Numerical Solutions to Partial Differential Equations

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First Strang Lemma — Abstract Error Estimate Including Consistency Error

Theorem

Let $\mathbb{V}_h \subset \mathbb{V}$*, and let the bilinear form* $a_h(\cdot, \cdot)$ *defined on* $\mathbb{V}_h \times \mathbb{V}_h$ *be uniform* $\mathbb{V}_h$*-elliptic,* i.e. *there exists a constant* $\hat{\alpha} > 0$ *independent of h such that*

$$a_h(v_h, v_h) \geq \hat{\alpha}\|v_h\|^2, \quad \forall v_h \in \mathbb{V}_h.$$

Then, there exists a constant C independent h such that

$$\|u - u_h\| \leq C\Big(\inf_{v_h \in \mathbb{V}_h} \Big\{ \|u - v_h\| + \sup_{w_h \in \mathbb{V}_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|} \Big\} + \sup_{w_h \in \mathbb{V}_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|} \Big).$$

Second Strang Lemma — Abstract Error Estimate for Non-Conforming FE

Theorem

Let the bilinear form $a_h(\cdot,\,\cdot)$ *be uniformly bounded on* $(\mathbb{V}+\mathbb{V}_h)\times(\mathbb{V}+\mathbb{V}_h)$*, and be uniformly* $\mathbb{V}_h$*-elliptic,* i.e. *there exist constants* $\hat{M}$ *and* $\hat{\alpha}>0$ *independent of* h *such that*

$$|a_h(u_h,\,v_h)| \le \hat{M}\|u_h\|_h\,\|v_h\|_h, \quad \forall u_h,\ v_h \in \mathbb{V}+\mathbb{V}_h,$$

$$a_h(v_h,\,v_h) \ge \hat{\alpha}\|v_h\|_h^2, \quad \forall v_h \in \mathbb{V}_h.$$

Then, the error of the solution u_h *of the corresponding approximation variational problem with respect to the solution* u *of the original variational problem satisfies*

$$\|u-u_h\|_h \cong \Big(\inf_{v_h\in\mathbb{V}_h}\|u-v_h\|_h + \sup_{w_h\in\mathbb{V}_h}\frac{|a_h(u,\,w_h)-f_h(w_h)|}{\|w_h\|_h}\Big).$$

Here $A_h(u)\cong B_h(u)$ *means that there exist positive constants* C_1 *and* C_2 *independent of* u *and* h *s.t.* $C_1B_h(u)\le A_h(u)\le C_2B_h(u)$*, for all* $h>0$ *sufficiently small.*

The Bramble-Hilbert Lemma —

— An Abstract Estimate on Polynomial Vanishing Linear Forms

Theorem

Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. For some $p \in [1, \infty]$ and some integer $k \geq 0$, let the bounded linear form f defined on $\mathbb{W}^{k+1,p}(\Omega)$ be such that

$$f(w) = 0, \qquad \forall w \in \mathbb{P}_k(\Omega).$$

Then, there exists a constant $C(\Omega)$ such that

$$|f(v)| \leq C(\Omega) \, \|f\|^*_{k+1,p,\Omega} |v|_{k+1,p,\Omega},$$

where $\|\cdot\|^*_{k+1,p,\Omega}$ is the norm on the dual space of $\mathbb{W}^{k+1,p}(\Omega)$.

The Bilinear Lemma —

— An Abstract Estimate on Polynomial Vanishing Bilinear Forms

Theorem

Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. For some $p, q \in [1, \infty]$, some integers $k, l \geq 0$ and a subspace $\mathbb{W}$ which satisfies the inclusion relation $\mathbb{P}_l(\Omega) \subset \mathbb{W} \subset \mathbb{W}^{l+1,q}(\Omega)$ and is endowed with the norm $\|\cdot\|_{l+1,q,\Omega}$, let the bounded bilinear form b defined on $\mathbb{W}^{k+1,p}(\Omega) \times \mathbb{W}$ be such that

$$\begin{aligned} b(r, w) &= 0, \qquad \forall r \in \mathbb{P}_k(\Omega), \ \forall w \in \mathbb{W}, \\ b(v, r) &= 0, \qquad \forall v \in \mathbb{W}^{k+1,p}(\Omega), \ \forall r \in \mathbb{P}_l(\Omega). \end{aligned}$$

Then, there exists a constant $C(\Omega)$ such that

$$|b(v, w)| \leq C(\Omega)\|b\||v|_{k+1,p,\Omega}|w|_{l+1,q,\Omega}, \ \ \forall v \in \mathbb{W}^{k+1,p}(\Omega), \ \forall w \in \mathbb{W},$$

where $\|b\|$ is the norm of the bilinear form b on $\mathbb{W}^{k+1,p}(\Omega) \times \mathbb{W}$.

Approximate $a(u, v)$ by $a_h(u, v)$ Using Numerical Quadratures

1. Let Ω be a polygonal region, and $\mathfrak{T}_h(\Omega)$ be a family of regular affine equivalent finite element triangulations of Ω.
2. $F_K : \hat{x} \in \hat{K} \to B_K \hat{x} + b_K \in K$: the corresponding affine equivalent mappings.
3. Let $a_{ij} \in \mathbb{W}^{1,\infty}(\Omega)$, $i, j = 1, \dots, n$, and
$$a(u, v) = \int_\Omega \sum_{i,j=1}^n a_{ij} \partial_i u \, \partial_j v \, dx = \sum_{K \in \mathfrak{T}_h(\Omega)} \sum_{i,j=1}^n \int_K a_{ij} \partial_i u \, \partial_j v \, dx.$$
4. $\{\hat{b}_l\}_{l=1}^L$, $\{b_{l,K} = F_K(\hat{b}_l)\}_{l=1}^L$: quadrature nodes on $\hat{K}$, K.
5. $\{\hat{\omega}_l\}_{l=1}^L$, $\{\omega_{l,K} = \det(B_K)\hat{\omega}_l\}_{l=1}^L$, quadrature weights on $\hat{K}$, K.

Approximate $a(u, v)$ by $a_h(u, v)$ Using Numerical Quadratures

6. Approximate $a(u, v)$ by numerical quadrature

$$a_h(u,\ v) = \sum_{K \in \mathfrak{T}_h(\Omega)} \sum_{i,j=1}^{n} \sum_{l=1}^{L} \omega_{l,K}\, a_{ij}(b_{l,K}) \partial_i u(b_{l,K})\, \partial_j v(b_{l,K}).$$

7. Denote the errors of integrals and numerical integrals of φ and $\hat{\varphi}$ on K and $\hat{K}$ by

$$E_K(\varphi) = \int_K \varphi(x)\, dx - \sum_{l=1}^{L} \omega_{l,K}\, \varphi(b_{l,K}),$$

$$\hat{E}(\hat{\varphi}) = \int_{\hat{K}} \hat{\varphi}(\hat{x})\, d\hat{x} - \sum_{l=1}^{L} \hat{\omega}_l\, \hat{\varphi}(\hat{b}_l).$$

Error on K of a Numerical Quadrature with Algebraic Accuracy $2k-2$

Lemma

Let $a_{ij} \in \mathbb{W}^{k,\infty}(\Omega)$, $i, j = 1, \dots, n$, for some integer $k \geq 1$. Let the reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ and the numerical quadrature satisfy

$$\hat{P} = \mathbb{P}_k(\hat{K}) \quad \text{and} \quad \hat{E}(\hat{\varphi}) = 0, \ \ \forall \hat{\varphi} \in \mathbb{P}_{2k-2}(\hat{K}).$$

Then, there exists a constant C independent of K and h, such that

$$|E_K(a_{ij}\partial_i \tilde{v}\, \partial_j \tilde{w})| \leq C\, h_K^k\, \|\tilde{v}\|_{k,K}\, |\tilde{w}|_{1,K}, \ \ \forall \tilde{v} \in \mathbb{P}_k(K), \ \forall \tilde{w} \in \mathbb{P}_k(K).$$

Proof of the Lemma — Key: Bounds the Error in Semi-Norms

1. $a \in \mathbb{W}^{k,\infty}(K)$ and v, $w \in \mathbb{P}_{k-1}(K) \Rightarrow \hat{a} \in \mathbb{W}^{k,\infty}(\hat{K})$ and $\hat{v}$, $\hat{w} \in \mathbb{P}_{k-1}(\hat{K})$, and we have $E_K(a\,v\,w) = \det(B_K)\,\hat{E}(\hat{a}\,\hat{v}\,\hat{w})$.
2. For a given $\hat{w} \in \mathbb{P}_{k-1}(\hat{K})$ and an arbitrary $\hat{\varphi} \in \mathbb{W}^{k,\infty}(\hat{K})$, $|\hat{E}(\hat{\varphi}\,\hat{w})| \le \hat{C}\,\|\hat{\varphi}\,\hat{w}\|_{0,\infty,\hat{K}} \le \hat{C}\,\|\hat{\varphi}\|_{0,\infty,\hat{K}}\,\|\hat{w}\|_{0,\infty,\hat{K}}$.
3. Since norms on the finite dimensional space $\mathbb{P}_{k-1}(\hat{K})$ are equivalent, we have
$$|\hat{E}(\hat{\varphi}\,\hat{w})| \le \hat{C}\,\|\hat{\varphi}\|_{0,\infty,\hat{K}}\,\|\hat{w}\|_{0,\hat{K}} \le \hat{C}\,\|\hat{\varphi}\|_{k,\infty,\hat{K}}\,\|\hat{w}\|_{0,\hat{K}}.$$
4. Therefore, for a fixed $\hat{w} \in \mathbb{P}_{k-1}(\hat{K})$, $\hat{E}(\cdot\,\hat{w})$ is a bounded linear form on $\mathbb{W}^{k,\infty}(\hat{K})$ with its norm $\le \hat{C}\,\|\hat{w}\|_{0,\hat{K}}$.

Proof of the Lemma — Key: Bounds the Error by Semi-Norms

5. In addition, $\hat{E}(\hat{\varphi}\,\hat{w}) = 0$, $\forall \hat{\varphi} \in \mathbb{P}_{k-1}(\hat{K})$. Consequently, by the Bramble-Hilbert lemma (see Theorem 7.15), we have

$$|\hat{E}(\hat{\varphi}\,\hat{w})| \leq \hat{C}|\hat{\varphi}|_{k,\infty,\hat{K}}\|\hat{w}\|_{0,\hat{K}}, \quad \forall \hat{\varphi} \in \mathbb{W}^{k,\infty}(\hat{K}),\ \forall \hat{w} \in \mathbb{P}_{k-1}(\hat{K}).$$

6. On the other hand, for $\hat{a} \in \mathbb{W}^{k,\infty}(\hat{K})$ and $\hat{v} \in \mathbb{P}_{k-1}(\hat{K})$, by the chain rule of the derivatives of composition functions and the equivalence of norms in the finite dimensional space $\mathbb{P}_{k-1}(\hat{K})$, we have

$$|\hat{a}\hat{v}|_{k,\infty,\hat{K}} \leq \hat{C}\sum_{j=0}^{k-1} |\hat{a}|_{k-j,\infty,\hat{K}}\,|\hat{v}|_{j,\infty,\hat{K}} \leq \hat{C}\sum_{j=0}^{k-1} |\hat{a}|_{k-j,\infty,\hat{K}}\,|\hat{v}|_{j,\hat{K}}.$$

Proof of the Lemma — Key: Bounds the Error in Semi-Norms

7. ⑤ and ⑥ yield that, $\forall \hat{a} \in \mathbb{W}^{k,\infty}(\hat{K})$,

$$|\hat{E}(\hat{a}\,\hat{v}\,\hat{w})| \leq \hat{C}\Big(\sum_{j=0}^{k-1} |\hat{a}|_{k-j,\infty,\hat{K}} |\hat{v}|_{j,\hat{K}}\Big) \|\hat{w}\|_{0,\hat{K}}, \quad \forall \hat{v},\ \hat{w} \in \mathbb{P}_{k-1}(\hat{K}).$$

8. By the relations of the semi-norms on K and $\hat{K}$, this yields

$$|E_K(avw)| \leq Ch_K^k\Big(\sum_{j=0}^{k-1} |a|_{k-j,\infty,K} |v|_{j,K}\Big) \|w\|_{0,K}, \quad \forall v, w \in \mathbb{P}_{k-1}(K).$$

9. Thus, the lemma follows by taking $a = a_{ij}$, $v = \partial_i \tilde{v}$ and $w = \partial_j \tilde{w}$. ∎

Consistency Error Estimate of Bilinear Forms

1. As a consequence of Lemma 7.2, for numerical quadrature with algebraic accuracy $2k-2$, we see that

$$|a(\Pi_h u,\, w_h) - a_h(\Pi_h u,\, w_h)| \leq Ch^k \Big(\sum_{K\in\mathfrak{T}_h(\Omega)} \|\Pi_h u\|_{k,K}^2 \Big)^{1/2} |w_h|_{1,\Omega}.$$

2. On the other hand, by the interpolation error estimates of the finite element solutions and the Cauchy-Schwarz inequality, we have

$$\Big(\sum_{K\in\mathfrak{T}_h(\Omega)} \|\Pi_h u\|_{k,K}^2 \Big)^{1/2} \leq \|u\|_{k,\Omega} + \Big(\sum_{K\in\mathfrak{T}_h(\Omega)} \|u-\Pi_h u\|_{k,K}^2 \Big)^{1/2} \leq C\|u\|_{k+1,\Omega}.$$

Consistency Error Estimate of Bilinear Forms

3. Consequently, we obtain the following consistency error estimate:

$$\sup_{w_h \in \mathbb{V}_h} \frac{|a(\Pi_h u,\, w_h) - a_h(\Pi_h u,\, w_h)|}{\|w_h\|_{1,\Omega}} \leq C\, h^k\, \|u\|_{k+1,\Omega}.$$

which is the same order as the interpolation error estimate.

Consistency Error Estimate of Linear Forms

Similarly, the consistency error estimation can be carried out for the numerical integration of $f(v) = \int_\Omega fv\,dx$. However, to reach accuracy of order h^k, the algebraic accuracy of the numerical quadrature needs to be $2k-1$.

1. Consider the linear form $f(v) = \int_\Omega fv\,dx$, and numerical quadrature $f_h(v) = \sum_{K\in\mathfrak{T}_h(\Omega)} \sum_{l=1}^{L} \omega_{l,K} f(b_{l,K}) v(b_{l,K})$.

2. Assume $\mathbb{H}^k(\Omega) \hookrightarrow \mathbb{C}(\bar{\Omega})$, and the finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ and the numerical quadrature satisfy
$$\hat{P} = \mathbb{P}_k(\hat{K}), \quad \text{and} \quad \hat{E}(\hat{\varphi}) = 0, \ \ \forall \hat{\varphi} \in \mathbb{P}_{2k-1}(\hat{K}).$$

Consistency Error Estimate of Linear Forms

3. Then, follows a similar argument as above, we can obtain the error estimate (see Exercise 7.9)

$$\sup_{w_h \in \mathbb{V}_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|_{1,\Omega}} \leq C\, h^k |f|_{k,\Omega}.$$

Consistency Error Estimate of Linear Forms

There is another approach, in which, to reach the accuracy of the same order h^k, the algebraic accuracy of the numerical quadrature needed goes back to $2k-2$. In fact, we have the following result:

1. Let $q \geq 2$ and assume $kq > n$ (so $\mathbb{W}^{k,q}(\Omega) \hookrightarrow \mathbb{C}(\bar{\Omega})$).
2. The reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ and the numerical quadrature satisfy

$$\hat{P} = \mathbb{P}_k(\hat{K}), \quad \text{and} \quad \hat{E}(\hat{\varphi}) = 0, \ \forall \hat{\varphi} \in \mathbb{P}_{2k-2}(\hat{K}).$$

Consistency Error Estimate of Linear Forms

3. Introduce a $\mathbb{P}_0(\hat{K})$ invariant orthogonal projection operator $\hat{\Pi} : \mathbb{L}^2(\hat{K}) \to \mathbb{P}_1(\hat{K})$ induced by the $\mathbb{L}^2(\hat{K})$ inner product.

4. Rewrite the error as $E(fw_h) = E(f(w_h - \Pi w_h)) + E(f\Pi w_h)$.

5. Then, we have

$$\sup_{w_h \in \mathbb{V}_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|_{1,\Omega}} \le C\, h^k \|f\|_{k,q,\Omega}.$$

Summary of the a Priori Finite Element Error Estimates

The a priori finite element error estimates basically consist of the following main parts

1. Abstract error estimate — transform the problem to the errors of subspace approximation and consistency of discrete operators.
2. Error estimates of subspace approximation — polynomial invariant interpolation operators, polynomial invariant projection operators, polynomial quotient spaces, relations of semi-norms on affine equivalent open sets, etc..
3. Error estimates of consistency of discrete operators — polynomial invariant interpolation or projection operators and linear, bilinear forms, polynomial quotient spaces, relations of semi-norms on affine equivalent open sets, etc..

Scaling Technique is the Key

Remark:

An important technique: scaling

- polynomial invariant + polynomial quotient space — show the required inequality in semi-norms in the function spaces on the reference finite element;
- the relations of semi-norms on affine equivalent open sets — bring out the power of h (scaling).

h^α appeared in the error estimates can usually be efficiently derived by the scaling technique.

A Priori and A Posteriori Error Estimates of Finite Element Solutions

1. A Priori Error Estimate: error bounds given by known information on the solution of the variational problem and the finite element function spaces. For example, for the second order elliptic problems, the error estimate is given by $\|u - u_h\|_{1,\Omega} \leq C\, h^k\, |u|_{k+1,\Omega}$ (see Theorem 7.10).

2. A Posteriori Error Estimate: error bounds given by information on the numerical solutions obtained on the finite element function spaces.

A Priori and A Posteriori Error Estimates of Finite Element Solutions

Remarks:

- In applications, $|u|_{k+1,\Omega}$ is generally not known a priori.
- A priori error estimate doesn't give a clue on how the mesh should be distributed to balance the cost and accuracy.
- Compared with the extrapolation technique (see § 1.5), the a posteriori local error estimator can be used to refine or coarsen the mesh wherever necessary locally.

Mixed BVP of Poisson Equation on Polygonal Region in $\mathbb{R}^2$

- Consider the boundary value problem of the Poisson equation

$$\begin{cases} -\triangle u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega_0, \qquad \dfrac{\partial u}{\partial \nu} = g, \quad x \in \partial\Omega_1, \end{cases}$$

where Ω is a polygonal region in $\mathbb{R}^2$, $\partial\Omega_0$ is a relatively closed subset in $\partial\Omega$ with positive 1-dimensional measure,

$$\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1, \;\; \partial\Omega_0 \cap \partial\Omega_1 = \emptyset,$$

$$f \in \mathbb{L}^2(\Omega), \; g \in \mathbb{L}^2(\partial\Omega_1).$$

Mixed BVP of Poisson Equation on Polygonal Region in $\mathbb{R}^2$

- consider the standard weak form of the problem:

$$\begin{cases} \text{Find} \quad u \in \mathbb{V} \text{ such that} \\ \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f \, v \, dx + \int_{\partial\Omega_1} g \, v \, ds, \quad \forall v \in \mathbb{V}, \end{cases}$$

where $\mathbb{V} = \left\{ v \in \mathbb{H}^1(\Omega) : v|_{\partial\Omega_0} = 0 \right\}$;

- consider the conforming finite element method based on a family of regular class C^0 type (1) Lagrange triangular elements.

A Theorem on the Relation of Residual and Error of a FE Solution

1. Define the residual operator $R : \mathbb{V} \to \mathbb{V}^*$ of the problem by
$$R(v)(w) = \int_\Omega f\, w\, dx + \int_{\partial\Omega_1} g\, w\, ds - \int_\Omega \nabla v \cdot \nabla w\, dx, \quad \forall w \in \mathbb{V}.$$
2. The dual norm of the residual of a finite element solution u_h:

$$\|R(u_h)\|_{\mathbb{V}^*} = \sup_{\substack{w \in \mathbb{V} \\ \|w\|_{1,2,\Omega}=1}} \left\{ \int_\Omega fw\, dx + \int_{\partial\Omega_1} gw\, ds - \int_\Omega \nabla u_h \cdot \nabla w\, dx \right\}$$

Theorem

Let $u \in \mathbb{V}$, $u_h \in \mathbb{V}_h$ be the weak solution and the finite element solution of the problem respectively. Then, there exists a constant $C(\Omega)$, which depends only on Ω, such that
$$\|R(u_h)\|_{\mathbb{V}^*} \leq \|u - u_h\|_{1,2,\Omega} \leq C(\Omega)\, \|R(u_h)\|_{\mathbb{V}^*}.$$

Proof of Residual Dual Norm ≅ Error of a FE Solution in $\mathbb{H}^1$-Norm

1. Since u is the weak solution of the problem, we have

$$\int_\Omega \nabla(u-u_h)\cdot\nabla w\,dx = \int_\Omega fw\,dx + \int_{\partial\Omega_1} gw\,ds - \int_\Omega \nabla u_h\cdot\nabla w\,dx, \quad \forall w \in \mathbb{V}.$$

2. Hence, by the Cauchy-Schwarz inequality, we have

$$\int_\Omega \nabla(u-u_h)\cdot\nabla w\,dx \le |u-u_h|_{1,2,\Omega}|w|_{1,2,\Omega} \le \|u-u_h\|_{1,2,\Omega}\|w\|_{1,2,\Omega}.$$

3. Thus, the first inequality follows directly from the definition.

Proof of Residual Dual Norm $\cong$ Error of a FE Solution in $\mathbb{H}^1$-Norm

4. On the other hand, by the Poincaré-Friedrichs inequality (see Exercise 5.6), $\exists$ constant $\gamma_0(\Omega) > 0$, s.t.

$$\gamma_0 \|v\|_{1,2,\Omega} \leq |v|_{1,2,\Omega}, \quad \forall v \in \mathbb{V}.$$

5. Thus, by taking $w = u - u_h$ in ①, the second inequality follows for $C(\Omega) = \gamma_0^{-2}$. ■

Remarks on Residual Dual Norm Estimation

1. We hope to develop a formula, which is easily computed and involves only available data such as f, g, u_h and geometric parameters of the triangulation and thus is usually called an a posteriori error estimator, to evaluate the dual norm of the residual.
2. Recall that in the a priori error estimates, the polynomial invariant interpolation operator plays an important role. For example, write w as $(w - \Pi_h w) + \Pi_h w$ can have some advantage.
3. However, the Lagrange nodal type interpolation operators require the function to be at least in $\mathbb{C}^0$.
4. Here, we need to introduce a polynomial invariant interpolation operator for functions in $\mathbb{H}^1$.

Notations on a Family of Regular Triangular Triangulations $\{\mathfrak{T}_h(\Omega)\}_{h>0}$

1. $\mathcal{E}(K)$, $\mathcal{N}(K)$: the sets of all edges and vertices of $K \in \mathfrak{T}_h(\Omega)$.
2. Denote $\mathcal{E}_h := \bigcup_{K \in \mathfrak{T}_h(\Omega)} \mathcal{E}(K)$, $\mathcal{N}_h := \bigcup_{K \in \mathfrak{T}_h(\Omega)} \mathcal{N}(K)$.
3. $\mathcal{N}(E)$: the sets of all vertices of an edge $E \in \mathcal{E}_h$.
4. $\mathcal{E}_{h,i} := \left\{ E \in \mathcal{E}_h : \overset{\circ}{E} \subset \partial\Omega_i \right\}$, $\mathcal{N}_{h,i} := \mathcal{N}_h \cap \partial\Omega_i$, $i = 0, 1$.
5. $\mathcal{E}_{h,\Omega} = \mathcal{E}_h \setminus (\mathcal{E}_{h,0} \cup \mathcal{E}_{h,1})$, $\mathcal{N}_{h,\Omega} = \mathcal{N}_h \setminus (\mathcal{N}_{h,0} \cup \mathcal{N}_{h,1})$.

Notations on a Family of Regular Triangular Triangulations $\{\mathfrak{T}_h(\Omega)\}_{h>0}$

6. $\omega_K := \bigcup_{\mathcal{E}(K)\cap\mathcal{E}(K')\neq\emptyset} K'$, $\omega_E := \bigcup_{E\in\mathcal{E}(K')} K'$, $\omega_x := \bigcup_{x\in\mathcal{N}(K')} K'$.

7. $\tilde{\omega}_K := \bigcup_{\mathcal{N}(K)\cap\mathcal{N}(K')\neq\emptyset} K'$, $\quad \tilde{\omega}_E := \bigcup_{\mathcal{N}(E)\cap\mathcal{N}(K')\neq\emptyset} K'$.

8. The corresponding finite element function space:

$$\mathbb{V}_h = \{v \in \mathbb{C}(\bar{\Omega}) : v|_K \in \mathbb{P}_1(K), \forall K \in \mathfrak{T}_h(\Omega), v(x) = 0, \forall x \in \mathcal{N}_{h,0}\}.$$

The Clément Interpolation Operator $I_h : \mathbb{V} \to \mathbb{V}_h$

Definition

For any $v \in \mathbb{V}$ and $x \in \mathcal{N}_h$, denote $\pi_x v$ as the $\mathbb{L}^2(\omega_x)$ projection of v on $\mathbb{P}_1(\omega_x)$, meaning $\pi_x v \in \mathbb{P}_1(\omega_x)$ satisfies

$$\int_{\omega_x} v\, p\, dx = \int_{\omega_x} (\pi_x v)\, p\, dx, \quad \forall p \in \mathbb{P}_1(\omega_x).$$

The Clément interpolation operator $I_h : \mathbb{V} \to \mathbb{V}_h$ is defined by

$$I_h v(x) = (\pi_x v)(x), \;\; \forall x \in \mathcal{N}_{h,\Omega} \cup \mathcal{N}_{h,1}; \qquad I_h v(x) = 0, \;\; \forall x \in \mathcal{N}_{h,0}.$$

The Clément Interpolation Operator $I_h : \mathbb{V} \to \mathbb{V}_h$

1. The Clément interpolation operator is well defined on $\mathbb{L}^1(\Omega)$.
2. If $v \in \mathbb{P}_1(\omega_x)$, then $(\pi_x)v(x) = v(x)$, $\forall x \in \omega_x$.
3. If $v \in \mathbb{P}_1(\tilde{\omega}_K)$, then $I_h v(x) = v(x)$, $\forall x \in K$.
4. It is in the above sense that the Clément interpolation operator is polynomial (more precisely $\mathbb{P}_1$) invariant.

Error Estimates of the Clément Interpolation Operator I_h

Lemma

There exist constants $C_1(\theta_{\min})$ and $C_2(\theta_{\min})$, which depend only on the smallest angle $\theta_{\min}$ of the triangular elements in the triangulation $\mathfrak{T}_h(\Omega)$, such that, for any given $K \in \mathfrak{T}_h(\Omega)$, $E \in \mathcal{E}_h$ and $v \in \mathbb{V}$,

$$\begin{aligned} \|v - I_h v\|_{0,2,K} &\le C_1(\theta_{\min})\, h_K\, |v|_{1,2,\tilde{\omega}_K}, \\ \|v - I_h v\|_{0,E} := \|v - I_h v\|_{0,2,E} &\le C_2(\theta_{\min})\, h_K^{1/2}\, |v|_{1,2,\tilde{\omega}_E}. \end{aligned}$$

Error Estimates of the Clément Interpolation Operator I_h

- More general properties and proofs on the Clément interpolation operator may be found in [8, 31].
- The basic ingredients of the proof are the scaling techniques (which include the polynomial quotient space and equivalent quotient norms, the relations of semi-norms on affine equivalent open sets), and the inverse inequality.

习题 8：1，2，3.

Thank You!