# AN $O(h)$ DISCRETE NEUMANN BOUNDARY CONDITION LEADS TO AN $O(h)$ SCHEME — AN EXAMPLE

Consider

$$\begin{cases} u''(x) = 2, & 0 < x < 1, \\ u'(0) = 0, \\ u(1) = 0. \end{cases}$$

The exact solution is $u(x) = x^2 - 1$, for which we have $u(0) = -1$, $u'(x) = 2x$.

Let $h = 1/N$. Consider the following scheme

$$(*) \qquad \begin{cases} U_{j-1} - 2U_j + U_{j+1} = 2h, & 1 \le j \le N, \\ U_0 = U_1, \\ U_N = 0, \end{cases}$$

which has second order truncation error for the equation and 1st order truncation error for the Neumann boundary condition.

Let $v_h(x) = (x - h/2)^2 - (1 - h/2)^2$. Then, $v_h(0) = v_h(h) = -1 + h$, and we have

$$\begin{cases} v_h''(x) = 2, & 0 < x < 1, \\ v_h(0) = -1 + h, \\ v_h(1) = 0. \end{cases}$$

On the other hand, let $V_j = v_h(jh)$, $j = 0, 1, \cdots, N$, we have

$$\begin{cases} V_{j-1} - 2V_j + V_{j+1} = 2h, & 1 \le j \le N, \\ V_0 = V_1, \\ V_N = 0. \end{cases}$$

In other words, $\{V_j\}_{j=0}^N$ is the solution to the scheme (*).

Since $v_h(x) - u(x) = (x - h/2)^2 - (1 - h/2)^2 - x^2 + 1 = (1 - x)h$, we have $\|U - u\|_\infty = h$, and $\|U - u\|_2 \approx h/\sqrt{2}$. Hence, the scheme (*) is of 1st order accurate.