Numerical Solutions to Partial Differential Equations

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Finite Difference Schemes for the Advection Equation

Performance of Difference Schemes on Initial Value Problems

Things to Check for the Overall Performance of a Scheme

Under the CFL condition,

- Local truncation error and consistency;
- 2 Dissipation and global error on the amplitude;
- Dispersion and relative error on the phase angle (dispersion relation: ω(k) = -ak, ω_h(k)τ = arg λ_k = -akτ(1+···));
- **3** The error on the group speed $(C(k) = \omega'(k), C_h(k) = \omega'_h(k))$, for large k;
- **5** Spurious modes must be under control.



Finite Difference Schemes for the Advection Equation

Stability Analysis of Numerical Boundary Conditions

Vertical Fourier Modes and Amplification Factors

- Separation of variables of difference solutions: $U_i^m = \lambda^m \mu^j$.
- **2** Standard Fourier modes: $U_j^m = \lambda_k^m e^{ikjh}$. $(\mu_k = e^{ikh})$
- **3** Vertical Fourier modes: $U_j^m = e^{\pm iamk\tau} \mu_k^j$. $(\lambda_k = e^{\pm iak\tau})$
- **④** Fourier mode solutions of the advection equation: $e^{\pm ik(x-at)}$.
- **(3)** The amplification factors of the advection equation: $e^{\pm ikh}$.
- Substitute the vertical Fourier mode U^m_j = e^{±iamkτ}μ^j_k into a difference scheme to get the amplification factor μ[±]_k.
- For real solution modes, we expect $\mu_k^{\pm} \approx e^{\mp i k h}$.



-Finite Difference Schemes for the Advection Equation

Stability Analysis of Numerical Boundary Conditions

Strong Damping Effect of the Lax-Wendroff Scheme to Spurious Modes

In general, If the CFL condition is satisfied, the Lax-Wendroff scheme has very strong damping effect to the reversely propagating waves, so the additional numerical boundary conditions will not cause too much error pollution to the numerical results.



Finite Difference Schemes for the Advection Equation

Stability Analysis of Numerical Boundary Conditions

Amplification Factor of Vertical Fourier Mode of the Leap-frog Scheme

- The leap-frog scheme: $U_j^{m+1} = U_j^{m-1} \nu (U_{j+1}^m U_{j-1}^m).$
- **2** Substitute $U_j^m = \lambda_k^m \mu_k^j = e^{\pm i amk\tau} \mu_k^j$ into the leap-frog scheme yields the characteristic equation:

$$\nu\lambda_k\mu_k^2-(1-\lambda_k^2)\mu_k-\nu\lambda_k=0.$$

③ The amplification factor of the vertical Fourier mode:

$$\mu_{k} = \frac{\left(1 - \lambda_{k}^{2}\right) \pm \sqrt{\left(1 - \lambda_{k}^{2}\right)^{2} + 4\nu^{2}\lambda_{k}^{2}}}{2\nu\lambda_{k}}.$$

$$(\mu_{k}^{\pm} = \pm (1 - (\frac{\sin\theta}{\nu})^{2})^{1/2} - i\frac{\sin\theta}{\nu}, \theta = \pm ak\tau = \pm\nu kh).$$

$$\lambda_{k}^{\pm} = e^{\pm iak\tau} \approx 1 \pm iak\tau = 1 \pm i\nu kh.$$



Finite Difference Schemes for the Advection Equation

Stability Analysis of Numerical Boundary Conditions

Amplification Factor of Vertical Fourier Mode of the Leap-frog Scheme

For the two real solution modes
$$U_j^{\pm m} = (\lambda_k^{\pm})^m (\mu_k^{r\pm})^j$$
:
 $\mu_k^{r\pm} = \mu_k^r (\lambda_k^{\pm}) \approx \begin{cases} 1 - \mathrm{i}kh \approx e^{-\mathrm{i}kh}, \\ 1 + \mathrm{i}kh \approx e^{\mathrm{i}kh}, \end{cases}$

So For the two spurious solution modes $V_j^{\pm m} = (\lambda_k^{\pm})^m (\mu_k^{s\pm})^j$: $\mu_k^{s\pm} = \mu_k^s (\lambda_k^{\pm}) \approx \begin{cases} -(1 + ikh) \approx -e^{ikh}; \\ -(1 - ikh) \approx -e^{-ikh}. \end{cases}$

() The leap-frog scheme may also have no damp to some high frequency vertical spurious modes (In fact, whenever $|\sin \nu kh| \le |\nu|$ holds. On the other hand, if $|\sin \nu kh| > |\nu|$ holds, then $|\mu_k^s| > 1$, then the corresponding frequency modes are damped sharply. For example, if $\nu = 1/2$, then $|\sin \nu \pi| = 2|\nu|$, and the highest frequency mode is damped very sharply.);



Special Cares Needed to Limit Spurious Modes of the Leap-frog Scheme

- General solution of the leap-frog scheme can be written as $U_j^m = \sum_k (\lambda_k^+)^m \Big[R_k^+ (\mu_k^{r+})^j + S_k^+ (\mu_k^{s+})^j \Big] + (\lambda_k^-)^m \Big[R_k^- (\mu_k^{r-})^j + S_k^- (\mu_k^{r-})^j \Big] \,.$
 - Start up scheme at m = 0, the data on m = 1 and numerical outflow boundary conditions should be properly provided and coupled, so that S⁺_k = h.o.t, S⁻_k = h.o.t., for all k.
 - The zero order extrapolation numerical boundary condition on the outflow boundary is a non-reflection boundary condition, which means no obvious new grid size spurious Fourier modes will be produced at the boundary.
 - It is important that U¹ contains as less as possible the spurious component, while has a real component with reasonable approximate accuracy.

Conservation Law and Consistent Conservative Schemes

-Numerical Flux and Consistent Conservative Scheme

Conservation Law and Its Cell Average Form

 $\begin{array}{l} \textbf{O} \quad \text{Differential form of 1D first order scalar conservation law:} \\ \quad \frac{\partial u(x,t)}{\partial t} + \frac{\partial f(x,t,u(x,t))}{\partial x} = 0, \qquad x \in I \subset \mathbb{R}, \ t > 0, \end{array}$

2 Integral form of 1D first order scalar conservation law:

$$\int_{x_l}^{x_r} u(x, t_a) dx = \int_{x_l}^{x_r} u(x, t_b) dx - \left[\int_{t_b}^{t_a} f(x_r, t, u(x_r, t)) dt - \int_{t_b}^{t_a} f(x_l, t, u(x_l, t)) dt \right], \quad \forall x_l < x_r, \ 0 \le t_b < t_a.$$

3 \bar{u}_j^m : the integral average of u on $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ at t_m ;

i \$\vec{t}{j_{j-\frac{1}{2}}^{m+\frac{1}{2}}\$: the integral average of the flux \$f\$ on \$(t_m, t_{m+1})\$ at \$x_{j-\frac{1}{2}}\$;

 i \$\vec{u}{j}^{m+1} = \vec{u}{j}^m - \frac{\tau{m}}{h_j} \$\vec{t}{f_{j+\frac{1}{2}}^{m+\frac{1}{2}}} - \vec{t}{f_{j-\frac{1}{2}}^{m+\frac{1}{2}}}\$], \$\forall \$j\$, \$\forall \$m \geq 0\$.

Conservation Law and Consistent Conservative Schemes

-Numerical Flux and Consistent Conservative Scheme

Numerical Flux and Discrete Conservation Law

• $U_j^m \sim \bar{u}_j^m$, numerical flux $F_{j\pm\frac{1}{2}}^{m+\frac{1}{2}} \sim \bar{f}_{j\pm\frac{1}{2}}^{m+\frac{1}{2}}$;

2 Numerical flux is of the following (translation invariant) form:

$$F_{j+\frac{1}{2}}^{m+\frac{1}{2}} = F\left([x_{j-p}, U_{j-p}^{m}, U_{j-p}^{m+1}], \cdots, [x_{j+q}, U_{j+q}^{m}, U_{j+q}^{m+1}]; t_{m}, t_{m+1}\right),$$

Onservative difference scheme for the conservation law:

$$U_{j}^{m+1} = U_{j}^{m} - \frac{\tau_{m}}{h_{j}} \left[F_{j+\frac{1}{2}}^{m+\frac{1}{2}} - F_{j-\frac{1}{2}}^{m+\frac{1}{2}} \right]$$

• Discrete conservation law: for all $j_l < j_r$, $k > l \ge 0$,

$$\sum_{j=j_l}^{j_r} h_j U_j^k = \sum_{j=j_l}^{j_r} h_j U_j^l - \left[\sum_{m=l}^{k-1} \tau_m F_{j_r+\frac{1}{2}}^{m+\frac{1}{2}} - \sum_{m=l}^{k-1} \tau_m F_{j_l-\frac{1}{2}}^{m+\frac{1}{2}}\right]$$



Consistent Conservative Scheme

Definition

A difference scheme is said to be a consistent conservative scheme for the conservation law

$$\begin{split} \int_{x_l}^{x_r} u(x, t_a) \, dx &= \int_{x_l}^{x_r} u(x, t_b) \, dx - \left[\int_{t_b}^{t_a} f(x_r, t, u(x_r, t)) \, dt \right. \\ &- \int_{t_b}^{t_a} f(x_l, t, u(x_l, t)) \, dt \right], \ \forall x_l < x_r, \ 0 \le t_b < t_a. \end{split}$$

if its numerical flux is of the form

$$F_{j+\frac{1}{2}}^{m+\frac{1}{2}} = F\left([x_{j-p}, U_{j-p}^{m}, U_{j-p}^{m+1}], \cdots, [x_{j+q}, U_{j+q}^{m}, U_{j+q}^{m+1}]; t_{m}, t_{m+1}\right),$$

and (1) F([x, u, u], · · · , [x, u, u]; t, t) = f(x, t, u), ∀ (x, t, u);
(2) F is continuous, and is Lipschitz continuous with respect to the unknown U.

Conservation Law and Consistent Conservative Schemes

-Numerical Flux and Consistent Conservative Scheme

Conservative is Crucial to a Difference Scheme for a Conservation Law

() Consider the initial value problem of the Burgers equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} = 0, \quad u(x,0) = \begin{cases} 1, & \text{if } x < 0; \\ 0, & \text{if } x \ge 0; \end{cases}$$

2 The upwind scheme based on the equivalent nonconservative equation $u_t + uu_x = 0$:

$$U_{j}^{m+1} = \begin{cases} U_{j}^{m} - \frac{\tau}{h} U_{j}^{m} \left(U_{j}^{m} - U_{j-1}^{m} \right), & \text{if } U_{j}^{m} \ge 0, \\ U_{j}^{m} - \frac{\tau}{h} U_{j}^{m} \left(U_{j+1}^{m} - U_{j}^{m} \right), & \text{if } U_{j}^{m} < 0. \end{cases}$$

3 Truncation error $O(\tau + h)$, and stable, in fact satisfies the maximum principle, if CFL condition holds.



Conservation Law and Consistent Conservative Schemes

Numerical Flux and Consistent Conservative Scheme

Conservative is Crucial to a Difference Scheme for a Conservation Law

4 Numerical solution:

$$U_j^m = U_j^0 = egin{cases} 1, & ext{if} \ j < 0; \ 0, & ext{if} \ j \geq 0, \end{cases} \quad orall m \geq 0.$$

Solution converges to $\tilde{u}(x,t) = u(x,0), \forall t > 0, \forall x$.

○ $0 = s[\tilde{u}] \neq [f] = 1/2$, Rankine-Hugoniot jump condition is not satisfied, therefore, \tilde{u} is not a weak solution.



Conservation Law and Consistent Conservative Schemes

-Numerical Flux and Consistent Conservative Scheme

Conservative is Crucial to a Difference Scheme for a Conservation Law

For a nonlinear conservation law

- Non-conservative schemes can converge to a function, which is not a weak solution.
- Consistency and stability of a finite difference scheme do not necessarily lead to the convergence of the scheme.
- On the other hand, it can be shown, for a consistent conservative scheme to a conservation law, if the numerical solutions converge in certain specified sense, then the limit function must be a weak solution to the conservation law.



Conservation Law and Consistent Conservative Schemes

Finite Volume Schemes

Upwind Numerical Flux and Conservative Upwind Scheme

The finite volume method is a natural way to establish conservative difference schemes. For simplicity, we consider $u_t + f(u)_x = 0$ with f smooth, and uniform grids in tensor product form.

() Rewrite the conservation law in the form $u_t + f'(u)u_x = 0$.

2 The numerical characteristic on the cell interface $x_{i+\frac{1}{2}}$:

$$a_{j+\frac{1}{2}}^{m} = \begin{cases} \frac{f(U_{j+1}^{m}) - f(U_{j}^{m})}{U_{j+1}^{m} - U_{j}^{m}}, & \text{if } U_{j+1}^{m} \neq U_{j}^{m}; \\ 0, & \text{if } U_{j+1}^{m} = U_{j}^{m}, \end{cases}$$

3 CFL condition: $\max_{u \in U} |f'(u)| \tau \le h$, (\mathcal{U} : the solution set).



Conservation Law and Consistent Conservative Schemes

Finite Volume Schemes

Upwind Numerical Flux and Conservative Upwind Scheme

Of Define the numerical flux by (the method of characteristics)

$$\mathcal{F}_{j+rac{1}{2}}^{m+rac{1}{2}} = egin{cases} f(U_j^m), & ext{if} \ a_{j+rac{1}{2}}^m \geq 0; \ f(U_{j+1}^m), & ext{if} \ a_{j+rac{1}{2}}^m \leq 0. \end{cases}$$

This leads to the upwind scheme for the scalar conservation law:

$$\begin{aligned} U_j^{m+1} &= U_j^m - \frac{\tau}{2h} \Big\{ \left[\left(1 + \operatorname{sign} \left(a_{j+\frac{1}{2}}^m \right) \right) f(U_j^m) + \left(1 - \operatorname{sign} \left(a_{j+\frac{1}{2}}^m \right) \right) f(U_{j+1}^m) \right] \\ &- \left[\left(1 + \operatorname{sign} \left(a_{j-\frac{1}{2}}^m \right) \right) f(U_{j-1}^m) + \left(1 - \operatorname{sign} \left(a_{j-\frac{1}{2}}^m \right) \right) f(U_j^m) \right] \Big\}. \end{aligned}$$



Conservation Law and Consistent Conservative Schemes

Finite Volume Schemes

Upwind Numerical Flux and Conservative Upwind Scheme

- So For f(u) = au (a = constant), the scheme is upwind.
- **()** The scheme is a consistent conservative difference scheme.
- **7** The integral averages \bar{u}_j^{m+1} and \bar{u}_j^m are calculated by the middle point quadrature rule.
- **③** The integral averages of the flux $\overline{f}_{j\pm\frac{1}{2}}^{m+\frac{1}{2}}$ are calculated by a one point (generally not the middle point) quadrature rule.
- Onsequently, the truncation error of the upwind scheme is generally only first order.



Conservation Law and Consistent Conservative Schemes

Finite Volume Schemes

Middle Point Quadrature and a Conservative Lax-Wendroff Scheme

To achieve higher order accuracy, the middle point quadrature rule can be used to calculate the integral averages of the flux, so that

$$F_{j+\frac{1}{2}}^{m+\frac{1}{2}} = f(U_{j+\frac{1}{2}}^{m+\frac{1}{2}})$$

with a properly defined finite difference operator $R_{j+\frac{1}{2}}$ satisfying

$$U_{j+rac{1}{2}}^{m+rac{1}{2}} = R_{j+rac{1}{2}}(U^m), \quad u_{j+rac{1}{2}}^{m+rac{1}{2}} = R_{j+rac{1}{2}}(u^m) + O(au^2 + h^2).$$

1 The two steps Richtmyer scheme:

$$U_{j+\frac{1}{2}}^{m+\frac{1}{2}} = \frac{1}{2} \left(U_{j}^{m} + U_{j+1}^{m} \right) - \frac{\tau}{2h} \left[f(U_{j+1}^{m}) - f(U_{j}^{m}) \right] \triangleq R_{j+\frac{1}{2}}(U^{m}),$$

$$U_{j}^{m+1} = U_{j}^{m} - \frac{\tau}{h} \left[f(U_{j+\frac{1}{2}}^{m+\frac{1}{2}}) - f(U_{j-\frac{1}{2}}^{m+\frac{1}{2}}) \right].$$

Conservation Law and Consistent Conservative Schemes

Finite Volume Schemes

Middle Point Quadrature and a Conservative Lax-Wendroff Scheme

2 Let the integral averages \bar{u}_j^{m+1} and \bar{u}_j^m be calculated by the 2nd order accurate middle point quadrature rule.

3 for piecewise constant \bar{u}_j^m , $\bar{u}(x_j, t) = \bar{u}_j^m$, $t_m \le t \le t_{m+\frac{1}{2}}$,

9 by Taylor expansion,
$$u_{j+\frac{1}{2}}^{m+\frac{1}{2}} = u_{j+\frac{1}{2}}^m - \frac{\tau}{2}f(u_{j+\frac{1}{2}}^m)_x + O(\tau^2).$$

3
$$f(u_{j+\frac{1}{2}}^m)_x = \frac{f(u_{j+1}^m) - f(u_j^m)}{h} + O(h^2) \Rightarrow u_{j+\frac{1}{2}}^{m+\frac{1}{2}} = R_{j+\frac{1}{2}}(u^m) + O(\tau^2 + h^2).$$

O Consequently, the truncation error of the scheme is 2nd order.

For f(u) = au (a = constant), the scheme reduces to the Lax-Wendroff scheme.



Conservation Law and Consistent Conservative Schemes

Finite Volume Schemes

Another Consistent Conservative Lax-Wendroff Scheme

There are many ways to extend the Lax-Wendroff scheme to a conservative scheme for the scalar conservation law.

For example, if f is smooth, by $u_t = -f(u)_x$ and $u_{tt} = -(f(u)_t)_x = (f'(u)f(u)_x)_x$, we can establish the Lax-Wendroff scheme for the scalar conservation law

$$U_{j}^{m+1} = U_{j}^{m} - \frac{\tau}{2h} \left[f(U_{j+1}^{m}) - f(U_{j-1}^{m}) \right] + \frac{\tau^{2}}{2h^{2}} \left[a_{j+\frac{1}{2}}^{m} \left(f(U_{j+1}^{m}) - f(U_{j}^{m}) \right) - a_{j-\frac{1}{2}}^{m} \left(f(U_{j}^{m}) - f(U_{j-1}^{m}) \right) \right],$$

where $a_{j\pm\frac{1}{2}}^m = f'(\frac{1}{2}(U_j^m + U_{j\pm 1}^m))$. It can be shown (see Exercise 3.10):



Conservation Law and Consistent Conservative Schemes

Finite Volume Schemes

Another Consistent Conservative Lax-Wendroff Scheme

- For f(u) = au (constant a), the scheme reduces to the Lax-Wendroff scheme.
- It is a consistent conservative scheme for the scalar conservation law.
- **③** The truncation error of the scheme is 2nd order.



Conservation Law and Consistent Conservative Schemes

Initial Data and Boundary Conditions

Numerical Initial-Boundary Conditions To Maintain Global Conservation

An initial-boundary value problem of a conservation law (f' > 0):

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \ x \in (0,1), \ t > 0, \begin{cases} u(x,0) = u^0(x), & x \in (0,1); \\ u(0,t) = u_0(t), & t \ge 0. \end{cases}$$

2 Global conservation (or balance) of weak solutions:

$$\int_0^1 u(x,t) \, dx = \int_0^1 u(x,0) \, dx - \left[\int_0^t f(u(1,t)) \, dt - \int_0^t f(u(0,t)) \, dt \right]$$

3 Conservative scheme: $U_j^{m+1} = U_j^m - \tau h^{-1} \left[F_{j+\frac{1}{2}}^m - F_{j-\frac{1}{2}}^m \right].$



Conservation Law and Consistent Conservative Schemes

Initial Data and Boundary Conditions

Numerical Initial-Boundary Conditions To Maintain Global Conservation

4 Numerical initial and boundary conditions (f' > 0):

$$U_{j}^{0} = \frac{1}{h} \int_{j-\frac{1}{2}}^{j+\frac{1}{2}} u^{0}(x) dx, \ \forall \ 1 \leq j \leq N; \ F_{\frac{1}{2}}^{m} = \frac{1}{\tau} \int_{t_{m}}^{t_{m+1}} f(u_{0}(t)) dt, \ \forall \ m \geq 0.$$

We can also define the value of U₀^m on the ghost node x₀ by making use of the relation

$$F(U_0^m, U_1^m) = F_{\frac{1}{2}}^m = \frac{1}{\tau} \int_{t_m}^{t_{m+1}} f(u_0(t)) dt$$



Conservation Law and Consistent Conservative Schemes

Initial Data and Boundary Conditions

Numerical Initial-Boundary Conditions To Maintain Global Conservation

6 Global property of the numerical solution:

$$h\sum_{j=1}^{N} U_{j}^{0} = \int_{0}^{1} u^{0}(x) \, dx, \quad \tau \sum_{l=m}^{m+k} F_{\frac{1}{2}}^{l} = \int_{t_{m}}^{t_{m+k}} f(u_{0}(t)) \, dt.$$

Numerical boundary condition on outflow boundary construction of upwind numerical flux:

- (1) 1st-order upwind numerical flux $F_{N+\frac{1}{2}}^{m} = f(U_{N}^{m})$, (may also be viewed as resulted from the zero order extrapolation $U_{N+1}^{m} = U_{N}^{m}$);
- (2) 2nd-order upwind numerical flux $\hat{F}_{N+\frac{1}{2}}^{m} = \hat{F}(U_{N-1}^{m}, U_{N}^{m}).$ (say the Beam-Warming Flux; may also be viewed as resulted from the 1st-order extrapolation.)



Finite Difference Schemes for Advection-Diffusion Equations

A Model Problem of the Advection-Diffusion Equation — Diffusion along Characteristics

Advection-Diffusion Equation — A Model Problem

- An initial value problem of a 1D constant-coefficient advection-diffusion equation (a > 0, c > 0): u_t + au_x = cu_{xx}, x ∈ ℝ, t > 0; u(x, 0) = u⁰(x), x ∈ ℝ.
- By a change of variables y = x at and $v(y, t) \triangleq u(y + at, t)$, $v_t = cv_{yy}$, $y \in \mathbb{R}$, t > 0; $v(x, 0) = u^0(x)$, $x \in \mathbb{R}$.

Characteristic global properties of the solution *u*:

- There is a characteristic speed as in the advection equation, which plays an important role to the solution, especially when $|a| \gg c$ (advection dominant).
- Along the characteristic, the solution behaves like a parabolic solution (dissipation and smoothing).



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Finite Difference Schemes for Advection-Diffusion Equations

Central Explicit and Modified Central Explicit Schemes

Central Explicit Scheme for the Advection-Diffusion Equation

The simplest difference scheme is the central explicit scheme

$$\frac{U_j^{m+1} - U_j^m}{\tau} + a \frac{U_{j+1}^m - U_{j-1}^m}{2h} = c \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{h^2}.$$

r equivalently $(\nu = a\tau/h, \ \mu = c\tau/h^2)$
 $U_j^{m+1} = (\mu - \frac{1}{2}\nu)U_{j+1}^m + (1 - 2\mu)U_j^m + (\mu + \frac{1}{2}\nu)U_{j-1}^m,$

() Condition for the maximum principle: $\mu = \frac{c\tau}{h^2} \le \frac{1}{2}$, $h \le \frac{2c}{a}$.

2 Amplification factor $\lambda_k = 1 - 2\mu(1 - \cos kh) - i\nu \sin kh$.

Note: L^2 -stable $\Leftrightarrow |\lambda_k| \le 1 + O(\tau)$; L^2 -strongly stable $\Leftrightarrow |\lambda_k| \le 1$.



Finite Difference Schemes for Advection-Diffusion Equations

Central Explicit and Modified Central Explicit Schemes

\mathbb{L}^2 Strong Stability Conditions of the Central Explicit Scheme

3
$$|\lambda_k|^2 = 1 - (1 - \cos kh)[4\mu - 4\mu^2(1 - \cos kh) - \nu^2(1 + \cos kh)].$$

3 Strongly stable
$$\Leftrightarrow 4\mu - 4\mu^2(1 - \cos kh) - \nu^2(1 + \cos kh) \ge 0$$

 $\Leftrightarrow 4\mu - 2\nu^2 + (\nu^2 - 4\mu^2)(1 - \cos kh) \ge 0$, if $\cos kh \ne 1$.

$$\textbf{ 3 } \mathbb{L}^2 \text{ strongly stable } \Leftrightarrow 4\mu - 2\nu^2 + (\nu^2 - 4\mu^2)(1 - \cos kh) \geq 0.$$

() Take
$$k = \pi h^{-1}$$
, we are lead to a necessary condition:
 $4\mu(1-2\mu) = 4\mu - 2\nu^2 + 2(\nu^2 - 4\mu^2) \ge 0 \Leftrightarrow \mu \le \frac{1}{2}$.



Finite Difference Schemes for Advection-Diffusion Equations

Central Explicit and Modified Central Explicit Schemes

\mathbb{L}^2 Strong Stability Conditions of the Central Explicit Scheme

() Let $kh \to 0 \Rightarrow \nu^2 \le 2\mu \Leftrightarrow \tau \le \frac{2c}{a^2}$, a necessary condition.

3 $g(\xi) = 4\mu - 2\nu^2 + (\nu^2 - 4\mu^2)\xi$ is monotonic with respect to *ξ*, and g(0) ≥ 0, g(2) ≥ 0, if the two necessary conditions hold.

 ${\it O} \ g(\xi)\geq {\it 0}, \ \forall \xi\in [0,2], \ {\it if the two necessary conditions hold}.$

 $\textcircled{0} \ \mathbb{L}^2 \text{ strongly stable } \Leftrightarrow \mu \leq \frac{1}{2} \text{ and } \tau \leq \frac{2c}{a^2}.$



Finite Difference Schemes for Advection-Diffusion Equations

Central Explicit and Modified Central Explicit Schemes

Péclet Number and the Stability of the Central Explicit Scheme

Under the condition $\mu \leq 1/2$,

- **1** the maximum principal holds if and only if the Péclet number $\frac{|a|h}{c}$ satisfies $\frac{|a|h}{c} \leq 2$.
- $\textcircled{2} \quad \frac{|a|h}{c} \leq 2 \Rightarrow \tau \leq h^2/(2c) \leq 4c^2/(2ca^2) = 2c/a^2 \Rightarrow \text{ the second condition for } \mathbb{L}^2 \text{ strong stability holds automatically.}$
- **When** $\frac{|a|h}{c} > 2$, the scheme does not satisfy the maximum principle, and the condition $\tau \leq \frac{2c}{a^2}$ substantially restricts the time step size for the scheme to be \mathbb{L}^2 strongly stable (see Example 4.1).

Note: The truncation error of the Central Explicit Scheme is $O(\tau + h^2)$



Finite Difference Schemes for Advection-Diffusion Equations

Central Explicit and Modified Central Explicit Schemes

Modified Central Explicit Scheme for the Advection-Diffusion Equation

By
$$u_t = -au_x + cu_{xx}$$
, $u_{tt} = a^2u_{xx} - 2ac\partial_x^3u + c^2\partial_x^4u$, for $c \ll |a|$:

$$u_{j}^{m+1} = \left[u + \tau u_{t} + \frac{\tau^{2}}{2}u_{tt}\right]_{j}^{m} + O(\tau^{3})$$
$$= \left[u - a\tau u_{x} + \left(c + \frac{a^{2}\tau}{2}\right)\tau u_{xx}\right]_{j}^{m} + O(c\tau^{2} + \tau^{3}).$$

This leads to the modified central explicit scheme

$$\frac{U_{j}^{m+1} - U_{j}^{m}}{\tau} + a \frac{U_{j+1}^{m} - U_{j-1}^{m}}{2h} = \left(c + \frac{1}{2}a^{2}\tau\right) \frac{U_{j+1}^{m} - 2U_{j}^{m} + U_{j-1}^{m}}{h^{2}}$$



Finite Difference Schemes for Advection-Diffusion Equations

Central Explicit and Modified Central Explicit Schemes

Modified Central Explicit Scheme for the Advection-Diffusion Equation

- Local truncation error $O(c\tau + \tau^2 + h^2)$.
- **2** By the results of the central explicit scheme (with $\tilde{c} = c + \frac{1}{2}a^2\tau$):

$$\begin{cases} \left(c + \frac{1}{2}a^{2}\tau\right)\frac{\tau}{h^{2}} \leq \frac{1}{2}, \quad h \leq \frac{2c + a^{2}\tau}{|a|}, \quad \Leftrightarrow \text{ maximum principle;} \\ \left(c + \frac{1}{2}a^{2}\tau\right)\frac{\tau}{h^{2}} \leq \frac{1}{2}, \quad \tau \leq \frac{2c + a^{2}\tau}{a^{2}}, \quad \Leftrightarrow \mathbb{L}^{2} \text{ strong stability.} \end{cases}$$

Compared with the central explicit scheme, to satisfy the maximum principle, the condition on the spatial grid size is relaxed, but the condition on the grid ratio is more restricted.



Finite Difference Schemes for Advection-Diffusion Equations

Central Explicit and Modified Central Explicit Schemes

Stability Conditions for the Modified Central Explicit Scheme

The second condition for the L² strong stability holds automatically.

(5) The necessary and sufficient stability conditions:

$$\begin{cases} \left(c + \frac{1}{2}a^{2}\tau\right)\frac{\tau}{h^{2}} \leq \frac{1}{2}, \quad h \leq \frac{2c + a^{2}\tau}{a}, \quad \Leftrightarrow \text{ maximum principle;} \\ \left(c + \frac{1}{2}a^{2}\tau\right)\frac{\tau}{h^{2}} \leq \frac{1}{2}, \qquad \qquad \Leftrightarrow \mathbb{L}^{2} \text{ strong stability.} \end{cases}$$



Finite Difference Schemes for Advection-Diffusion Equations

Upwind Scheme for Advection-Diffusion Equation

The Upwind Scheme for the Advection-Diffusion Equation

The upwind scheme (a > 0)

$$\frac{U_{j}^{m+1} - U_{j}^{m}}{\tau} + a \frac{U_{j}^{m} - U_{j-1}^{m}}{h} = c \frac{U_{j+1}^{m} - 2U_{j}^{m} + U_{j-1}^{m}}{h^{2}},$$

or equivalently (compare the central explicit scheme)

$$\frac{U_{j}^{m+1} - U_{j}^{m}}{\tau} + a \frac{U_{j+1}^{m} - U_{j-1}^{m}}{2h} = \left(c + \frac{1}{2}ah\right) \frac{U_{j+1}^{m} - 2U_{j}^{m} + U_{j-1}^{m}}{h^{2}}$$



Finite Difference Schemes for Advection-Diffusion Equations

Upwind Scheme for Advection-Diffusion Equation

The Upwind Scheme for the Advection-Diffusion Equation

- 1 Local truncation error $O(\tau + h)$.
- **2** By the results of the central explicit scheme (with $\tilde{c} = c + \frac{1}{2}ah$):

$$\begin{cases} (2c+ah)\frac{\tau}{h^2} \leq 1, & h \leq \frac{2c+ah}{a}, & \Leftrightarrow \text{maximum principle}; \\ (2c+ah)\frac{\tau}{h^2} \leq 1, & \tau \leq \frac{2c+ah}{a^2}, & \Leftrightarrow \mathbb{L}^2 \text{ strong stability}. \end{cases}$$

The second condition for the maximum principle holds automatically.



Finite Difference Schemes for Advection-Diffusion Equations

Upwind Scheme for Advection-Diffusion Equation

The Stability Condition for the Upwind Scheme

The second condition for the L² strong stability is a consequence of the first, since

$$\frac{a^2\tau}{2c+ah}=\frac{a^2h^2}{2c+ah}\frac{\tau}{h^2}\leq\frac{(2c+ah)^2}{2c+ah}\frac{\tau}{h^2}=(2c+ah)\frac{\tau}{h^2},$$

The necessary and sufficient stability condition:

$$\begin{cases} (2c+ah)\frac{\tau}{h^2} \leq 1, & \Leftrightarrow \text{maximum principle;} \\ (2c+ah)\frac{\tau}{h^2} \leq 1, & \Leftrightarrow \mathbb{L}^2 \text{ strong stability.} \end{cases}$$



Finite Difference Schemes for Advection-Diffusion Equations

A Crank-Nicolson Type Implicit Scheme

A Crank-Nicolson Type Scheme for the Advection-Diffusion Equation

The Crank-Nicolson scheme for the advection-diffusion equation:

$$\frac{U_{j}^{m+1} - U_{j}^{m}}{\tau} + \frac{a}{2} \left[\frac{U_{j+1}^{m} - U_{j-1}^{m}}{2h} + \frac{U_{j+1}^{m+1} - U_{j-1}^{m+1}}{2h} \right]$$
$$= \frac{c}{2} \left[\frac{U_{j+1}^{m} - 2U_{j}^{m} + U_{j-1}^{m}}{h^{2}} + \frac{U_{j+1}^{m+1} - 2U_{j}^{m+1} + U_{j-1}^{m+1}}{h^{2}} \right].$$

or

$$(1+\mu)U_{j}^{m+1} = (1-\mu)U_{j}^{m+1} + \frac{1}{2}(\mu - \frac{\nu}{2})\left(U_{j+1}^{m} + U_{j+1}^{m+1}\right) + \frac{1}{2}(\mu + \frac{\nu}{2})\left(U_{j-1}^{m} + U_{j-1}^{m+1}\right).$$



Finite Difference Schemes for Advection-Diffusion Equations

└─A Crank-Nicolson Type Implicit Scheme

A Crank-Nicolson Type Scheme for the Advection-Diffusion Equation

1 Amplification factor:
$$\lambda_k = \frac{1 - \mu(1 - \cos kh) - i\frac{1}{2}\nu \sin kh}{1 + \mu(1 - \cos kh) + i\frac{1}{2}\nu \sin kh}$$
.

2 $|\lambda_k| \leq 1$, for all k, unconditionally \mathbb{L}^2 strongly stable.

Solution for the maximum principle µ ≤ 1, h ≤ 2c/|a|, i.e. the condition on µ is the same as for the diffusion equation, provided h is less than the Péclet number |a|h/c ≤ 2.



习题 3: 9, 11; 上机作业 1 Thank You!

