Numerical Solutions to Partial Differential Equations

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Mixed BVP of Poisson Equation on Polygonal Region in $\mathbb{R}^2$

- Consider the boundary value problem of the Poisson equation

$$\begin{cases} -\triangle u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega_0, \qquad \dfrac{\partial u}{\partial \nu} = g, \quad x \in \partial\Omega_1, \end{cases}$$

where Ω is a polygonal region in $\mathbb{R}^2$, $\partial\Omega_0$ is a relative closed subset in $\partial\Omega$ with positive 1-dimensional measure,

$$\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1, \;\; \partial\Omega_0 \cap \partial\Omega_1 = \emptyset,$$

$$f \in \mathbb{L}^2(\Omega), \; g \in \mathbb{L}^2(\partial\Omega_1).$$

Mixed BVP of Poisson Equation on Polygonal Region in $\mathbb{R}^2$

- consider the standard weak form of the problem:

$$\begin{cases} \text{Find} \quad u \in \mathbb{V} \text{ such that} \\ \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f \, v \, dx + \int_{\partial\Omega_1} g \, v \, ds, \quad \forall v \in \mathbb{V}, \end{cases}$$

where $\mathbb{V} = \left\{ v \in \mathbb{H}^1(\Omega) : v|_{\partial\Omega_0} = 0 \right\}$;

- consider the conforming finite element method based on a family of regular class C^0 type (1) Lagrange triangular elements.

A Theorem on the Relation of Residual and Error of a FE Solution

1. Define the residual operator $R : \mathbb{V} \to \mathbb{V}^*$ of the problem by
$$R(v)(w) = \int_\Omega f\, w\, dx + \int_{\partial\Omega_1} g\, w\, ds - \int_\Omega \nabla v \cdot \nabla w\, dx, \quad \forall w \in \mathbb{V}.$$
2. The dual norm of the residual of a finite element solution u_h:

$$\|R(u_h)\|_{\mathbb{V}^*} = \sup_{\substack{w \in \mathbb{V} \\ \|w\|_{1,2,\Omega}=1}} \left\{ \int_\Omega fw\, dx + \int_{\partial\Omega_1} gw\, ds - \int_\Omega \nabla u_h \cdot \nabla w\, dx \right\}$$

Theorem

Let $u \in \mathbb{V}$, $u_h \in \mathbb{V}_h$ be the weak solution and the finite element solution of the problem respectively. Then, there exists a constant $C(\Omega)$, which depends only on Ω, such that
$$\|R(u_h)\|_{\mathbb{V}^*} \leq \|u - u_h\|_{1,2,\Omega} \leq C(\Omega)\, \|R(u_h)\|_{\mathbb{V}^*}.$$

Remarks on Residual Dual Norm Estimation

1. We hope to develop a formula, which is easily computed and involves only available data such as f, g, u_h and geometric parameters of the triangulation and thus is usually called an a posteriori error estimator, to evaluate the dual norm of the residual.
2. Recall that in the a priori error estimates, the polynomial invariant interpolation operator plays an important role. For example, write w as $(w - \Pi_h w) + \Pi_h w$ can have some advantage.
3. However, the Lagrange nodal type interpolation operators require the function to be at least in $\mathbb{C}^0$.
4. Here, we need to introduce a polynomial invariant interpolation operator for functions in $\mathbb{H}^1$.

Notations on a Family of Regular Triangular Triangulations $\{\mathfrak{T}_h(\Omega)\}_{h>0}$

1. $\mathcal{E}(K)$, $\mathcal{N}(K)$: the sets of all edges and vertices of $K \in \mathfrak{T}_h(\Omega)$.
2. Denote $\mathcal{E}_h := \bigcup_{K \in \mathfrak{T}_h(\Omega)} \mathcal{E}(K)$, $\mathcal{N}_h := \bigcup_{K \in \mathfrak{T}_h(\Omega)} \mathcal{N}(K)$.
3. $\mathcal{N}(E)$: the sets of all vertices of an edge $E \in \mathcal{E}_h$.
4. $\mathcal{E}_{h,i} := \left\{ E \in \mathcal{E}_h : \overset{\circ}{E} \subset \partial\Omega_i \right\}$, $\mathcal{N}_{h,i} := \mathcal{N}_h \cap \partial\Omega_i$, $i = 0, 1$.
5. $\mathcal{E}_{h,\Omega} = \mathcal{E}_h \setminus (\mathcal{E}_{h,0} \cup \mathcal{E}_{h,1})$, $\mathcal{N}_{h,\Omega} = \mathcal{N}_h \setminus (\mathcal{N}_{h,0} \cup \mathcal{N}_{h,1})$.

Notations on a Family of Regular Triangular Triangulations $\{\mathfrak{T}_h(\Omega)\}_{h>0}$

6. $\omega_K := \bigcup\limits_{\mathcal{E}(K)\cap\mathcal{E}(K')\neq\emptyset} K'$, $\omega_E := \bigcup\limits_{E\in\mathcal{E}(K')} K'$, $\omega_x := \bigcup\limits_{x\in\mathcal{N}(K')} K'$.

7. $\tilde{\omega}_K := \bigcup_{\mathcal{N}(K)\cap\mathcal{N}(K')\neq\emptyset} K'$, $\quad \tilde{\omega}_E := \bigcup_{\mathcal{N}(E)\cap\mathcal{N}(K')\neq\emptyset} K'$.

8. The corresponding finite element function space:

$$\mathbb{V}_h = \{v \in \mathbb{C}(\bar{\Omega}) : v|_K \in \mathbb{P}_1(K), \forall K \in \mathfrak{T}_h(\Omega), v(x) = 0, \forall x \in \mathcal{N}_{h,0}\}.$$

The Clément Interpolation Operator $I_h : \mathbb{V} \to \mathbb{V}_h$

Definition

For any $v \in \mathbb{V}$ and $x \in \mathcal{N}_h$, denote $\pi_x v$ as the $\mathbb{L}^2(\omega_x)$ projection of v on $\mathbb{P}_1(\omega_x)$, meaning $\pi_x v \in \mathbb{P}_1(\omega_x)$ satisfies

$$\int_{\omega_x} v\, p\, dx = \int_{\omega_x} (\pi_x v)\, p\, dx, \quad \forall p \in \mathbb{P}_1(\omega_x).$$

The Clément interpolation operator $I_h : \mathbb{V} \to \mathbb{V}_h$ is defined by

$$I_h v(x) = (\pi_x v)(x), \ \ \forall x \in \mathcal{N}_{h,\Omega} \cup \mathcal{N}_{h,1}; \qquad I_h v(x) = 0, \ \ \forall x \in \mathcal{N}_{h,0}.$$

The Clément Interpolation Operator $I_h : \mathbb{V} \to \mathbb{V}_h$

1. The Clément interpolation operator is well defined on $\mathbb{L}^1(\Omega)$.
2. If $v \in \mathbb{P}_1(\omega_x)$, then $(\pi_x)v(x) = v(x)$, $\forall x \in \omega_x$.
3. If $v \in \mathbb{P}_1(\tilde{\omega}_K)$, then $I_h v(x) = v(x)$, $\forall x \in K$.
4. It is in the above sense that the Clément interpolation operator is polynomial (more precisely $\mathbb{P}_1$) invariant.

Error Estimates of the Clément Interpolation Operator I_h

Lemma

There exist constants $C_1(\theta_{\min})$ and $C_2(\theta_{\min})$, which depend only on the smallest angle $\theta_{\min}$ of the triangular elements in the triangulation $\mathfrak{T}_h(\Omega)$, such that, for any given $K \in \mathfrak{T}_h(\Omega)$, $E \in \mathcal{E}_h$ and $v \in \mathbb{V}$,

$$\begin{aligned} \|v - I_h v\|_{0,2,K} &\leq C_1(\theta_{\min})\, h_K\, |v|_{1,2,\tilde{\omega}_K}, \\ \|v - I_h v\|_{0,E} := \|v - I_h v\|_{0,2,E} &\leq C_2(\theta_{\min})\, h_K^{1/2}\, |v|_{1,2,\tilde{\omega}_E}. \end{aligned}$$

Error Estimates of the Clément Interpolation Operator I_h

- More general properties and proofs on the Clément interpolation operator may be found in [8, 31].

- The basic ingredients of the proof are the scaling techniques (which include the polynomial quotient space and equivalent quotient norms, the relations of semi-norms on affine equivalent open sets), and the inverse inequality.

An Upper Bound for the Dual Norm of the Residual $R(u_h)$

Lemma

There exists a constant $C(\theta_{\min})$, where $\theta_{\min}$ is the smallest angle of the triangular elements in the triangulation $\mathfrak{T}_h(\Omega)$, such that

$$\int_\Omega f\, w\, dx + \int_{\partial\Omega_1} g\, w\, ds - \int_\Omega \nabla u_h \cdot \nabla w\, dx$$
$$\leq C(\theta_{\min}) \|w\|_{1,2,\Omega} \Big\{ \sum_{K \in \mathfrak{T}_h(\Omega)} h_K^2 \, \|f\|_{0,2,K}^2 + \sum_{E \in \mathcal{E}_{h,1}} h_E \, \|g - \nu_E \cdot \nabla u_h\|_{0,E}^2$$
$$+ \sum_{E \in \mathcal{E}_{h,\Omega}} h_E \, \|[\nu_E \cdot \nabla u_h]_E\|_{0,E}^2 \Big\}^{1/2}, \quad \forall w \in \mathbb{V},$$

Proof of the Lemma — An Upper Bound for the Residual $R(u_h)$

where in the theorem, ν_E is an arbitrarily given unit normal of E if $E \in \mathcal{E}_{h,\Omega}$, and is the unit exterior normal of Ω if $E \in \mathcal{E}_{h,1}$, $[\varphi]_E$ is the jump of φ across E in the direction of ν_E, *i.e.*

$$[\varphi]_E(x) = \lim_{t\to 0+} \varphi(x + t\nu_E) - \lim_{t\to 0+} \varphi(x - t\nu_E), \quad \forall x \in E.$$

Proof:

1. Since u_h is the finite element solution, we have

$$R(u_h)(v_h) := \int_\Omega f v_h \, dx + \int_{\partial\Omega_1} g v_h \, ds - \int_\Omega \nabla u_h \cdot \nabla v_h \, dx = 0, \quad \forall v_h \in \mathbb{V}_h.$$

   In particular, $R(u_h)(w) = R(u_h)(w - I_h w)$, for all $w \in \mathbb{V}$.

Proof of the Lemma — An Upper Bound for the Residual $R(u_h)$

2. Applying the Green's formula on every element K, denoting ν_K as the unit exterior normal of ∂K, and noticing $u_h|_K \in \mathbb{P}_1(K)$ and thus $\triangle u_h = 0$ on each element, we obtain

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = \sum_{K \in \mathfrak{T}_h(\Omega)} \int_K \nabla u_h \cdot \nabla v \, dx$$

$$= \sum_{K \in \mathfrak{T}_h(\Omega)} \left\{ -\int_K \triangle u_h \, v \, dx + \int_{\partial K} \nu_K \cdot \nabla u_h \, v \, dx \right\}$$

$$= \sum_{E \in \mathcal{E}_{h,1}} \int_E \nu_E \cdot \nabla u_h \, v \, ds + \sum_{E \in \mathcal{E}_{h,\Omega}} \int_E [\nu_E \cdot \nabla u_h]_E \, v \, ds, \quad \forall v \in \mathbb{V}.$$

Proof of the Lemma — An Upper Bound for the Residual $R(u_h)$

3. Thus, recall $R(u_h)(w) = R(u_h)(w - I_h w)$, we are lead to

$$\int_\Omega f\, w\, dx + \int_{\partial\Omega_1} g\, w\, ds - \int_\Omega \nabla u_h \cdot \nabla w\, dx = \sum_{K \in \mathfrak{T}_h(\Omega)} \int_K f\, (w - I_h w)\, dx$$

$$+ \sum_{E \in \mathcal{E}_{h,1}} \int_E (g - \nu_E \cdot \nabla u_h)(w - I_h w)\, ds - \sum_{E \in \mathcal{E}_{h,\Omega}} \int_E [\nu_E \cdot \nabla u_h]_E\, (w - I_h w)\, ds,$$

Proof of the Lemma — An Upper Bound for the Residual $R(u_h)$

4. By the Cauchy-Schwarz inequality and Lemma 8.1, we have

$$\int_K f\,(w-I_h w)\,dx \le \|f\|_{0,2,K}\|w-I_h w\|_{0,2,K} \le C_1\,h_K\,\|f\|_{0,2,K}\|w\|_{1,2,\tilde{\omega}_K},$$

$$\begin{aligned}\int_E (g-\nu_E\cdot\nabla u_h)(w-I_h w)ds &\le \|g-\nu_E\cdot\nabla u_h\|_{0,E}\|w-I_h w\|_{0,E}\\ &\le C_2 h_E^{1/2}\|g-\nu_E\cdot\nabla u_h\|_{0,E}\|w\|_{1,2,\tilde{\omega}_E},\end{aligned}$$

$$\begin{aligned}\int_E [\nu_E\cdot\nabla u_h]_E\,(w-I_h w)\,ds &\le \|[\nu_E\cdot\nabla u_h]_E\|_{0,E}\|w-I_h w\|_{0,E}\\ &\le C_2 h_E^{1/2}\|[\nu_E\cdot\nabla u_h]_E\|_{0,E}\|w\|_{1,2,\tilde{\omega}_E}.\end{aligned}$$

Proof of the Lemma — An Upper Bound for the Residual $R(u_h)$

5. The number of element in ω_x, $\sharp\omega_x \le C_3 = 2\pi/\theta_{\min}$, $\forall x \in \mathcal{N}_h$.
6. Each element K has three vertices, each edge E has two vertices.
7. Therefore, $\sharp\tilde{\omega}_K \le 3C_3$, $\forall K \in \mathfrak{T}_h$, $\sharp\tilde{\omega}_E \le 2C_3$, $\forall E \in \mathcal{E}_h$.
8. Thus, we have

$$\Big[\sum_{K\in\mathfrak{T}_h(\Omega)} \|w\|^2_{1,2,\tilde{\omega}_K} + \sum_{E\in\mathcal{E}_{h,\Omega}\cup\mathcal{E}_{h,1}} \|w\|^2_{1,2,\tilde{\omega}_E} \Big]^{1/2} \le \sqrt{5\,C_3}\,\|w\|_{1,2,\Omega}.$$

9. The conclusion of the lemma follows as a consequence of ③, ④ and ⑧ with $C(\theta_{\min}) = \sqrt{5C_3}\max\{C_1(\theta_{\min}), C_2(\theta_{\min})\}$. ■

A Theorem on the a Posteriori Error Estimate

Theorem

As a corollary of Theorem 8.1 and Lemma 8.2, we have the following a posteriori error estimate of the finite element solution:

$$\|u - u_h\|_{1,2,\Omega} \leq C \Big\{ \sum_{K \in \mathfrak{T}_h(\Omega)} h_K^2 \, \|f\|_{0,2,K}^2 +$$

$$\sum_{E \in \mathcal{E}_{h,1}} h_E \, \|g - \nu_E \cdot \nabla u_h\|_{0,E}^2 + \sum_{E \in \mathcal{E}_{h,\Omega}} h_E \, \|[\nu_E \cdot \nabla u_h]_E\|_{0,E}^2 \Big\}^{1/2},$$

where $C = C(\theta_{\min})\, C(\Omega)$ is a constant depending only on Ω and the smallest angle $\theta_{\min}$ of the triangulation $\mathfrak{T}_h(\Omega)$.

- The righthand side term above essentially gives an upper bound estimate for the $\mathbb{V}^*$-norm of the residual $R(u_h)$, which can be directly used as a posteriori error estimator for the upper bound of the error of the finite element solution.

A Practical a Posteriori Error Estimator

1. For convenience of analysis and practical computations, f and g are usually replaced by some approximation functions, say by $f_K = \frac{1}{|K|}\int_K f\,dx$ and $g_E = h_E^{-1}\int_E g\,ds$.

2. A practical a posteriori error estimator of residual type:

$$\eta_{R,K} = \Big\{ h_K^2 \|f_K\|_{0,2,K}^2 + \sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,1}} h_E \|g_E - \nu_E \cdot \nabla u_h\|_{0,E}^2 + \frac{1}{2} \sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,\Omega}} h_E \|[\nu_E \cdot \nabla u_h]_E\|_{0,E}^2 \Big\}^{1/2}.$$

3. In applications, f_K and g_E can be further replaced by the numerical quadratures of the corresponding integrals.

Error Estimate Based on the Practical a Posteriori Error Estimator

Theorem

For the constant $C = C(\theta_{\min})\, C(\Omega)$ in Theorem 8.2, the following a posteriori error estimate holds:

$$\|u - u_h\|_{1,2,\Omega} \le C \Big\{ \sum_{K \in \mathfrak{T}_h(\Omega)} \eta_{R,K}^2 + \sum_{K \in \mathfrak{T}_h(\Omega)} h_K^2 \, \|f - f_K\|_{0,2,K}^2 + \sum_{E \in \mathcal{E}_{h,1}} h_E \, \|g - g_E\|_{0,E}^2 \Big\}^{1/2}.$$

Remark: Generally speaking, for h sufficiently small, the first term on the righthand side represents the leading part of the error. Therefore, in practical computations, $\eta_{R,K}$ alone is often used to estimate the local error, particularly in a mesh adaptive process.

Reliability of an a Posteriori Error Estimator

1. The a posteriori error estimators given in Theorem 8.2 and 8.3 provide upper bounds for the error of the finite element solution u_h in the $\mathbb{V}$-norm.

2. Such a property is called the reliability of the a posteriori error estimator.

3. In general, the reliability of an a posteriori error estimator can be understood in the sense of a constant times

Reliability of an a Posteriori Error Estimator

Definition

Let u and u_h be the solution and the finite element solution of the variational problem. Let η_h be an a posteriori error estimator. If there exists a constant $\widehat{C}$ independent of h such that

$$\|u - u_h\|_{1,2,\Omega} \leq \widehat{C}\,\eta_h.$$

Then, the a posteriori error estimator η_h is said to be reliable, or has reliability.

1. Reliability guarantees the accuracy.
2. To avoid mesh being unnecessarily refined and have the computational cost under control, efficiency is required.

Efficiency of an a Posteriori Error Estimator

Definition

Let u and u_h be the solution and the finite element solution of the variational problem. Let η_h be an a posteriori error estimator. If, for any given $h_0 > 0$, there exists a constant $\widetilde{C}(h_0)$ such that

$$\widetilde{C}(h_0)^{-1}\, \|u - u_h\|_{1,2,\Omega} \le \eta_h \le \widetilde{C}(h_0)\, \|u - u_h\|_{1,2,\Omega}, \quad \forall h \in (0,\ h_0),$$

Then, the a posteriori error estimator η_h is said to be efficient, or has efficiency. In addition, if the constant $\widetilde{C}(h_0)$ is such that

$$\lim_{h_0 \to 0+} \widetilde{C}(h_0) = 1,$$

Then, the a posteriori error estimator η_h is said to be asymptotically exact.

a Posteriori Local Error Estimator and the Local Error

1. In applications, to efficiently control the error, we hope to refine the mesh only on the regions where the local error is relatively large.

2. Therefore, in addition to a good estimate of the global error of a finite element solution, what we expect more on an a posteriori error estimator is that it can efficiently evaluate the local error.

a Posteriori Local Error Estimator and the Local Error

3. Recall $\int_\Omega \nabla(u-u_h)\cdot\nabla w\,dx = R(u_h)(w)$, and the a posteriori local error estimator of residual type is given as

$$\eta_{R,K} = \Big\{ h_K^2 \|f_K\|_{0,2,K}^2 + \sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,1}} h_E \|g_E - \nu_E\cdot\nabla u_h\|_{0,E}^2 + \frac{1}{2}\sum_{E\in\mathcal{E}(K)\cap\mathcal{E}_{h,\Omega}} h_E \|[\nu_E\cdot\nabla u_h]_E\|_{0,E}^2 \Big\}^{1/2}.$$

4. We hope, by choosing proper test functions w, to establish relationship between the local error of $u-u_h$ and the three terms in $\eta_{R,K}$.

Relate Terms in $\eta_{R,K}$ to the Local Errors of $u - u_h$

1. Notice that f_K is piecewise constant, $\forall w \in \mathbb{V}$, we have

$$\int_K f_K\,(f_K w)\,dx = |f_K|^2 \int_K w\,dx = |K|^{-1}\left(\int_K w\,dx\right)\|f_K\|_{0,2,K}^2,$$

2. If we take a positive $w \in \mathbb{H}_0^1(K)$, called a bubble function on K, then, the above equation will establish a relation between $\|f_K\|_{0,2,K}^2$ and the local error of $(u - u_h)|_K$ through

$$\int_K \nabla(u - u_h)\cdot\nabla(f_K w)\,dx = R(u_h)(f_K w) = \int_K f\,(f_K w)\,dx$$

$$= \int_K f_K\,(f_K w)\,dx + \int_K (f - f_K)\,(f_K w)\,dx.$$

Relate Terms in $\eta_{R,K}$ to the Local Errors of $u - u_h$

3. Similarly, by taking proper bubble functions, we can also establish the relationship between the local error of $u - u_h$ and the terms

$$\|g_E - \nu_E \cdot \nabla u_h\|_{0,E}^2, \qquad \|[\nu_E \cdot \nabla u_h]_E\|_{0,E}^2,$$

which are also piecewise constant functions.

Triangular Element Bubble Functions and Edge Bubble Functions

1. Let $\lambda_{K,i}$, $i = 1, 2, 3$ be the area coordinates of $K \in \mathfrak{T}_h(\Omega)$, define the triangular bubble function $\mathfrak{b}_K$ as

$$\mathfrak{b}_K(x) = \begin{cases} 27\, \lambda_{K,1}(x)\, \lambda_{K,2}(x)\, \lambda_{K,3}(x), & \forall x \in K; \\ 0, & \forall x \in \Omega \setminus K. \end{cases}$$

Triangular Element Bubble Functions and Edge Bubble Functions

2. For a given edge $E \in \mathcal{E}_{h,\Omega}$, let $\omega_E = K_1 \cup K_2$, let $\lambda_{K_i,j}$, $j = 1, 2, 3$ be the area coordinates of K_i, denote the vertex of K_i which is not on E as the third vertex of K_i, define the edge bubble function $\mathfrak{b}_E$ as

$$\mathfrak{b}_E(x) = \begin{cases} 4\,\lambda_{K_i,1}(x)\,\lambda_{K_i,2}(x), & \forall x \in K_i, \quad i = 1, 2; \\ 0, & \forall x \in \Omega \setminus \omega_E. \end{cases}$$

Triangular Element Bubble Functions and Edge Bubble Functions

3. For a given edge $E \in \mathcal{E}_{h,\partial\Omega}$, let $\omega_E = K'$, denote the vertex of K' which is not on E as the third vertex of K', define the edge bubble function $\mathfrak{b}_E$ as

$$\mathfrak{b}_E(x) = \begin{cases} 4\,\lambda_{K',1}(x)\,\lambda_{K',2}(x), & \forall x \in K'; \\ 0, & \forall x \in \Omega \setminus K'. \end{cases}$$

Properties of the Bubble Functions

Lemma

For any given $K \in \mathfrak{T}_h(\Omega)$ and $E \in \mathcal{E}_h$, the bubble functions $\mathfrak{b}_K$ and $\mathfrak{b}_E$ have the following properties:

$$\operatorname{supp} \mathfrak{b}_K \subset K, \quad 0 \le \mathfrak{b}_K \le 1, \quad \max_{x \in K} \mathfrak{b}_K(x) = 1;$$

$$\operatorname{supp} \mathfrak{b}_E \subset \omega_E, \quad 0 \le \mathfrak{b}_E \le 1, \quad \max_{x \in E} \mathfrak{b}_E(x) = 1;$$

$$\int_E \mathfrak{b}_E \, ds = \frac{2}{3} h_E;$$

Properties of the Bubble Functions

Lemma

there exists a constant $\hat{c}_i$, $i = 1, \ldots, 6$, which depends only on the smallest angle of the triangular triangulation $\mathfrak{T}_h(\Omega)$, such that

$$\hat{c}_1\, h_K^2 \le \int_K \mathfrak{b}_K\, dx = \frac{9}{20}\,|K| \le \hat{c}_2\, h_K^2;$$

$$\hat{c}_3\, h_E^2 \le \int_{K'} \mathfrak{b}_E\, dx = \frac{1}{3}\,|K'| \le \hat{c}_4\, h_E^2, \;\; \forall K' \subset \omega_E;$$

$$\|\nabla \mathfrak{b}_K\|_{0,2,K} \le \hat{c}_5\, h_K^{-1} \|\mathfrak{b}_K\|_{0,2,K};$$

$$\|\nabla \mathfrak{b}_E\|_{0,2,K'} \le \hat{c}_6\, h_E^{-1} \|\mathfrak{b}_E\|_{0,2,K'}, \;\; \forall K' \subset \omega_E.$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|f_K\|_{0,2,K}$

1. For any given $K \in \mathfrak{T}_h(\Omega)$, set $w_K := f_K \mathfrak{b}_K$. Then, by the properties of $\mathfrak{b}_K$ (see Lemma 8.3), we have

$$\int_K f_K\, w_K\, dx = \frac{9}{20}\,|K|\,|f_K|^2 = \frac{9}{20}\,\|f_K\|^2_{0,2,K}.$$

2. Since $\operatorname{supp} w_K \subset K$, it follows

$$\int_{\partial\Omega_1} g\, w_K\, ds - \int_\Omega \nabla u_h \cdot \nabla w_K\, dx = -\nabla u_h|_K \int_K \nabla w_K\, dx = 0.$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|f_K\|_{0,2,K}$

3. Thus, by $\int_\Omega \nabla(u-u_h)\cdot\nabla w_K\,dx = R(u_h)(w_K)$, we obtain

$$
\begin{aligned}
\int_K f_K\, w_K\, dx &= \int_K f\, w_K\, dx + \int_K (f-f_K)\, w_K\, dx \\
&= \int_K \nabla(u-u_h)\cdot\nabla w_K\, dx + \int_K (f-f_K)\, w_K\, dx \\
&\le \|u-u_h\|_{1,2,K}\|\nabla w_K\|_{0,2,K} + \|f-f_K\|_{0,2,K}\|w_K\|_{0,2,K}.
\end{aligned}
$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|f_K\|_{0,2,K}$

4. On the other hand, since f_K is a constant, by the properties of $\mathfrak{b}_K$ (see Lemma 8.3), we have

$$\|w_K\|_{0,2,K} = |f_K|\,\|\mathfrak{b}_K\|_{0,2,K} \leq |f_K| \left(\int_K \mathfrak{b}_K \, dx \right)^{1/2} = \sqrt{\frac{9}{20}}\, \|f_K\|_{0,2,K};$$

$$\|\nabla w_K\|_{0,2,K} \leq \hat{c}_5\, h_K^{-1} \|w_K\|_{0,2,K}.$$

Proof of the Efficiency of $\eta_{R,K}$ — Estimate of $\|f_K\|_{0,2,K}$

5. Combining the three inequalities obtained in ③ and ④ with $\int_K f_K\, w_K\, dx = \frac{9}{20}\,\|f_K\|_{0,2,K}^2$ (see (8.2.3)) leads to

$$\|f_K\|_{0,2,K} \le \sqrt{\frac{20}{9}}\;\hat{c}_5\, h_K^{-1}\|u - u_h\|_{1,2,K} + \sqrt{\frac{20}{9}}\;\|f - f_K\|_{0,2,K}.$$

Similar techniques can be applied to estimate the other terms in $\eta_{R,K}$.

习题 8：4.

Thank You!