(K)$ is defined by (2.6).

By lemma 2.1, $E_M(K) \to 0$ uniformly for M as $K \to \infty$. Thus it follows from (b) that there exists $K(\epsilon, A) > 1$ such that

$$
\begin{aligned}
I_{22} \leq \;& \textstyle\int_{E_M(K)}(A - f_M^-(x,u_M,P_M))\,d\mu \\
< \;& \epsilon/4, \qquad \text{if} \quad K \geq K(\epsilon, A).
\end{aligned}
$$

Let $\bar{K} = K(\epsilon, A)$, then

$$
G(K) = \{u \in R^m : |u| \leq \bar{K}\} \times \{P \in R^n : |P| \leq \bar{K}\}
$$

is a compact set in $R^m \times R^n$. It follows from (iv), (b), (2.14) and the boundedness of $\int_\Omega f_M(x,u_M,P_M)\,d\mu$ that there exists $M(\epsilon, A, l) > 0$ such that

$$
|I_{21}| < \epsilon/4, \qquad \forall M \geq M(\epsilon, A, l).
$$

Thus, we have

$$
|I_2| < \epsilon/2, \qquad \forall M \geq M(\epsilon, A, l). \tag{2.17}
$$

Thus (2.15) follows from (2.16) and (2.17). □

Lemma 2.3 *: Let $f, \{f_M\}$ satisfy the hypotheses in theorem 2.1. Let $\{u_M\}, u \in U$ and $\{P_M\}, P \in V$ satisfy (2.1) and (2.2) respectively. Let*

$$F(M, A) = \{x \in \Omega : f(x, u_M(x), P_M(x)) > A\}. \tag{2.18}$$

Suppose

$$\int_\Omega f_M(x, u_M, P_M)\, d\mu \le C, \tag{2.19}$$

for some constant $C > 0$.

Then, for any $\epsilon > 0$ and $K \ge 1$, there exist $A(\epsilon) > 1$ and $M(\epsilon, K) > 1$ such that

$$\begin{aligned} \mu\, F(M, A) \;\; & \le \mu\, E_M(K) + \epsilon, \\ & \textit{if } \; A \ge A(\epsilon) \quad \textit{and} \quad M \ge M(\epsilon, K), \end{aligned} \tag{2.20}$$

where $E_M(K)$ is defined by (2.6).

Proof: By (c), there is a sequence of measurable subsets $\{\Omega_l\}$ in Ω such that

$$\mu\,(\Omega \setminus \Omega_l) \longrightarrow 0, \qquad \text{as } l \to \infty, \tag{2.21}$$

and

$$f_M \longrightarrow f, \qquad \text{uniformly on } \Omega_l \times G, \tag{2.22}$$

for each l and any compact set $G \subset R^m \times R^n$.

For any $\epsilon > 0$, by (2.21), there is $l_1(\epsilon) \ge 1$ such that

$$\mu\,(\Omega \setminus \Omega_l) < \epsilon/2, \qquad \text{if } l \ge l_1(\epsilon). \tag{2.23}$$

By (2.19),

$$\begin{aligned} & \textstyle\int_{\Omega_l \setminus E_M(K)} f(x, u_M, P_M)\, d\mu \\ = \;\; & \textstyle\int_{\Omega_l \setminus E_M(K)} (f(x, u_M, P_M) - f_M(x, u_M, P_M))\, d\mu \\ & + \textstyle\int_{\Omega \setminus (\Omega_l \setminus E_M(K))} (-f_M(x, u_M, P_M))\, d\mu + C \\ = \;\; & I_1 + I_2 + C, \end{aligned}$$

It follows from (b) that

$$\begin{aligned} I_2 \;\; & \le \;\; \int_{\Omega \setminus (\Omega_l \setminus E_M(K))} (-f_M^-(x, u_M, P_M))\, d\mu \\ & \le \;\; \int_\Omega (-f_M^-(x, u_M, P_M))\, d\mu \\ & \le \;\; C_1, \end{aligned} \tag{2.24}$$

for some constant $C_1 > 0$.

It follows from (2.22), (iv), (b) and (2.19) that there exists $M(l, K) > 1$ such that

$$\begin{aligned} |I_1| &\leq \textstyle\int_{\Omega_l \setminus E_M(K)} \quad |f(x, u_M, P_M) - f_M(x, u_M, P_M)|\, d\mu \\ &\leq 1, \qquad\qquad \forall M \geq M(l, K). \end{aligned} \tag{2.25}$$

Thus we have

$$\int_{\Omega_l \setminus E_M(K)} f(x, u_M, P_M)\, d\mu \leq C_2, \quad \forall M \geq M(l, K), \tag{2.26}$$

where $C_2 = C + C_1 + 1$ is a constant.

Denote $\Omega_M^- = \{x \in \Omega : f(x, u_M(x), P_M(x)) < 0\}$ and $f^+ = \max\{f, 0\}$; then, by (2.26)

$$\begin{aligned} & \textstyle\int_{\Omega_l \setminus E_M(K)} f^+(x, u_M, P_M)\, d\mu \\ \leq\ & \textstyle\int_{\Omega_l \setminus E_M(K)} (-f^-(x, u_M, P_M))\, d\mu + C_2 \\ =\ & \textstyle\int_{(\Omega_l \setminus E_M(K)) \cap \Omega_M^-} (-f(x, u_M, P_M))\, d\mu + C_2 \\ =\ & \textstyle\int_{(\Omega_l \setminus E_M(K)) \cap \Omega_M^-} (f_M(x, u_M, P_M) - f(x, u_M, P_M))\, d\mu \\ & \quad + \textstyle\int_{(\Omega_l \setminus E_M(K)) \cap \Omega_M^-} (-f_M(x, u_M, P_M))\, d\mu + C_2 \\ \leq\ & \textstyle\int_{\Omega_l \setminus E_M(K)} |f(x, u_M, P_M) - f_M(x, u_M, P_M)|\, d\mu \\ & \quad + \textstyle\int_{\Omega} (-f_M^-(x, u_M, P_M))\, d\mu + C_2. \end{aligned}$$

It follows from this and (2.24), (2.25) that

$$\int_{\Omega_l \setminus E_M(K)} f^+(x, u_M, P_M)\, d\mu \leq C_3, \quad \forall M \geq M(l, K), \tag{2.27}$$

where $C_3 = C_1 + C_2 + 1$ is a constant.

Now (2.27) implies that there exists $A(\epsilon) > 0$ such that

$$\begin{aligned} \mu \{x \in \Omega_l \setminus E_M(K) : \quad & f(x, u_M(x), P_M(x)) > A\} < \epsilon/2, \\ & \text{if } A \geq A(\epsilon) \text{ and } M \geq M(l, K). \end{aligned} \tag{2.28}$$

Since

$$F(M, A) \subset E_M(K) \cup (\Omega \setminus \Omega_l) \cup F(l, K, M, A),$$

where $F(l, K, M, A) = \{x \in \Omega_l \setminus E_M(K) : f(x, u_M(x), P_M(x)) > A\}$, we have

$$\mu\, F(M, A) \leq \mu\, E_M(K) + \mu\, (\Omega \setminus \Omega_l) + \mu\, F(l, K, M, A).$$

Taking $l = l_1(\epsilon)$ and $M(\epsilon, K) = M(l_1(\epsilon), K)$, by (2.23) and (2.28), we conclude that (2.20) is true. $\square$

Proof of Theorem 2.1:
Without loss of generality, we assume that

$$\int_\Omega f_M(x, u_M, P_M)\, d\mu \le C,$$

for some constant $C > 0$. It follows from (c) that there exists a sequence of measurable subsets $\{\Omega_l\}$ of Ω such that

$$\Omega_l \subset \Omega_{l+1}, \ \ \forall l, \quad \text{and} \quad \lim_{l\to\infty} \mu\,(\Omega \setminus \Omega_l) = 0, \tag{2.29}$$

and

$$f_M \longrightarrow f, \qquad \text{uniformly on} \quad \Omega_l \times G, \tag{2.30}$$

for each l and any compact set $G \subset R^m \times R^n$.

Let $E_M(K)$ be defined by (2.6). It follows from lemma 2.1 that there exists an increasing sequence $\{K_i\}$ such that

$$\sum_{i=1}^{\infty} \sup_{1\le M<\infty} \{\mu\, E_M(K_i)\} < \infty. \tag{2.31}$$

Let $\epsilon_i > 0, i = 1, 2, \cdots$ be a decreasing sequence of numbers satisfying $\lim_{i\to\infty} \epsilon_i = 0$. Let $A_i = A(\epsilon_i/2^i)$, $\ l_i = l(\epsilon_i)$, $\ M_i = \max\{M(l_i, K_i),$ $M(\epsilon_i, A_i, l_i)\}$ and $F_i = F(M_i,\ A_i) = \{x \in \Omega : f(x, u_{M_i}(x), P_{M_i}(x)) > A_i\}$ where $A(\cdot)$, $\ M(\cdot,\ \cdot)$ are defined by lemma 2.3 and $l(\cdot)$, $\ M(\cdot,\ \cdot,\ \cdot)$ are defined by lemma 2.2. Then, by lemma 2.2, we have

$$\int_{\Omega_{l_i}} \bar f_{A_i}(x, u_{M_i}, P_{M_i})\, d\mu \le \int_\Omega f_{M_i}(x, u_{M_i}, P_{M_i})\, d\mu + \epsilon_i \quad \forall i, \tag{2.32}$$

and by lemma 2.3, we have

$$\sum_{i=1}^{\infty} \mu\ F_i \le \sum_{i=1}^{\infty} (\mu\ E_{M_i}(K_i) + \epsilon_i/2^i) < \infty. \tag{2.33}$$

Let $H_j = ((\Omega \setminus \Omega_{l_j}) \cup (\cup_{i\ge j} F_i))$ and $G_j = \Omega \setminus H_j$. It follows from (2.28) and (2.32) that

$$G_j \subset G_{j+1}, \ \ \forall j, \quad \text{and} \quad \lim_{j\to\infty} (\Omega \setminus G_j) = 0. \tag{2.34}$$

Thus, by the definition of $\bar{f}_{A_i}$ and F_i, we have

$$\begin{array}{ll} & \int_{G_j} f(x, u_{M_i}, P_{M_i})\, d\mu \\ = & \int_{\Omega_{l_i}\setminus F_i} f(x, u_{M_i}, P_{M_i})\, d\mu \quad + \int_{(\Omega_{l_i}\setminus F_i)\setminus G_j} (-f(x, u_{M_i}, P_{M_i}))\, d\mu \\ \leq & \int_{\Omega_{l_i}\setminus F_i} f(x, u_{M_i}, P_{M_i})\, d\mu \quad + \int_{\Omega\setminus G_j} (-f^-(x, u_{M_i}, P_{M_i}))\, d\mu \\ \leq & \int_{\Omega_{l_i}\setminus F_i} \bar{f}_{A_i}(x, u_{M_i}, P_{M_i})\, d\mu \quad + \int_{H_j} (-f^-(x, u_{M_i}, P_{M_i}))\, d\mu, \quad \forall i \geq j. \end{array}$$

It follows from this and (2.32) that

$$\begin{array}{ll} & \int_{G_j} f(x, u_{M_i}, P_{M_i})\, d\mu \\ \leq & \int_{\Omega} f_{M_i}(x, u_{M_i}, P_{M_i})\, d\mu \\ & + \int_{H_j} (-f^-(x, u_{M_i}, P_{M_i}))\, d\mu \quad +\epsilon_i, \quad \forall i \geq j. \end{array} \tag{2.35}$$

Let $i \to \infty$ in (2.35). By (i) – (iv), (2.1), (2.2) and theorem 1.1, we have

$$\begin{array}{ll} & \int_{G_j} f(x, u, P)\, d\mu \\ \leq & \underline{\lim}_{i\to\infty} \int_{\Omega} f_{M_i}(x, u_{M_i}, P_{M_i})\, d\mu \\ & + \overline{\lim}_{i\to\infty} \int_{H_j} (-f^-(x, u_{M_i}, P_{M_i}))\, d\mu. \end{array} \tag{2.36}$$

By (iv) and (2.34), we have

$$\lim_{j\to\infty} (\sup_{i\geq 1} \int_{H_j} (-f^-(x, u_{M_i}, P_{M_i}))\, d\mu) = 0.$$

It follows from this and (2.34), (2.36) that

$$\begin{array}{ll} \int_{\Omega} f(x, u, P)\, d\mu & = \lim_{j\to\infty} \int_{G_j} f(x, u, P)\, d\mu \\ & \leq \underline{\lim}_{i\to\infty} \int_{\Omega} f_{M_i}(x, u_{M_i}, P_{M_i})\, d\mu. \end{array}$$

This completes the proof. □

Notice that the hypothesis (c) for f_M was only used to show that there exists $M(\epsilon, l, K) > 0$ such that

$$\left| \int_{\Omega_l \setminus E_M(K)} f(x, u_M, P_M) - f_M(x, u_M, P_M)\, d\mu \right| < \epsilon/4,$$

for $M \geq M(\epsilon, l, K)$. We may replace it with a weaker hypothesis

(c') There exists a sequence of measurable subsets Ω_l in Ω such that

$$\lim_{l\to\infty} \mu(\Omega \setminus \Omega_l) = 0$$

and

$$\int_{\Omega_l \setminus E(u,P,K)} f_M(x, u(x), P(x))\, d\mu \to \int_{\Omega_l \setminus E(u,P,K)} f(x, u(x), P(x))\, d\mu$$

uniformly in $U \times V$ for each l and any fixed $K > 0$, where

$$E(u, P, K) = \{x \in \Omega : |u(x)| > K \quad \text{or} \quad |P(x)| > K\}.$$

Theorem 2.2 *: Let U and V satisfy (H1) and (H2). Let $f : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ satisfy (i) – (iv) in theorem 2.1. Let $f_M : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ satisfy (a), (b) in theorem 2.1 and (c$'$) above. Let $\{u_M\}, u \in U$ and $\{P_M\}, P \in V$ satisfy (2.1) and (2.2) respectively. Then*

$$\int_\Omega f(x, u, P)\; d\mu \leq \underline{\lim}_{M\to\infty} \int_\Omega f_M(x, u_M, P_M)\; d\mu.$$

3 Some Corollaries

In this section, some more usable criteria, which are the generalizations of Ioffe's corresponding theorems [2], as corollaries of theorem 2.1, for lower semicontinuity of integral functionals of the form (1.3) are given. In fact, all the lower semicontinuity theorems which can be covered by theorem 1.1, in the case $f_M \equiv f$, can be generalized to the form (1.3) and can be covered by theorem 2.1 and theorem 2.2.

Theorem 3.1 *: Let $U = L^1(\Omega;\ R^m)$ with the topology induced by the L^1-norm and $V = L^1(\Omega;\ R^n)$ with the weak topology. Let $f : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ satisfy*

(i) *$f(\cdot, \cdot, \cdot)$ is $\mathbf{L} \otimes \mathbf{B}$- measurable,*

(ii) *$f(x, \cdot, \cdot)$ is lower semicontinuous,*

(iii) *$f(x, u, \cdot)$ is convex,*

(iv) *$f(x, u, P) \geq -c(|u| + |P|) + b(x)$ for some $c \in R$, and $b(\cdot) \in L^1(\Omega)$.*

Let $f_M : \Omega \times R^m \times R^n \to R \cup \{+\infty\}$ satisfy

(a) *$f_M(\cdot, \cdot, \cdot)$ are $\mathbf{L} \otimes \mathbf{B}$- measurable,*

(b) *$f_M \to f$ locally uniformly in $\Omega \times R^m \times R^n$,*

(c) *$f_M(x, u, P) \geq -c(|u|+|P|)+b(x)$ for some $c \in R$, and $b(\cdot) \in L^1(\Omega)$.*

Let $\{u_M\}, u \in L^1(\Omega;\ R^m)$ and $\{P_M\}, P \in L^1(\Omega;\ R^n)$ be such that

$$u_M \to u, \qquad in \quad L^1(\Omega;\ R^m), \tag{3.1}$$

and

$$P_M \rightharpoonup P, \qquad in \quad L^1(\Omega;\ R^n), \tag{3.2}$$

Then

$$\int_\Omega f(x, u, P)\ d\mu \leq \underline{\lim}_{M\to\infty} \int_\Omega f_M(x, u_M, P_M)\ d\mu. \tag{3.3}$$

Proof: It is obvious that U and V taken in the theorem satisfy (H1) and (H2).

Since $u_M(\cdot)$ converge to $u(\cdot)$ in U implies that $|u_M(\cdot)|$ converge strongly in $L^1(\Omega;\ R^m)$ and $P_M(\cdot)$ converge to $P(\cdot)$ in V implies that $|P_M(\cdot)|$ is weakly precompact in $L^1(\Omega;\ R^n)$, by (iv) and (c), we conclude that f has the lower compactness property, and f_M have uniform lower compactness property. Thus (3.3) follows from theorem 2.1. □

Now, we consider that $\Omega \subset R^k$ is a bounded open set, which is the case in most applications.

Definition 3.1 *: A function $f : \Omega \times R^m \times R^n \to R$ is said to be a Carathéodory function if*

(i) *$f(\cdot, u, P)$ is measurable for every $u \in R^m$ and $P \in R^n$,*

(ii) *$f(x, \cdot, \cdot)$ is continuous for almost every $x \in \Omega$.*

It is easy to see that Carathéodory functions are $\mathbf{L} \otimes \mathbf{B}$- measurable and lower semicontinuous

Theorem 3.2 *: Let $U = L^1(\Omega;\ R^m)$ with the topology induced by the L^1-norm and $V = L^1(\Omega;\ R^n)$ with the weak topology. Let $f : \Omega \times R^m \times R^n \to R$ satisfy*

(i) *$f(\cdot,\cdot,\cdot)$ is a Carathéodory function,*

(ii) *$f(x,u,\cdot)$ is convex,*

(iii) *$f(x,u,P) \geq -c(|u|+|P|)+b(x)$ for some $c \in R$, and $b(\cdot) \in L^1(\Omega)$.*

Let $f_M : \Omega \times R^m \times R^n \to R$ satisfy

(a) *$f_M(\cdot,\cdot,\cdot)$ are $\mathbf{L} \otimes \mathbf{B}$- measurable,*

(b) *$f_M \to f$ locally uniformly in $\Omega \times R^m \times R^n$,*

(c) *$f_M(x,u,P) \geq -c(|u|+|P|)+b(x)$ for some $c \in R$, and $b(\cdot) \in L^1(\Omega)$.*

Let $\{u_M\}, u \in L^1(\Omega;\ R^m)$ and $\{P_M\}, P \in L^1(\Omega;\ R^n)$ be such that

$$u_M \to u, \qquad in \quad L^1(\Omega;\ R^m), \tag{3.4}$$

and

$$P_M \rightharpoonup P, \qquad in \quad L^1(\Omega;\ R^n), \tag{3.5}$$

Then

$$\int_\Omega f(x,u,P)\ d\mu \leq \underline{\lim}_{M\to\infty} \int_\Omega f_M(x,u_M,P_M)\ d\mu. \tag{3.6}$$

Proof: The theorem is a direct corollary of theorem 3.1. □

Theorem 3.3 *: Let $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Let $U = L^p(\Omega;\ R^m)$ with the topology induced by the L^p-norm and $V = L^q(\Omega;\ R^n)$ with the weak topology. Let $f : \Omega \times R^m \times R^n \to R$ satisfy*

(i) *$f(\cdot,\cdot,\cdot)$ is a Carathéodory function,*

(ii) *$f(x,u,\cdot)$ is convex,*

(iii) *$f(x,u,P) \geq b(x) \quad for \quad b(\cdot) \in L^1(\Omega)$.*

Let $f_M : \Omega \times R^m \times R^n \to R$ satisfy

(a) *$f_M(\cdot,\cdot,\cdot)$ are $\mathbf{L} \otimes \mathbf{B}$- measurable,*

(b) *$f_M \to f$ locally uniformly in $\Omega \times R^m \times R^n$,*

(c) *$f_M(x,u,P) \geq b(x)$ for some $b(\cdot) \in L^1(\Omega)$.*

Let $\{u_M\}, u \in L^p(\Omega;\ R^m)$ and $\{P_M\}, P \in L^q(\Omega;\ R^n)$ be such that

$$u_M \to u, \qquad in \quad L^p(\Omega;\ R^m), \tag{3.7}$$

and

$$P_M \rightharpoonup P, \qquad in \quad L^q(\Omega;\ R^n), \tag{3.8}$$

Then

$$\int_\Omega f(x,u,P)\, d\mu \leq \underline{\lim}_{M\to\infty} \int_\Omega f_M(x,u_M,P_M)\, d\mu. \tag{3.9}$$

Proof: It is obvious that U and V taken in the theorem satisfy (H1) and (H2).

It follows from (iii) and (c) that f has the lower compactness property, and f_M have uniform lower compactness property. Thus, the theorem follows from theorem 2.1. □

As a corollary of theorem 3.3, we have

Theorem 3.4 *: The conclusion of theorem 3.3 remains true if the hypothesis (a) for f_M is replaced by*

(a′) *f_M are Carathéodory functions.*

Remark 3.1 *: As corollaries of theorem 2.2, theorem 3.1 — theorem 3.4 still hold if the hypothesis (b) for f_M is replaced by the hypothesis (c′) in §2.*

Remark 3.2 *: As an application, theorem 3.3 can be used to simplify the proofs in [6] where the element removal method for singular minimizers is proved to be able to overcome the Lavrentiev phenomenon. An application of theorem 3.4 can be found in [7].*

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