Numerical Solutions to Partial Differential Equations

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a Posteriori Local Error Estimator

1. Recall $\int_\Omega \nabla(u-u_h)\cdot\nabla w\,dx = R(u_h)(w)$, and the a posteriori local error estimator of residual type is given as

$$\eta_{R,K} = \Big\{ h_K^2 \, \|f_K\|_{0,2,K}^2 + \sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,1}} h_E \, \|g_E - \nu_E\cdot\nabla u_h\|_{0,E}^2 + \frac{1}{2}\sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,\Omega}} h_E \, \|[\nu_E\cdot\nabla u_h]_E\|_{0,E}^2 \Big\}^{1/2}.$$

2. We hope, by choosing proper bubble functions w, to establish relationship between the local error of $u-u_h$ and the three terms in $\eta_{R,K}$.

Triangular Element Bubble Functions and Edge Bubble Functions

1. For $K \in \mathfrak{T}_h(\Omega)$, the element bubble function $\mathfrak{b}_K$ is
$$\mathfrak{b}_K(x) = \begin{cases} 27\,\lambda_{K,1}(x)\,\lambda_{K,2}(x)\,\lambda_{K,3}(x), & \forall x \in K; \\ 0, & \forall x \in \Omega \setminus K. \end{cases}$$

2. For a given edge $E \in \mathcal{E}_{h,\Omega}$, the edge bubble function $\mathfrak{b}_E$ is
$$\mathfrak{b}_E(x) = \begin{cases} 4\,\lambda_{K_i,1}(x)\,\lambda_{K_i,2}(x), & \forall x \in K_i, \quad i = 1, 2; \\ 0, & \forall x \in \Omega \setminus \omega_E. \end{cases}$$

3. For a given edge $E \in \mathcal{E}_{h,\partial\Omega}$, the edge bubble function $\mathfrak{b}_E$ is
$$\mathfrak{b}_E(x) = \begin{cases} 4\,\lambda_{K',1}(x)\,\lambda_{K',2}(x), & \forall x \in K'; \\ 0, & \forall x \in \Omega \setminus K'. \end{cases}$$

Properties of the Bubble Functions

Lemma

For any given $K \in \mathfrak{T}_h(\Omega)$ and $E \in \mathcal{E}_h$, the bubble functions $\mathfrak{b}_K$ and $\mathfrak{b}_E$ have the following properties:

$$\operatorname{supp} \mathfrak{b}_K \subset K, \quad 0 \le \mathfrak{b}_K \le 1, \quad \max_{x \in K} \mathfrak{b}_K(x) = 1;$$

$$\operatorname{supp} \mathfrak{b}_E \subset \omega_E, \;\; 0 \le \mathfrak{b}_E \le 1, \;\; \max_{x \in E} \mathfrak{b}_E(x) = 1; \;\; \int_E \mathfrak{b}_E \, ds = \frac{2}{3} h_E;$$

there exists a constant $\hat{c}_i$, $i = 1, \ldots, 6$, which depends only on the smallest angle of the triangular triangulation $\mathfrak{T}_h(\Omega)$, such that

$$\hat{c}_1 \, h_K^2 \le \int_K \mathfrak{b}_K \, dx = \frac{9}{20} \, |K| \le \hat{c}_2 \, h_K^2;$$

$$\hat{c}_3 \, h_E^2 \le \int_{K'} \mathfrak{b}_E \, dx = \frac{1}{3} \, |K'| \le \hat{c}_4 \, h_E^2, \;\; \forall K' \subset \omega_E;$$

$$\|\nabla \mathfrak{b}_K\|_{0,2,K} \le \hat{c}_5 \, h_K^{-1} \|\mathfrak{b}_K\|_{0,2,K};$$

$$\|\nabla \mathfrak{b}_E\|_{0,2,K'} \le \hat{c}_6 \, h_E^{-1} \|\mathfrak{b}_E\|_{0,2,K'}, \;\; \forall K' \subset \omega_E.$$

The Efficiency of the Residual a Posteriori Local Error Estimator

Theorem

Let u and u_h be the solution and the finite element solution of the variational problem. Let η_h be an residual type local a posteriori error estimator given above (see (8.1.8)). Then, there exists a constant $\widetilde{C}$, which depends only on the smallest angle of the triangular triangulation $\mathfrak{T}_h(\Omega)$, such that

$$\eta_{R,K} \leq \widetilde{C} \left\{ \|u - u_h\|^2_{1,2,\omega_K} + \sum_{K' \in \omega_K} h^2_{K'} \|f - f_{K'}\|^2_{0,2,K'} + \sum_{E \in \mathcal{E}(K) \cap \mathcal{E}_{h,1}} h_E \|g - g_E\|^2_{0,E} \right\}^{1/2}, \quad \forall K \in \mathfrak{T}_h(\Omega).$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|f_K\|_{0,2,K}$

1. For any given $K \in \mathfrak{T}_h(\Omega)$, set $w_K := f_K \mathfrak{b}_K$. Then, by the properties of $\mathfrak{b}_K$ (see Lemma 8.3), we have

$$\int_K f_K \, w_K \, dx = \frac{9}{20} \, |K| \, |f_K|^2 = \frac{9}{20} \, \|f_K\|_{0,2,K}^2.$$

2. Since $\operatorname{supp} w_K \subset K$, it follows

$$\int_{\partial\Omega_1} g \, w_K \, ds - \int_\Omega \nabla u_h \cdot \nabla w_K \, dx = -\nabla u_h|_K \int_K \nabla w_K \, dx = 0.$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|f_K\|_{0,2,K}$

3. Thus, by $\int_\Omega \nabla(u-u_h)\cdot\nabla w_K\,dx = R(u_h)(w_K)$, we obtain

$$\begin{aligned}\int_K f_K\,w_K\,dx &= \int_K f\,w_K\,dx + \int_K (f-f_K)\,w_K\,dx \\ &= \int_K \nabla(u-u_h)\cdot\nabla w_K\,dx + \int_K (f-f_K)\,w_K\,dx \\ &\le \|u-u_h\|_{1,2,K}\|\nabla w_K\|_{0,2,K} + \|f-f_K\|_{0,2,K}\|w_K\|_{0,2,K}.\end{aligned}$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|f_K\|_{0,2,K}$

4. On the other hand, since f_K is a constant, by the properties of $\mathfrak{b}_K$ (see Lemma 8.3), we have

$$\|w_K\|_{0,2,K} = |f_K| \, \|\mathfrak{b}_K\|_{0,2,K} \le |f_K| \left(\int_K \mathfrak{b}_K \, dx \right)^{1/2} = \sqrt{\frac{9}{20}} \, \|f_K\|_{0,2,K};$$

$$\|\nabla w_K\|_{0,2,K} \le \hat{c}_5 \, h_K^{-1} \|w_K\|_{0,2,K}.$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|[\nu_E \cdot \nabla u_h]_E\|_{0,E}$

5. Combining the three inequalities obtained in ③ and ④ with $\int_K f_K\, w_K\, dx = \frac{9}{20}\,\|f_K\|_{0,2,K}^2$ (see (8.2.3)) leads to

$$\|f_K\|_{0,2,K} \le \sqrt{\frac{20}{9}}\;\hat{c}_5\, h_K^{-1}\|u - u_h\|_{1,2,K} + \sqrt{\frac{20}{9}}\;\|f - f_K\|_{0,2,K}.$$

6. For any given $E \in \mathcal{E}_{h,\Omega}$, set $w_E := [\nu_E \cdot \nabla u_h]_E\, \mathfrak{b}_E$, by the properties of $\mathfrak{b}_E$ (see Lemma 8.3), we have

$$\int_E [\nu_E \cdot \nabla u_h]_E\, w_E\, ds = \frac{2}{3}\, h_E\, \left|[\nu_E \cdot \nabla u_h]_E\right|^2 = \frac{2}{3}\, \|[\nu_E \cdot \nabla u_h]_E\|_{0,E}^2.$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|[\nu_E \cdot \nabla u_h]_E\|_{0,E}$

7. Since $\operatorname{supp} w_E \subset \omega_E$, it follows from u is the solution and $\int_\Omega \nabla(u-u_h)\cdot\nabla w_E\,dx = R(u_h)(w_E)$ (see (8.1.4)) that

$$
\begin{aligned}
\int_E [\nu_E \cdot \nabla u_h]_E\, w_E\, ds &= \int_\Omega \nabla u_h \cdot \nabla w_E\, dx \\
&= \sum_{K\subset\omega_E} \int_K f\, w_E\, dx - \int_{\omega_E} \nabla(u-u_h)\cdot\nabla w_E\, dx \\
&\le \|f\|_{0,2,\omega_E}\|w_E\|_{0,2,\omega_E} + \|u-u_h\|_{1,2,\omega_E}\|\nabla w_E\|_{0,2,\omega_E}.
\end{aligned}
$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|[\nu_E \cdot \nabla u_h]_E\|_{0,E}$

8. Since $[\nu_E \cdot \nabla u_h]_E$ is a constant, and $w_E := [\nu_E \cdot \nabla u_h]_E \, \mathfrak{b}_E$, by the properties of $\mathfrak{b}_E$ (see Lemma 8.3), we have

$$\|\nabla w_E\|_{0,2,\omega_E} \le \hat{c}_6 \, h_E^{-1} \|w_E\|_{0,2,\omega_E},$$

$$\begin{aligned}\|w_E\|_{0,2,\omega_E} &= |[\nu_E \cdot \nabla u_h]_E| \, \|\mathfrak{b}_E\|_{0,2,\omega_E} \le |[\nu_E \cdot \nabla u_h]_E| \left(\int_{\omega_E} \mathfrak{b}_E \, dx \right)^{1/2} \\ &\le \sqrt{2\,\hat{c}_4} \; h_E \, |[\nu_E \cdot \nabla u_h]_E| \le \sqrt{2\,\hat{c}_4} \; h_E^{1/2} \, \|[\nu_E \cdot \nabla u_h]_E\|_{0,E}.\end{aligned}$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|[\nu_E \cdot \nabla u_h]_E\|_{0,E}$

9. Combining the three inequalities obtained in ⑦ and ⑧ with

$$\int_E [\nu_E \cdot \nabla u_h]_E \, w_E \, ds = \frac{2}{3} \|[\nu_E \cdot \nabla u_h]_E\|_{0,E}^2$$

(see (8.2.5)), and $\|f\|_{0,2,\omega_E} \le \sum_{K\subset\omega_E} \|f - f_K\|_{0,2,K} + \sum_{K\subset\omega_E} \|f_K\|_{0,2,K}$

and ⑤ (see (8.2.4)), we are leads to

$$\|[\nu_E \cdot \nabla u_h]_E\|_{0,E} \le \frac{3}{2} \, c \, h_E^{-1/2} \left\{ h_E \sum_{K\subset\omega_E} \|f - f_K\|_{0,2,K} + \|u - u_h\|_{1,2,\omega_E} \right\},$$

where $c = \sqrt{2\,\hat{c}_4} \; \max\left\{ 1 + \sqrt{\frac{20}{9}}, \; \hat{c}_6 + \sqrt{\frac{20}{9}} \, \hat{c}_5 \; \max_{K\subset\omega_E} h_E/h_K \right\}$.

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|g_E - \nu_E \cdot \nabla u_h\|_{0,E}$

10. For any given $E \in \mathcal{E}_{h,\partial\Omega_1}$, set $w_E := (g_E - \nu_E \cdot \nabla u_h)\, \mathfrak{b}_E$, by the properties of $\mathfrak{b}_E$ (see Lemma 8.3), we have

$$\int_E (g_E - \nu_E \cdot \nabla u_h) w_E \, ds = \frac{2}{3} h_E |g_E - \nu_E \cdot \nabla u_h|^2 = \frac{2}{3} \|g_E - \nu_E \cdot \nabla u_h\|_{0,E}^2.$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|g_E - \nu_E \cdot \nabla u_h\|_{0,E}$

10. Since E is on the boundary and $\operatorname{supp} w_E \subset \omega_E$, it follows from (8.1.4) that

$$
\begin{aligned}
&\int_E (g_E - \nu_E \cdot \nabla u_h) w_E \, ds = \int_E (g - \nu_E \cdot \nabla u_h) w_E \, ds + \int_E (g_E - g) w_E \, ds \\
&= \int_\Omega f w_E \, dx + \int_{\partial\Omega_1} g w_E \, ds - \int_\Omega \nabla u_h \cdot \nabla w_E \, dx - \int_{\omega_E} f w_E \, dx + \int_E (g_E - g) w_E \, ds \\
&= \int_{\omega_E} \nabla(u - u_h) \cdot \nabla w_E \, dx - \int_{\omega_E} f \, w_E \, dx + \int_E (g_E - g) \, w_E \, ds \\
&\leq \|u - u_h\|_{1,2,\omega_E} \|\nabla w_E\|_{0,2,\omega_E} + \|f\|_{0,2,\omega_E} \|w_E\|_{0,2,\omega_E} + \|g - g_E\|_{0,E} \|w_E\|_{0,E}.
\end{aligned}
$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|g_E - \nu_E \cdot \nabla u_h\|_{0,E}$

12. Since $(g_E - \nu_E \cdot \nabla u_h)$ is a constant, $\omega_E = K'$ and $\|f\|_{0,2,K'} \leq \|f - f_{K'}\|_{0,2,K'} + \|f_{K'}\|_{0,2,K'}$, by Lemma 8.3 and (8.2.4), this combined with $\int_E (g_E - \nu_E \cdot \nabla u_h) w_E \, ds = \frac{2}{3}\|g_E - \nu_E \cdot \nabla u_h\|_{0,E}^2$ (see (8.2.7)) yields

$$\|g_E - \nu_E \cdot \nabla u_h\|_{0,E} \leq \frac{3}{2}\, c'\, h_E^{-1/2} \{\|u - u_h\|_{1,2,K'} + h_E \,\|f - f_{K'}\|_{0,2,K'} + h_E^{1/2}\,\|g - g_E\|_{0,E}\},$$

where

$$c' = \max\left\{\sqrt{\frac{2}{3}},\ \sqrt{\hat{c}_4}\left(1+\sqrt{\frac{20}{9}}\right),\ \sqrt{\hat{c}_4}\left(\hat{c}_6+\sqrt{\frac{20}{9}}\,\hat{c}_5\, h_E/h_{K'}\right)\right\}.$$

13. The theorem follows from the three parts with $\widetilde{C} =$

$$\max\left\{3\sqrt{2}, 3\sqrt{2}\hat{c}_5, 3\sqrt{3\hat{c}_4}\left(1+\sqrt{\tfrac{20}{9}}\right), 3\sqrt{3\hat{c}_4}\left(\hat{c}_6+\sqrt{\tfrac{20}{9}}\hat{c}_5\right)\right\}. \blacksquare$$

The a Posteriori Local Error Estimator $\eta_{R,K}$ is Reliable and Efficient

1. By Theorem 8.3, we have

$$\|u-u_h\|_{1,2,\Omega} \le C\left\{\sum_{K\in\mathfrak{T}_h(\Omega)} \eta_{R,K}^2 + \sum_{K\in\mathfrak{T}_h(\Omega)} h_K^2\,\|f-f_K\|_{0,2,K}^2 + \sum_{E\in\mathcal{E}_{h,1}} h_E\,\|g-g_E\|_{0,E}^2\right\}^{1/2}.$$

2. By Theorem 8.4, we have

$$\eta_{R,K} \le \widetilde{C}\left\{\|u-u_h\|_{1,2,\omega_K}^2 + \sum_{K'\in\omega_K} h_{K'}^2\|f-f_{K'}\|_{0,2,K'}^2 + \sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,1}} h_E\|g-g_E\|_{0,E}^2\right\}^{1/2}, \ \forall K\in\mathfrak{T}_h(\Omega).$$

The a Posteriori Local Error Estimator $\eta_{R,K}$ is Reliable and Efficient

3. In applications, if the mesh is reasonably fine,

$$\Big\{ \sum_{K \in \mathfrak{T}_h(\Omega)} h_K^2 \, \|f - f_K\|_{0,2,K}^2 \Big\}^{1/2} = o(h),$$

$$\Big\{ \sum_{E \in \mathcal{E}_{h,\partial\Omega_1}} h_E \, \|g - g_E\|_{0,2,E}^2 \Big\}^{1/2} = o(h),$$

and in nontrivial cases,

$$\|u - u_h\|_{1,2,\Omega} \geq C\, h.$$

In such cases, by Theorem 8.3 and Theorem 8.4, the a posteriori local error estimator $\eta_{R,K}$ is efficient as well as reliable.

What is an Adaptive Method

1. A practical computation start with an initial mesh, according to theory or experience, often quite coarse to reduce the cost, and reasonably fine so that the key features of the solution can be roughly captured.

2. Generally speaking, one can not guarantee that the initially obtained finite element solution has the required accuracy.

What is an Adaptive Method

3. By the properties of the current finite element solution, for example, the scale of the local a posteriori error estimator, and by certain strategy, the mesh is refined or coarsened (non-uniformly) so that a finite element solution with sufficiently high approximation accuracy can be obtained with reasonably low cost.

4. An adaptive method usually consists of many such mesh refining and coarsening processes according to the complexity of the problem, the cost and accuracy requirements, etc..

Adaptive methods of *h*-version, *p*-version and *h*-*p*-version

By Theorem 7.10, for second order elliptic problems, under certain general conditions, we have the a priori error estimate

$$\|u - u_h\|_{1,\Omega} \leq C\, h^k\, |u|_{k+1,\Omega}.$$

Obviously, the error of the finite element solution can be reduced by decreasing h or increasing k.

This observation leads to the following 3-versions of adaptive methods:

1. h-version: reduce h by refining the mesh.
2. p-version: increase k by including higher order polynomials.
3. h-p-version: the combination of h-version and p-version.

Adaptive methods of *h*-version, *p*-version and *h*-*p*-version

Remark 1: The general idea is to increase the local approximation accuracy by increasing the local degrees of freedom of the finite element function.

Remark 2: The p-version works only when u is sufficiently smooth.

Remark 3: May increase h and/or decrease k locally where the local error is much smaller than the average.

Adaptive Criterion Using a Posteriori Local Error Estimators

1. η_K: an efficient local a posteriori error estimator, $K \in \mathfrak{T}_h(\Omega)$.
2. Denote $\eta := \max_{K \in \mathfrak{T}_h(\Omega)} \eta_K$. Choose a threshold $\gamma \in (0,\ 1)$.
3. Mark all elements K, which satisfy $\eta_K \geq \gamma\, \eta$, as the candidates of mesh refinement.
4. An alternative way is to mark the edges instead, since, for most a posteriori error estimators, the leading term consists of the jumps across the edges.

Adaptive Criterion Using a Posteriori Local Error Estimators

Note: Another more complicated way is:

1. Let there be an asymptotic error estimate of the form $c\, h_K^{\lambda}$.
2. $\mathfrak{T}_h(\Omega)$ be a uniform refinement of $\mathfrak{T}_H(\Omega)$.
3. The constants c and λ can be roughly estimated by the a posteriori error estimators η_{H,K_H} and η_{h,K_h}.
4. Mark the elements and estimate the final local mesh size accordingly.

Basic requirements of the mesh refinement process

- No hanging nodes — the vertices of refined elements located on the middle of an edge of the neighboring element.

- The shape of the elements in the final mesh should maintain certain regularity, for example, the smallest angle should have a positive lower bound.

3 Mesh Refinement Methods for Triangular Element in $\mathbb{R}^2$

1. Longest edge bisection: divide a marked element into two elements by connecting the middle point of the longest edge of the element to its opposite vertex;

2. Marked edge bisection: divide an element into two by connecting the middle point of the marked edge of the element to its opposite vertex;

3. Regular refinement: divide a marked element into four by connecting the the middle points of the three edges.

Additional Rules to Remove Hanging Nodes

The two additional rules to the longest edge bisection refinement (green refinement, see Figure 8.1(b)):

(1a) If there are two hanging nodes not on the longest edge, then, divide the triangle into four by the regular refinement (red refinement, see Figure 8.1(a));

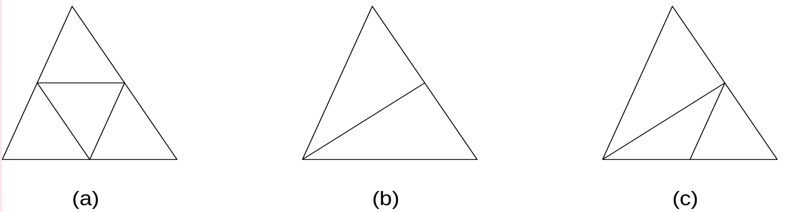
(1b) If there is only one hanging node not on the longest edge, then, divide the triangle into three by connecting the hanging node to the middle point of the longest edge (whether it is a hanging node or not) and the opposite vertex respectively (named as the blue refinement, see Figure 8.1(c)).

Regularity: the mesh produced by repeatedly applying the longest bisection refinement with the two additional rules (1a) and (1b) has the property that $\theta_{\mathbf{min}}^{\mathbf{final}} \geq \frac{1}{2}\theta_{\mathbf{min}}^{\mathbf{initial}}$ (cf. [26]).

The Red, Green and Blue Refinements

1. The longest edge bisection is also called the green refinement;
2. The regular refinement is also called the red refinement;
3. The blue refinement: if there is only one hanging node which is not on the longest edge of a triangle, then, divide the triangle into three triangles by connecting the middle point of the longest edge (whether it is a hanging node or not) to the hanging node and the opposite vertex respectively.

(a): red refinement; (b): green refinement; (c): blue refinement.



The Marked Edge (Newest Vertices) Refinement

The marked edge bisection refinement, also known as the newest vertex bisection, since the new marked edges are usually the opposite edges of the newest vertices.

The marked edge bisection refinement should follow the rules:

(2a) Each triangle has exactly one marked edge;

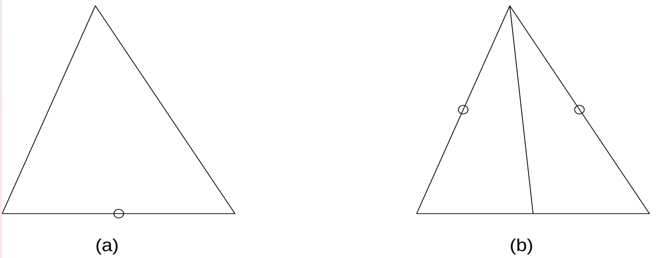
(2b) When a triangle is bisected, its two unmarked edges become the marked edges of its two sons (see Figure 8.2, where $\circ$ represents an marked edge);

(2c) If and only if an edge is a common marked edge of both of its adjacent triangles, the bisection will be carried out (on both triangles at the same time).

The Marked Edge (Newest Vertices) Refinement

The rules (2a) and (2c) ensure that the elements are only refined by bisection, while the rule (2b) ensures that an initial triangle will produce at most four different types of similar triangles [12,28].

(a): Father element; (b): Son elements and their marked edges.



The Strategies of Dealing with Hanging Nodes for Regular Refinement

The regular refinement always produces son triangles similar to their father ones, but usually also introduces hanging nodes, which can be treated in one of the following ways:

1. The hanging nodes are not regarded as degrees of freedom, instead the function values on the hanging nodes are calculated by the corresponding interpolation.

2. To apply the longest edge bisection (red, green and blue refinements) to eliminate the hanging nodes.

Finite Steps Termination and Regularity

Remark:

- The process of eliminating the hanging nodes by the longest edge bisection **always stops in finite steps**.

- The final mesh obtained by repeatedly applying the combination of the regular refinement and the longest edge bisection has the property that

$$\theta_{\text{min}}^{\text{final}} \geq \frac{1}{2}\theta_{\text{min}}^{\text{initial}}.$$

General Remarks on Nested Mesh Refinements

1. The meshes produced by the above mesh refinement methods are nested, *i.e.* an element of a coarser mesh is always a union of several elements of a finer mesh.

   The property makes it easier to coarsening the mesh whenever necessary.

2. The finite element function spaces defined on the nested meshes are also nested, *i.e.* $\mathbb{V}_H \subset \mathbb{V}_h$.

   The property makes it easier to apply fast solvers, such as the multi-grid methods.

Mesh Redistribution (*r*-Version) Adaptive Methods

1. Do not change the total number of the degrees of freedom.
2. Do not even change the topological structure of the mesh.
3. Redistribute the mesh according to the finite element solution obtained on the current mesh and certain mesh redistribution principles.

Mesh Redistribution by Error Equi-Distribution Principle

A popular mesh redistribution principle is the so called error equi-distribution principle, which assume among all meshes with the same topological structure, the mesh with uniformly distributed error must have the least global error.

The error equi-distribution principle is not easily implemented in applications. So, it is often relaxed by introducing certain monitor functions.

Mesh Redistribution According to Equi-Arc-Length Principle

1. To solve a two point BVP of $u'' + f = 0$ on $(0,\ 1)$ with a continuous piecewise affine finite element function space.
2. Let $\xi_i = i/N$, $i = 0, 1, \dots, N$, be a uniform partition Ξ_N of $[0,\ 1]$, let $x_h : [0,\ 1] \to [0,\ 1]$ be an arbitrarily given continuous piecewise affine and strictly monotonically increasing finite element function.
3. Then, $x_i = x_h(\xi_i)$, $i = 0, 1, \dots, N$, defines on $[0,\ 1]$ a finite element partition (*i.e.* a triangulation) X_h.
4. Let u_h be the finite element solution obtained on the mesh X_h.
5. Take the solution arc-length density $g(x) = \sqrt{1 + |u_h'(x)|^2}$ as a monitor function. Define $G_i = g(x)|_{x \in (x_{i-1},\ x_i)}$.
6. Equi-Arc-Length Principle:

$$G_i \cdot (x_i - x_{i-1}) = G_j \cdot (x_j - x_{j-1}), \qquad \forall\ 0 \le i, j \le N.$$

Moving Mesh According to Equi-Arc-Length Principle

1. Increase the cell length $\triangle x_i$, if G_i is less than the average.
2. Decrease the cell length $\triangle x_i$, if G_i is greater than the average.
3. The aim is $G_i \cdot (x_i - x_{i-1}) = G_j \cdot (x_j - x_{j-1})$; or

$$g(x_h(\xi))x_h'(\xi)|_{(\xi_{i-1},\xi_i)} = g(x_h(\xi))x_h'(\xi)|_{(\xi_{j-1},\xi_j)}, \quad 1 \le i,j \le N;$$

or $\quad g(x_h(\xi))x_h'(\xi) = \text{constant}, \ 1 \le i \le N.$

Moving Mesh According to Equi-Arc-Length Principle

4. So, the mesh redistribution can be realized by the finite element solution x_h of the harmonic equation

$$\begin{cases} (g(x(\xi))x'(\xi))' = 0, & \forall \xi \in (0,\ 1); \\ x(0) = 0,\ \ x(1) = 1, \end{cases}$$

on the uniform mesh Ξ_N. This is a nonlinear problem.

5. Start from an initial mesh, apply some iterative method, for example, the Picard iteration, to move the mesh, project the current u_h on the new mesh and solve for the new u_h, $\cdots$, until convergence.

Moving Mesh According to Energy Minimization Principle

1. Consider the energy minimization problem:

$$\begin{cases} \text{Find } u \in \mathbb{V} = \mathbb{H}_0^1(\Omega) \text{ such that} \\ F(u) = \inf_{v \in \mathbb{V}} F(v). \end{cases}$$

2. $F(u) = \frac{1}{2}a(u,\, u) - f(u)$; $a(\cdot,\, \cdot)$ bounded symmetric uniformly elliptic bilinear, $f(\cdot)$ bounded linear forms defined on $\mathbb{V}$.

3. Then, we have

$$\begin{aligned} F(v) - F(u) &= \frac{1}{2}a(v,\, v) - a(u,\, v) - \frac{1}{2}a(u,\, u) + a(u,\, u) \\ &= \frac{1}{2}a(v - u,\, v - u), \qquad \forall v \in \mathbb{V}. \end{aligned}$$

4. $\mathbb{T}_h(\Omega) = \{\hat{\mathfrak{T}}_h(\Omega) :$ which is topologically same as $\mathfrak{T}_h(\Omega)\}$.

5. Find a mesh $\hat{\mathfrak{T}}_h(\Omega) \in \mathbb{T}_h(\Omega)$ and $\hat{u}_h \in \hat{\mathbb{V}}_h$, such that $F(\hat{u}_h)$ minimize the energy.

Thank You!