LAMINATED MICROSTRUCTURE IN VARIATIONAL PROBLEMS WITH NON-RANK-ONE CONNECTED DOUBLE WELL POTENTIAL

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ABSTRACT. Variational problems with non-rank-one connected double well potential are considered. It is proved that the problem has a laminated microstructure solution which is uniquely determined by the two potential wells yet certainly not characterized by the fine oscillations between them because of the lack of rank-one connection. The laminated microstructure is explicitly worked out. It is also shown that an application of a nonconforming finite element method can cause over-relaxation and fail to approximate the right microstructure.

1. INTRODUCTION

Variational problems that are not quasiconvex can fail to attain a minimum value, and the minimizing sequences for such problems can consist of finer and finer oscillations and lead to microstructures [1, 2, 3, 4]. A typical example of such a problem is the double well system in which the two potential wells have a rank-one connection [2, 5]. It is well known that such problems can have a laminated microstructure solution which is characterized by the fine oscillations between the two potential wells [2]. Finite element methods, including nonconforming finite element methods, are known to be successfully applied to solve the so called double well problems (see [6]-[12] among many others).

In the present paper, we consider a double well problem in which the two potential wells are not in rank-one connection. Since the lack of rank-one connection, there can be no oscillations between the two potential wells. However, it is shown in Sec. 2 that there exist two states with rank-one connection such that the fine oscillations between the two states form a minimizing sequence for the problem. It is proved that such a laminated microstructure solution is uniquely determined by the two potential wells. An explicit formula of the microstructure for the problem in given in Sec. 2. In Sec. 3, a nonconforming finite element method is applied to solve the problem, and it turns out that the method can cause over-relaxation, that is to reach an energy less than the infimum, and fail to approximate the right microstructure.

2. The problem and its laminated microstructure

Let $B = (b_{ij}) \in R^{2\times 2}$ be a real symmetric matrix satisfying

$$b_{ii} > 0,\ i = 1, 2;\ \ b_{12} = b_{21} < 0;\ \text{and}\ \det B > 0. \tag{2.1}$$

Let $f : R^{2\times 2} \to R$ be given by

$$f(A) = \langle A - B,\ A - B\rangle \cdot \langle A + B,\ A + B\rangle, \quad \forall A \in R^{2\times 2}, \tag{2.2}$$

where $\langle A,\ C\rangle = \sum_{i,j=1}^{2} a_{ij}c_{ij}$ denotes the inner product in $R^{2\times 2}$. Let $\Omega \subset R^2$ be a bounded open set with Lipschitz continuous boundary. Consider the problem of minimizing the integral functional

$$F(\vec{u}) = \int_\Omega f(\nabla \vec{u}(x))\, dx \tag{2.3}$$

in the set of admissible functions

$$\mathbb{A} = \{\vec{u} \in W^{1,4}(\Omega; R^2) : \vec{u} = \vec{0} \text{ on } \partial\Omega\}, \tag{2.4}$$

where, and in what follows, $\vec{0}$ denotes the origin of the space in question.

It is easily seen that B and $-B$ are the only two potential wells of the energy density $f(\cdot)$, and there is no rank-one connection between the two potential wells B and $-B$, since by (2.1)

$$\det(B - (-B)) = \det(2B) = 4 \det B > 0.$$

Let $Qf(\cdot)$ be the quasiconvex envelope [1, 5] of $f(\cdot)$. It is well known [1, 5] that the problem of minimizing the relaxed integral functional

$$QF(\vec{u}) = \int_\Omega Qf(\nabla \vec{u}(x))\, dx \tag{2.5}$$

in $\mathbb{A}$ has a solution $\vec{u}(x) \equiv \vec{0}$ and

$$QF(\vec{0}) = Qf(\vec{0})\, \text{meas}(\Omega) = \inf_{\vec{u}\in\mathbb{A}} F(\vec{u}). \tag{2.6}$$

Thus, assuming that the measure of Ω is known, to calculate the infimum value of $F(\cdot)$ in $\mathbb{A}$ is equivalent to evaluate $Qf(\vec{0})$.

Let $Pf(\cdot)$ and $Rf(\cdot)$ be the polyconvex envelope and rank-one convex envelope of $f(\cdot)$ respectively [5].

Lemma 2.1. *We have*

$$\begin{aligned} Pf(\vec{0}) &= \sup_{\substack{Y\in R^{2\times 2}\\ \lambda\in R}} \inf_{A\in R^{2\times 2}} [f(A) - \langle Y,\ A\rangle - \lambda \det A] \\ &\geq \sup_{\lambda\in R} \inf_{A\in R^{2\times 2}} [f(A) - \lambda \det A], \end{aligned} \tag{2.7}$$

$$\begin{aligned} Rf(\vec{0}) &\leq \inf\{\lambda_1 f(A_1) + \lambda_2 f(A_2) : \lambda_1 A_1 + \lambda_2 A_2 = \vec{0}, \\ &\qquad\qquad \lambda_1 \geq 0, \lambda_2 \geq 0; \lambda_1 + \lambda_2 = 1\} \\ &\leq \inf\{\frac{1}{2} f(A) + \frac{1}{2} f(-A) : \det A = 0\}. \end{aligned} \tag{2.8}$$

Proof. The lemma follows directly from a theorem (theorem 1.1 of chapter 5) in [5]. □

Theorem 2.1. *We have*

$$\begin{aligned} &Pf(\vec{0}) = Qf(\vec{0}) = Rf(\vec{0}) \\ =&\sup_{\lambda\in R} \inf_{A\in R^{2\times 2}} [f(A) - \lambda \det A] \\ =&\inf\{f(A) : \det A = 0\}. \end{aligned} \tag{2.9}$$

Proof. Since $f(A) = f(-A)$, by (2.8), we have

$$Rf(\vec{0}) \leq \inf\{f(A) : \det A = 0\}.$$

Since

$$\inf\{f(A) : \det A = 0\} \leq \sup_{\lambda\in R} \inf_{A\in R^{2\times 2}} [f(A) - \lambda \det A]$$

and $Pf \leq Qf \leq Rf$ (see [5]), (2.9) follows from (2.7). □

In view of theorem 2.1, to evaluate $Qf(\vec{0})$ is equivalent to solve the problem

$$\begin{cases} \text{find } \hat{A} \in R^{2\times 2} \text{ such that } \det \hat{A} = 0 \text{ and} \\ f(\hat{A}) = \inf\{f(A) : \det A = 0\}. \end{cases} \tag{2.10}$$

Lemma 2.2. *Suppose $\hat{A}$ is a solution of the problem* (2.10)*. Then, there exists a $\hat{\lambda} \in R$ such that*

$$4(\langle \hat{A},\ \hat{A}\rangle + \langle B,\ B\rangle)\hat{A} - 8\langle \hat{A},\ B\rangle B - \hat{\lambda}\,\mathrm{adj}\,\hat{A} = 0, \tag{2.11}$$

$$\det \hat{A} = 0, \tag{2.12}$$

where $\mathrm{adj}\, A$ *is the adjoint matrix of* A*.*

Proof. By (2.2), we may rewrite

$$f(A) = (\langle A,\ A\rangle + \langle B,\ B\rangle)^2 - 4(\langle A,\ B\rangle)^2. \tag{2.13}$$

Hence,

$$\nabla f(A) = 4(\langle A,\ A\rangle + \langle B,\ B\rangle)A - 8\langle A,\ B\rangle B, \tag{2.14}$$

where ∇ is the gradient operator in $R^{2\times 2}$. We have also

$$\nabla(\det A) = \mathrm{adj}\, A, \quad \forall A \in R^{2\times 2}.$$

Thus, the lemma follows by applying the Lagrange multiplier theorem. $\square$

Lemma 2.3. *Suppose $\hat{A}$ is a solution of the problem* (2.10)*. Then,*

$$f(\hat{A}) = (\langle B,\ B\rangle)^2 - (\langle \hat{A},\ \hat{A}\rangle)^2. \tag{2.15}$$

Proof. By (2.11), we have

$$4(\langle \hat{A},\ \hat{A}\rangle + \langle B,\ B\rangle)\langle \hat{A},\ \hat{A}\rangle - 8(\langle \hat{A},\ B\rangle)^2 - \hat{\lambda}\langle \mathrm{adj}\,\hat{A},\ \hat{A}\rangle = 0.$$

Since, by (2.12), $\langle \mathrm{adj}\,\hat{A},\ \hat{A}\rangle = 2\det \hat{A} = 0$, we have

$$4(\langle \hat{A},\ B\rangle)^2 = 2(\langle \hat{A},\ \hat{A}\rangle + \langle B,\ B\rangle)\langle \hat{A},\ \hat{A}\rangle. \tag{2.16}$$

Thus, by (2.13),

$$\begin{aligned} f(\hat{A}) &= (\langle \hat{A},\ \hat{A}\rangle + \langle B,\ B\rangle)^2 - 2(\langle \hat{A},\ \hat{A}\rangle + \langle B,\ B\rangle)\langle \hat{A},\ \hat{A}\rangle \\ &= (\langle \hat{A},\ \hat{A}\rangle + \langle B,\ B\rangle)(\langle B,\ B\rangle - \langle \hat{A},\ \hat{A}\rangle) \\ &= (\langle B,\ B\rangle)^2 - (\langle \hat{A},\ \hat{A}\rangle)^2. \end{aligned}$$

This completes the proof. $\square$

Theorem 2.2. *The problem* (2.10) *has a unique pair of solutions $\hat{A}$ and $-\hat{A}$.*

Proof. By (2.2), we have $f(A) \geq 0$ for all $A \in R^{2\times 2}$ and $\lim_{\|A\|\to\infty} f(A) = +\infty$. Since $f(\cdot)$ and $\det(\cdot)$ are continuous on $R^{2\times 2}$ and the set $\{A \in R^{2\times 2} : \det A = 0,\ f(A) \leq f(\vec{0})\}$ is compact, we conclude that there exists at least one solution.

Since, by (2.2), $f(A) = f(-A)$, we see that if $\hat{A}$ is a solution of (2.10) so is $-\hat{A}$.

Suppose A_1 and A_2 are solutions of (2.10). By lemma 2.3, assuming $A_1 \neq \vec{0}$, we have

$$\langle A_1,\ A_1\rangle = \langle A_2,\ A_2\rangle = c^2 > 0 \tag{2.17}$$

where $c^2 = ((\langle B,\ B\rangle)^2 - \inf\{f(A) : \det A = 0\})^{1/2}$. Let $i = 1, 2$. It follows from (2.11) that

$$\begin{aligned} &4(\langle A_i,\ A_i\rangle + \langle B,\ B\rangle)\langle A_i,\ \operatorname{adj} A_i\rangle \\ &\qquad - 8\langle A_i,\ B\rangle\langle B,\ \operatorname{adj} A_i\rangle - \hat{\lambda}_i\langle \operatorname{adj} A_i,\ \operatorname{adj} A_i\rangle = 0. \end{aligned}$$

Since, by (2.12),

$$\begin{aligned} \langle A_i,\ \operatorname{adj} A_i\rangle &= 2\det A_i = 0, \\ \langle B,\ \operatorname{adj} A_i\rangle &= \langle A_i,\ \operatorname{adj} B\rangle, \\ \langle \operatorname{adj} A_i,\ \operatorname{adj} A_i\rangle &= \langle A_i,\ A_i\rangle, \end{aligned}$$

we have

$$\hat{\lambda}_i = \frac{-8\langle A_i,\ B\rangle\langle A_i,\ \operatorname{adj} B\rangle}{\langle A_i,\ A_i\rangle}. \tag{2.18}$$

It follows again from (2.11) that

$$\begin{aligned} &4(\langle A_i,\ A_i\rangle + \langle B,\ B\rangle)\langle A_i,\ \operatorname{adj} B\rangle \\ &\qquad - 8\langle A_i,\ B\rangle\langle B,\ \operatorname{adj} B\rangle - \hat{\lambda}_i\langle \operatorname{adj} A_i,\ \operatorname{adj} B\rangle = 0. \end{aligned}$$

Since, by (2.16) and (2.17), $\langle A_i,\ B\rangle \neq 0$, this gives

$$\hat{\lambda}_i = \frac{4(\langle A_i,\ A_i\rangle + \langle B,\ B\rangle)\langle A_i,\ \operatorname{adj} B\rangle - 16\det B\langle A_i,\ B\rangle}{\langle A_i,\ B\rangle} \tag{2.19}$$

It follows from (2.18), (2.19) and (2.16) that

$$\begin{aligned} \langle A_i,\ \operatorname{adj} B\rangle &= \frac{16\det B\langle A_i,\ B\rangle\langle A_i,\ A_i\rangle}{4(\langle A_i,\ A_i\rangle + \langle B,\ B\rangle)\langle A_i,\ A_i\rangle + 8(\langle A_i,\ B\rangle)^2} \\ &= \frac{\langle A_i,\ A_i\rangle}{\langle A_i,\ B\rangle}\det B. \end{aligned}$$

Substitute this into (2.18), we obtain

$$\hat{\lambda}_1 = \hat{\lambda}_2 = -8 \det B. \tag{2.20}$$

Now, by (2.16), (2.17) and (2.20), we can rewrite (2.11) into

$$(c^2 + \langle B,\ B\rangle)A_i + 2 \det B \operatorname{adj} A_i = 2c \left(\frac{c^2 + \langle B,\ B\rangle}{2} \right)^{1/2} B \tag{2.21}$$

for $i = 1, 2$.

Since, by (2.1),

$$-\langle B,\ B\rangle < 2 \det B < \langle B,\ B\rangle, \tag{2.22}$$

the system (2.21) is nonsingular and thus has one and only one solution for each of the two right hand side term which are distinguished by the sign of c. Hence, the conclusion of the theorem follows. □

Lemma 2.4. *Suppose $\hat{A} = (\hat{a}_{ij})$ with $\hat{a}_{11} + \hat{a}_{22} > 0$ is a solution of* (2.10). *Then, we have*

$$\hat{a}_{11} > 0,\ \hat{a}_{22} > 0,\ \hat{a}_{12} = \hat{a}_{21} = -\sqrt{\hat{a}_{11}\hat{a}_{22}} < 0, \tag{2.23}$$

$$\langle \hat{A},\ \hat{A}\rangle = (\hat{a}_{11} + \hat{a}_{22})^2. \tag{2.24}$$

Proof. By (2.12), we have

$$\hat{a}_{12}\hat{a}_{21} = \hat{a}_{11}\hat{a}_{22}. \tag{2.25}$$

By (2.21), (2.22), (2.1) and the assumption $\hat{a}_{11} + \hat{a}_{22} > 0$ which implies $c > 0$, we have

$$\hat{a}_{12} = \hat{a}_{21} < 0. \tag{2.26}$$

Since $\hat{a}_{11} + \hat{a}_{22} > 0$, (2.25) and (2.26) imply (2.23). (2.24) is a direct consequence of (2.23). □

Theorem 2.3. *Let $\hat{A}$ be a solution of* (2.10). *Then*

$$\langle \hat{A},\ \hat{A}\rangle = \sqrt{(\langle B,\ B\rangle)^2 - 4(\det B)^2}, \tag{2.27}$$

$$f(\hat{A}) = 4(\det B)^2 \tag{2.28}$$

and $\hat{A}$ is determined up to a sign by the system

$$(c^2 + \langle B,\ B\rangle)\hat{A} + 2 \det B \operatorname{adj} \hat{A} = 2c \left(\frac{c^2 + \langle B,\ B\rangle}{2} \right)^{1/2} B, \tag{2.29}$$

where $c = ((\langle B,\ B\rangle)^2 - 4(\det B)^2)^{1/4}$.

Proof. Let $c^2 = \langle \hat{A},\ \hat{A} \rangle$. By lemma 2.4, without loss of generality, we assume that the diagonal entries of $\hat{A} = (\hat{a}_{ij})$ are positive and $c > 0$. Thus, by taking the trace of (2.21), we have

$$\hat{a}_{11} + \hat{a}_{22} = \frac{2r(b_{11} + b_{22})}{c^2 + \langle B,\ B \rangle + 2 \det B}, \tag{2.30}$$

where

$$r = 2c \left(\frac{c^2 + \langle B,\ B \rangle}{2} \right)^{1/2}. \tag{2.31}$$

Since $(\hat{a}_{11} + \hat{a}_{22})^2 = \langle \hat{A},\ \hat{A} \rangle = c^2$ by (2.24), (2.30) and (2.31) give

$$2(b_{11} + b_{22})^2 (c^2 + \langle B,\ B \rangle) = (c^2 + \langle B,\ B \rangle + 2 \det B)^2. \tag{2.32}$$

Since B is symmetric, we have

$$\langle B,\ B \rangle + 2 \det B - (b_{11} + b_{22})^2 = 0. \tag{2.33}$$

(2.32) and (2.33) give that $c > 0$ is a solution of the equation

$$c^4 - [(\langle B,\ B \rangle)^2 - 4(\det B)^2] = 0. \tag{2.34}$$

this proves (2.27). (2.28) follows from (2.27) and lemma 2.3. Substitute (2.27) into (2.21), we obtain (2.29). $\square$

Next, we are going to construct the laminated microstructure from the solutions of (2.10).

Let A be a solution of the problem (2.10). By (2.1) and theorem 2.3 (see (2.27)), $\langle A,\ A \rangle > 0$. Thus, by lemma 2.2 (see (2.12)), A is rank-one. This implies that $A = (a_{ij})$ has the following decomposition

$$2A = \vec{\alpha} \otimes \vec{n}, \tag{2.35}$$

where $\otimes$ denotes the tensor product in R^2 and $\vec{n} = (n_1,\ n_2)^T \in R^2$ is a unit vector with $n_1 > 0$, this is possible since $a_{11} \neq 0$ by theorem 2.2 and lemma 2.4.

Denote, for $i = 1, 2, \dots$ and $\nu = 0, \pm 1, \pm 2, \dots$,

$$\Omega_{i,\nu} = \{x \in \Omega : x \cdot \vec{n} \in [\frac{\nu - 1}{i},\ \frac{\nu}{i})\}. \tag{2.36}$$

Define a sequence of functions $\{\vec{\tilde{u}}_{i,A}\}_{i=1}^{\infty}$ by

$$\vec{\tilde{u}}_{i,A}(x) = \begin{cases} A(x - \frac{2\nu-1}{i}\vec{n}), & \text{if } x \in \Omega_{i,2\nu}, \\ -A(x - \frac{2\nu+1}{i}\vec{n}), & \text{if } x \in \Omega_{i,2\nu+1}. \end{cases} \tag{2.37}$$

Then it is easily verified that

$$\vec{\tilde{u}}_{i,A} \in W^{1,\infty}(\Omega; R^2), \quad \forall i \tag{2.38}$$

and

$$\nabla\vec{\tilde{u}}_{i,A}(x) = \begin{cases} A, & \text{if } x \in \Omega_{i,A}, \\ -A, & \text{if } x \in \Omega_{i,-A}, \end{cases} \tag{2.39}$$

where

$$\Omega_{i,A} = \cup_{\nu=-\infty}^{+\infty}\Omega_{i,2\nu}, \tag{2.40}$$

$$\Omega_{i,-A} = \cup_{\nu=-\infty}^{+\infty}\Omega_{i,2\nu+1}. \tag{2.41}$$

Let $\phi_i(x) \in C_0^{\infty}(\Omega)$ be a sequence of functions satisfying

$$\phi_i(x) = \begin{cases} 1, & \text{if } x \in \Omega \text{ and } \operatorname{dist}(x, \partial\Omega) \geq i^{-1}, \\ 0, & \text{if } x \in \Omega \text{ and } \operatorname{dist}(x, \partial\Omega) \leq (2i)^{-1}, \end{cases} \tag{2.42}$$

and

$$\|\nabla\phi_i(x)\| \leq Ci^{-1}, \quad \forall x \in \Omega, \tag{2.43}$$

where C is a constant independent of i.

Defining a sequence of functions $\{\vec{u}_i\}_{i=1}^{\infty}$ by

$$\vec{u}_i(x) = \vec{\tilde{u}}_{i,A}(x)\phi_i(x), \quad i = 1, 2, \ldots, \tag{2.44}$$

we have the following result.

Theorem 2.4. *Let A be a solution of the problem* (2.10). *Let $\vec{u}_i$, $i = 1, 2, \ldots$ be a sequence of functions defined by* (2.44). *Then, $\vec{u}_i \in \mathbb{A}$ is a minimizing sequence of $F(\cdot)$ in $\mathbb{A}$ and represents a laminated microstructure.*

Proof. By the definition of $\vec{u}_i$, it is easily seen that $\vec{u}_i \in \mathbb{A}$ and

$$\nabla\vec{u}_i(x) = \begin{cases} A, & \text{if } x \in \Omega_{i,A} \cap \Omega(i^{-1}), \\ -A, & \text{if } x \in \Omega_{i,-A} \cap \Omega(i^{-1}), \\ O(1), & \text{if } x \in \Omega \setminus \Omega(i^{-1}), \end{cases} \tag{2.45}$$

where $\Omega(i^{-1}) = \{x \in \Omega : \mathrm{dist}(x,\ \partial\Omega) > i^{-1}\}$. Thus

$$\begin{aligned} F(\vec{u}_i) &= \int_{\Omega_{i,A}\cap\Omega(i^{-1})} f(A)\,dx + \int_{\Omega_{i,-A}\cap\Omega(i^{-1})} f(-A)\,dx + \int_{\Omega\setminus\Omega(i^{-1})} f(\nabla\vec{u}_i(x))\,dx \\ &\le f(A)\,\mathrm{meas}(\Omega) + O(i^{-1}) \\ &= \inf\{f(A') : \det(A') = 0\}\,\mathrm{meas}(\Omega) + O(i^{-1}). \end{aligned}$$

This, by (2.6) and (2.9), shows that $\{\vec{u}_i\}$ is a minimizing sequence of $F(\cdot)$ in $\mathbb{A}$. It is easily seen from (2.45) that $\vec{u}_i$ has a fine scaled laminated structure and

$$\nabla\vec{u}_i \rightharpoonup \frac{1}{2}\delta_A + \frac{1}{2}\delta_{-A} \quad \text{in the sense of measure,} \tag{2.46}$$

where δ_A is the Dirac measure in $R^{2\times 2}$ at A. This completes the proof. □

3. Application of a nonconforming finite element method and over-relaxation

Let $\Omega = (0,\ 1)\times(0,\ 1)$, and let $B \in R^{2\times 2}$ be given by

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{3.1}$$

Consider the problem of minimizing the integral functional $F(\cdot)$ in $\mathbb{A}$ (see (2.3) and (2.4) respectively) with $f(A)$ defined by (2.2).

It follows from (2.6), (2.9) and (2.28) that

$$\inf_{\vec{v}\in\mathbb{A}} F(\vec{v}) = \inf\{f(A) : \det A = 0\}\,\mathrm{meas}(\Omega) = 4(\det B)^2 = \frac{1}{4}, \tag{3.2}$$

and it follows from theorem 2.2, theorem 2.3 and theorem 2.4 that the laminated microstructure of the problem is determined by the unique solution $A \in R^{2\times 2}$ of the following system

$$\alpha A + \mathrm{adj}\,A = \beta B, \tag{3.3}$$

where

$$\alpha = \frac{7+\sqrt{45}}{2}, \quad \beta = (\frac{\sqrt{45}(7+\sqrt{45})}{2})^{1/2}. \tag{3.4}$$

A resulted fine scaled oscillation is shown in figure 1.

Next, we are going to solve the problem by applying a nonconforming finite element method utilizing the Crouzeix-Raviart piece wise linear, triangular

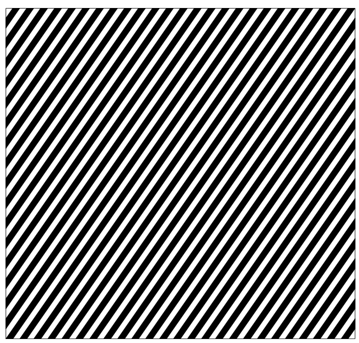

FIGURE 1. Fine scaled oscillation between A and $-A$.

element which is constrained to be continuous at the midpoints of line segments which are edges of adjacent triangles [14].

For each integer $M \geq 1$ and $0 \leq i, j \leq M-1$, define

$$K_M^+(i,j) = \{\text{the triangle in } R^2 \text{ with vertices}$$
$$a_1 = (\frac{i}{M}, \frac{j}{M}),\ a_2 = (\frac{i+1}{M}, \frac{j+1}{M}),\ a_3 = (\frac{i}{M}, \frac{j+1}{M})\},$$
$$K_M^-(i,j) = \{\text{the triangle in } R^2 \text{ with vertices}$$
$$a_1 = (\frac{i}{M}, \frac{j}{M}),\ a_2 = (\frac{i+1}{M}, \frac{j}{M}),\ a_3 = (\frac{i+1}{M}, \frac{j+1}{M})\}.$$

Let $\mathfrak{T}_M = (\cup_{i,j=0}^{M-1} K_M^+(i,j)) \cup (\cup_{i,j=0}^{M-1} K_M^-(i,j))$. Then $\{\mathfrak{T}\}_{M=1}^\infty$ is a regular family of triangulations of Ω [14].

Define

$$X_M = \{u : \prod_{K \in \mathfrak{T}_M} K \to R \mid u \text{ is affine on each } K \in \mathfrak{T}_M$$

$$\text{and } u \text{ is continuous at } b,\ \forall b \in \mathbb{N}_M\},$$

where $\mathbb{N}_M$ is the set of all nodes, or the degrees of freedom of the finite element function space [14], defined by

$$\mathbb{N}_M = \{a_{ij}^K = (a_i^K + a_j^K)/2,\ 1 \le i \ne j \le 3,\ \forall K \in \mathbb{T}_M\}.$$

Let

$$\mathbb{A}_M = \{\vec{u} \in X_M \times X_M : \vec{u} = \vec{0},\ \forall b \in \mathbb{N}_M \cap \partial\Omega\}. \tag{3.5}$$

Consider the finite problem of minimizing the integral functional $F(\cdot)$ in $\mathbb{A}_M$, that is

$$\begin{cases} \text{find } \vec{u} \in \mathbb{A}_M \text{ such that} \\ F(\vec{u}) = \inf_{\vec{v} \in \mathbb{A}_M} F(\vec{v}). \end{cases} \tag{3.6}$$

We claim that

$$\inf_{\vec{v} \in \mathbb{A}_M} F(\vec{v}) = O(M^{-1}). \tag{3.7}$$

In fact, we can construct a minimizing sequence $\vec{u}_M = (u_{M,1}, u_{M,2})$ by defining

$$u_{M,1}(x) = \begin{cases} (x_1 - \frac{i}{M}) - \frac{1}{2}(x_2 - \frac{j+\frac{1}{2}}{M}), & \text{if } x \in K_M^+(i,j), \text{ for } j \ne M-1; \\ -(x_1 - \frac{i+1}{M}) + \frac{1}{2}(x_2 - \frac{j+\frac{1}{2}}{M}), & \text{if } x \in K_M^-(i,j), \text{ for } j \ne 0; \\ (x_1 - \frac{i}{M}) - (x_2 - M + \frac{1}{2M}), & \text{if } x \in K_M^+(i, M-1); \\ -(x_1 - \frac{i+1}{M}) + (x_2 - \frac{1}{2M}), & \text{if } x \in K_M^+(i,j), \end{cases}$$

$$u_{M,2}(x) = \begin{cases} -\frac{1}{2}(x_1 - \frac{i}{M}) + \frac{1}{2}(x_2 - \frac{j+\frac{1}{2}}{M}), & \text{if } x \in K_M^+(i,j); \\ \frac{1}{2}(x_1 - \frac{i+1}{M}) - \frac{1}{2}(x_2 - \frac{j+\frac{1}{2}}{M}), & \text{if } x \in K_M^-(i,j). \end{cases}$$

It is easily verified that $\vec{u}_M$ thus defined is in $\mathbb{A}_M$ and satisfies

$$\nabla \vec{u}_M = \begin{cases} B, & \text{on } K_M^+(i,j) \text{ for } j \ne M-1; \\ -B, & \text{on } K_M^-(i,j) \text{ for } j \ne 0; \\ B - \frac{1}{2}E, & \text{on } K_M^+(i, M-1); \\ -B + \frac{1}{2}E, & \text{on } K_M^-(i,0), \end{cases} \tag{3.8}$$

where

$$E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus, by (2.2), (3.1) and (3.8),

$$f(\nabla \vec{u}_M(x)) = \begin{cases} 0, & \text{if } x \notin K_M^+(i, M-1) \cup K_M^-(i, 0); \\ 33/16, & \text{if } x \in K_M^+(i, M-1) \cup K_M^-(i, 0). \end{cases}$$

This gives

$$F(\vec{u}_M) = \int_\Omega f(\nabla \vec{u}_M(x)\, dx = \frac{33}{16} M^{-1}. \tag{3.9}$$

Since $F(\vec{u}) \geq 0$ for all $\vec{u} \in \mathbb{A}_M$, (3.7) follows from (3.9).

It follows from (3.7) that the application of the finite element method with Crouzeix-Raviart triangular element results in over-relaxation to the original problem. We see also that the finite element solutions of (3.6) present unrealistic oscillations between the two potential wells which have no rank-one connection and lead to a pseudo-microstructure (see figure 2).

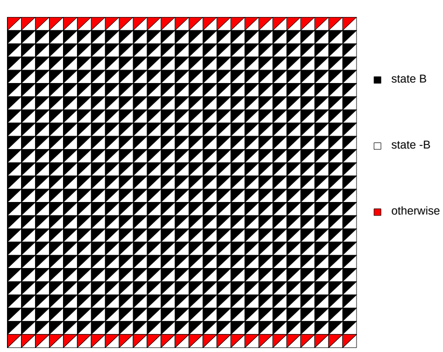

FIGURE 2. 25×25 nonconforming finite element pseudo-microstructure.

4. Conclusions

Conclusions are drawn from the results of this paper that laminated microstructures can occur even when there is no rank-one connected potential wells, and that a direct application of nonconforming finite element methods to such problems can cause over-relaxation and fail both to approximate the microstructure and to calculate the infimum value of the potential energy.

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