Numerical Solutions to Partial Differential Equations

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Numerical Methods for Partial Differential Equations

- Finite Difference Methods for Elliptic Equations
- Finite Difference Methods for Parabolic Equations
- Finite Difference Methods for Hyperbolic Equations
- Finite Element Methods for Elliptic Equations

Finite Difference Methods for Elliptic Equations

The definitions of the elliptic equations — 2nd order

A general second order linear elliptic partial differential equation with n independent variables has the following form:

$$\pm L(u) \triangleq \pm \left[\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c \right] u = f, \qquad (1)$$

with (the key point in the definition)

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha(x) \sum_{i=1}^{n} \xi_i^2, \;\; \alpha(x)>0, \; \forall\, \xi \in \mathbb{R}^n \setminus \{0\}, \; \forall x \in \Omega. \quad (2)$$

Note that (2) says the matrix $A = (a_{ij}(x))$ is positive definite.

The definitions of the elliptic equations — 2nd order

- L – the 2nd order linear elliptic partial differential operator;
- a_{ij}, b_i, c — coefficients, functions of $x = (x_1, \ldots, x_n)$;
- f — right hand side term, or source term, a function of x;

The operator L and the equation (1) are said to be uniformly elliptic, if

$$\inf_{x \in \Omega} \alpha(x) = \alpha > 0. \tag{3}$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2, \quad \alpha > 0, \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}, \ \ \forall x \in \Omega.$$

The definitions of the elliptic equations — 2nd order

For example, $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is a linear second order uniformly elliptic partial differential operator, since we have here

$$a_{ii} = 1, \ \forall i, \quad a_{ij} = 0, \ \forall i \neq j,$$

and the Poisson equation

$$-\triangle u(x) = f(x)$$

is a linear second order uniformly elliptic partial differential equation.

The definitions of the elliptic equations — $2m$-th order

A general linear elliptic partial differential equations of order $2m$ with n independent variables has the following form:

$$\pm L(u) \triangleq \pm \left[\sum_{k=1}^{2m} \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1,\ldots,i_k} \frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} + a_0 \right] u = f, \qquad (4)$$

with (the key point in the definition)

$$\sum_{i_1,\ldots,i_{2m}=1}^{n} a_{i_1,\ldots,i_{2m}}(x)\xi_{i_1} \cdots \xi_{i_{2m}} \geq \alpha(x) \sum_{i=1}^{n} \xi_i^{2m},$$

$$\alpha(x) > 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \ \forall x \in \Omega. \qquad (5)$$

Note that (5) says the $2m$ order tensor $A = (a_{i_1,\ldots,i_{2m}})$ is positive definite.

The definitions of the elliptic equations — $2m$-th order

- L – the $2m$-th order linear elliptic partial differential operator;
- $a_{i_1,\ldots,i_k}$, a_0 — coefficients, functions of $x = (x_1, \ldots, x_n)$;
- f — right hand side term, or source term, a function of x;

The operator L and the equation (4) are said to be uniformly elliptic, if

$$\inf_{x\in\Omega} \alpha(x) = \alpha > 0. \tag{6}$$

$$\sum_{i_1,\ldots,i_{2m}=1}^{n} a_{i_1,\ldots,i_{2m}}(x)\xi_{i_1}\cdots\xi_{i_{2m}} \geq \alpha \sum_{i=1}^{n} \xi_i^{2m},$$

$$\alpha > 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \ \forall x \in \Omega.$$

The definitions of the elliptic equations

As a typical example, the $2m$-th order harmonic equation

$$(-\triangle)^m u = f$$

is a linear $2m$-th order uniformly elliptic partial differential equation, and $\triangle^m$ is a linear $2m$-th order uniformly elliptic partial differential operator, since we have here

$$a_{i_1,\ldots,i_{2m}}(x) = 1, \quad \text{if the indexes appear in pairs;}$$

$$a_{i_1,\ldots,i_{2m}}(x) = 0, \quad \text{otherwise.}$$

In particular, the biharmonic equation $\triangle^2 u = f$ is a linear 4th order uniformly elliptic partial differential equation, and $\triangle^2$ is a linear 4-th order uniformly elliptic partial differential operator.

Steady state convection-diffusion equation

1. $x \in \Omega \subset \mathbb{R}^n$;
2. $\mathbf{v}(x)$: the velocity of the fluid at x;
3. $u(x)$: the density of certain substance in the fluid at x;
4. $a(x) > 0$: the diffusive coefficient;
5. $f(x)$: the density of the source or sink of the substance.
6. J: diffusion flux (measured by amount of substance per unit area per unit time)
7. Fick's law: $J = -a(x)\nabla u(x)$.

Steady state convection-diffusion equation

For an arbitrary open subset $\omega \subset \Omega$ with piecewise smooth boundary $\partial\omega$, Fick's law says the substance brought into ω by diffusion per unit time is given by

$$\int_{\partial\omega} J \cdot (-\nu(x))\, ds = \int_{\partial\omega} a(x)\nabla u(x) \cdot \nu(x)\, ds,$$

while the substance brought into ω by the flow per unit time is

$$\int_{\partial\omega} u(x)\mathbf{v}(x) \cdot (-\nu(x))\, ds$$

and the substance produced in ω by the source per unit time is

$$\int_{\omega} f(x)\, dx.$$

Steady state convection-diffusion equation

Therefore, the net change of the substance in ω per unit time is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega} u(x)\,dx = \int_{\partial\omega} a(x)\nabla u(x)\cdot\nu(x)\,ds - \int_{\partial\omega} u(x)\mathbf{v}(x)\cdot\nu(x)\,ds + \int_{\omega} f(x)\,dx.$$

By the steady state assumption, $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega} u(x)\,dx = 0$, for arbitrary ω, and by the divergence theorem (or Green's formula or Stokes formula), this leads to the steady state convection-diffusion equation in the integral form

$$\int_{\omega} \{\nabla\cdot(a\nabla u - u\,\mathbf{v}) + f\}\,dx = 0, \quad \forall\omega$$

Steady state convection-diffusion equation

The term $-[a(x)\nabla u(x) - u(x)\mathbf{v}(x)]$ is named as the substance flux, since it represents the speed that the substance flows.

Assume that $\nabla \cdot (a\nabla u - u\,\mathbf{v}) + f$ is smooth, then, we obtain the steady state convection-diffusion equation in the differential form

$$-\nabla \cdot (a(x)\nabla u(x) - u\,\mathbf{v}) = f(x), \quad \forall x \in \Omega.$$

In particular, if $\mathbf{v} = 0$ and $a = 1$, we have the steady state diffusion equation $-\triangle u = f$.

Boundary conditions for the elliptic equations

For a complete steady state convection-diffusion problem, or problems of elliptic equations in general, we also need to impose proper boundary conditions.

Three types of most commonly used boundary conditions:

First type $$u = u_D, \quad \forall x \in \partial\Omega;$$

Second type $$\frac{\partial u}{\partial \nu} = g, \quad \forall x \in \partial\Omega;$$

Third type $$\frac{\partial u}{\partial \nu} + \alpha u = g, \quad \forall x \in \partial\Omega;$$

where $\alpha \geq 0$, and $\alpha > 0$ at least on some part of the boundary (*physical meaning: higher density produces bigger outward diffusion flux*).

Boundary conditions for the steady state convection-diffusion equation

- 1st type boundary condition — Dirichlet boundary condition;
- 2nd type boundary condition — Neumann boundary condition;
- 3rd type boundary condition — Robin boundary condition;
- Mixed-type boundary conditions — different types of boundary conditions imposed on different parts of the boundary.

General framework of Finite Difference Methods

1. Discretize the domain Ω by introducing a grid;
2. Discretize the function space by introducing grid functions;
3. Discretize the differential operators by properly defined difference operators;
4. Solve the discretized problem to get a finite difference solution;
5. Analyze the approximate properties of the finite difference solution.

Dirichlet boundary value problem of the Poisson equation

$$\begin{cases} -\triangle u(x) = f(x), & \forall x \in \Omega, \\ \quad u(x) = u_D(x), & \forall x \in \partial\Omega, \end{cases}$$

where $\Omega = (0,\ 1) \times (0,\ 1)$ is a rectangular region.

Discretize Ω by introducing a grid

1. Space (spatial) step sizes: $\triangle x = \triangle y = h = 1/N$;
2. Index set of the grid nodes: $J = \{(i,j) : (x_i, y_j) \in \overline{\Omega}\}$;
3. Index set of grid nodes on the Dirichlet boundary: $J_D = \{(i,j) : (x_i, y_j) \in \partial\Omega\}$;
4. Index set of interior nodes: $J_\Omega = J \setminus J_D$.

For simplicity, both (i,j) and (x_i, y_j) are called grid nodes.

Discretize the function space by introducing grid functions

- $u_{i,j} = u(x_i, y_j)$, exact solution restricted on the grid;
- $f_{i,j} = f(x_i, y_j)$, source term restricted on the grid;
- $U_{i,j}$, numerical solution on the grid;
- $V_{i,j}$, a grid function.

Discretize differential operators by difference operators

- $\dfrac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\triangle x^2} \approx \partial_x^2 u;$
- $\dfrac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\triangle y^2} \approx \partial_y^2 u;$

The poisson equation $-\triangle u(x) = f(x)$ is discretized to the 5 point difference scheme

$$-L_h U_{i,j} \triangleq \frac{4U_{i,j} - U_{i-1,j} - U_{i,j-1} - U_{i+1,j} - U_{i,j+1}}{h^2} = f_{i,j}, \quad \forall (i,j) \in J_\Omega.$$

The Dirichlet boundary condition is discretized to

$$U_{i,j} = u_D(x_i, y_j), \quad \forall (i,j) \in J_D.$$

Solution of the discretized problem

The discrete system

$$-L_h U_{i,j} \triangleq \frac{4U_{i,j} - U_{i-1,j} - U_{i,j-1} - U_{i+1,j} - U_{i,j+1}}{h^2} = f_{i,j}, \quad \forall (i,j) \in J_\Omega,$$

$$U_{i,j} = u_D(x_i, y_j), \quad \forall (i,j) \in J_D,$$

is a system of linear algebraic equations, whose matrix is symmetric positive definite. Consequently, there is a unique solution.

Analyze the Approximate Property of the Discrete Solution

1. Approximation error: $e_{i,j} = U_{i,j} - u_{i,j}$;
2. The error equation:
$$-L_h e_{i,j} \triangleq \frac{4e_{i,j} - e_{i-1,j} - e_{i,j-1} - e_{i+1,j} - e_{i,j+1}}{h^2} = T_{i,j}, \ \forall (i,j) \in J_\Omega;$$
3. The local truncation error
$$T_{i,j} := [(L_h - L)u]_{i,j} = L_h u_{i,j} - (Lu)_{i,j} = L_h u_{i,j} + f_{i,j}, \quad \forall (i,j) \in J_\Omega.$$
4. $\|e_h\| = \|(-L_h)^{-1} T_h\| \leq \|(-L_h)^{-1}\| \|T_h\|.$

Truncation Error of the 5 Point Difference Scheme

Suppose that the function u is sufficiently smooth, then, by Taylor series expansion of u on the grid node (x_i, y_j), we have

$$u_{i\pm1,j} = \left[u \pm h\partial_x u + \frac{h^2}{2}\partial_x^2 u \pm \frac{h^3}{6}\partial_x^3 u + \frac{h^4}{24}\partial_x^4 u \pm \frac{h^5}{120}\partial_x^5 u + \cdots\right]_{i,j}$$

$$u_{i,j\pm1} = \left[u \pm h\partial_y u + \frac{h^2}{2}\partial_y^2 u \pm \frac{h^3}{6}\partial_y^3 u + \frac{h^4}{24}\partial_y^4 u \pm \frac{h^5}{120}\partial_y^5 u + \cdots\right]_{i,j}$$

Since $T_{i,j} = L_h u_{i,j} + f_{i,j}$ and $f_{i,j} = -\triangle u_{i,j}$, we obtain

$$T_{i,j} := \frac{1}{12}h^2(\partial_x^4 u + \partial_y^4 u)_{i,j} + \frac{1}{360}h^4(\partial_x^6 u + \partial_y^6 u)_{i,j} + O(h^6), \quad \forall (i,j) \in J_\Omega.$$

Consistency and Order of Accuracy of L_h

1. Consistent condition of the scheme (or L_h to L) in l^∞-norm:

$$\lim_{h\to 0} T_h = \lim_{h\to 0} \max_{(i,j)\in J_\Omega} |T_{i,j}| = 0,$$

2. The order of the approximation accuracy of the scheme (or L_h to L): 2nd order approximation accuracy, since $T_h = O(h^2)$

Stability of the Scheme

Remember that

$$\|e_h\|_\infty = \|(-L_h)^{-1} T_h\|_\infty \le \|(-L_h)^{-1}\|_\infty \|T_h\|_\infty$$

$$\lim_{h\to 0} T_h = \lim_{h\to 0} \max_{(i,j)\in J_\Omega} |T_{i,j}| = 0,$$

therefore $\lim_{h\to 0} \|e_h\|_\infty = 0$, if $\|(-L_h)^{-1}\|_\infty$ is uniformly bounded, *i.e.* there exists a constant C independent of h such that

$$\max_{(i,j)\in J} |U_{i,j}| \le C \left(\max_{(i,j)\in J_\Omega} |f_{i,j}| + \max_{(i,j)\in J_D} |(u_D)_{i,j}| \right).$$

$\|(-L_h)^{-1}\|_\infty \le C$ is the stability of the scheme in l^∞-norm.

Convergence and the Accuracy of the Scheme

Remember that

$$-L_h U_{i,j} = \frac{4U_{i,j} - U_{i-1,j} - U_{i,j-1} - U_{i+1,j} - U_{i,j+1}}{h^2} = f_{i,j}, \quad \forall (i,j) \in J_\Omega.$$

$$-L_h e_{i,j} \triangleq \frac{4e_{i,j} - e_{i-1,j} - e_{i,j-1} - e_{i+1,j} - e_{i,j+1}}{h^2} = T_{i,j}, \quad \forall (i,j) \in J_\Omega.$$

therefore, since $\max_{(i,j)\in J_D} |e_{i,j}| = 0$,

$$\max_{(i,j)\in J} |U_{i,j}| \le C \left(\max_{(i,j)\in J_\Omega} |f_{i,j}| + \max_{(i,j)\in J_D} |(u_D)_{i,j}| \right).$$

implies also

$$\max_{(i,j)\in J} |e_{i,j}| \le C \max_{(i,j)\in J_\Omega} |T_{i,j}| \le C\, T_h \le C\, h^2 \max_{(x,y)\in\overline{\Omega}} \left(M_{xxxx} + M_{yyyy} \right),$$

where $M_{xxxx} = \max_{(x,y)\in\overline{\Omega}} |\partial_x^4 u|$, $M_{yyyy} = \max_{(x,y)\in\overline{\Omega}} |\partial_y^4 u|$.

The Maximum Principle and Comparison Theorem

- **Maximum principle of** L_h: for any grid function Ψ, $L_h \Psi \geq 0$, *i.e* $4\Psi_{i,j} \leq \Psi_{i-1,j} + \Psi_{i+1,j} + \Psi_{i,j-1} + \Psi_{i,j+1}$, implies that Ψ can not assume nonnegative maximum in the set of interior nodes J_Ω, unless Ψ is a constant.
- **Comparison Theorem**: Let $F = \max_{(i,j)\in J_\Omega} |f_{i,j}|$ and $\Phi(x,y) = (x-1/2)^2 + (y-1/2)^2$, take a comparison function
$$\Psi^{\pm}_{i,j} = \pm U_{i,j} + \frac{1}{4} F \Phi_{i,j}, \quad \forall\, (i,j) \in J.$$
It is easily verified that $L_h \Psi^{\pm} \geq 0$. Thus, noticing that $\Phi \geq 0$ and by the maximum principle, we obtain
$$\pm U_{i,j} \leq \pm U_{i,j} + \frac{1}{4} F \Phi_{i,j} \leq \max_{(i,j)\in J_D} |(u_0)_{i,j}| + \frac{1}{8} F, \quad \forall\, (i,j) \in J_\Omega.$$

Consequently, $\|U\|_\infty \leq \frac{1}{8} \max_{(i,j)\in J_\Omega} |f_{i,j}| + \max_{(i,j)\in J_D} |(u_0)_{i,j}|$,

The Maximum Principle and Comparison Theorem

Apply the maximum principle and comparison theorem to the error equation

$$-L_h e_{i,j} \triangleq \frac{4e_{i,j} - e_{i-1,j} - e_{i,j-1} - e_{i+1,j} - e_{i,j+1}}{h^2} = T_{i,j}, \quad \forall (i,j) \in J_\Omega.$$

we obtain

$$\|e\|_\infty \le \max_{(i,j)\in J_D} |e_{i,j}| + \frac{1}{8} T_h,$$

where $T_h = \max_{(i,j)\in J_\Omega} |T_{i,j}|$ is the l^∞-norm of the truncation error.

Grid and multi-index of grid

1. Discretize $\Omega \subset \mathbb{R}^n$: introduce a grid, say by taking the step sizes $h_i = \triangle x_i$, $i = 1, \ldots, n$, for the corresponding coordinate components;
2. The set of multi-index: $J = \{\mathbf{j} = (j_1, \cdots, j_n) : x = x_{\mathbf{j}} \triangleq (j_1 h_1, \cdots, j_n h_n) \in \bar{\Omega}\}$;
3. The index set of Dirichlet boundary nodes: $J_D = \{\mathbf{j} \in J : x = (j_1 h_1, \cdots, j_n h_n) \in \partial\Omega_D\}$;
4. The index set of interior nodes: $J_\Omega = J \setminus J_D$.

For simplicity, both (i, j) and (x_i, y_j) are called grid nodes.

Regular and irregular interior nodes with respect to L_h

1. Adjacent nodes: $\mathbf{j}, \mathbf{j}' \in J$ are adjacent, if $\sum_{k=1}^{n} |j_k - j'_k| = 1$;
2. $D_{L_h}(\mathbf{j})$: the set of nodes other than $\mathbf{j}$ used in calculating $L_h U_{\mathbf{j}}$
3. Regular interior nodes (with respect to L_h): $\mathbf{j} \in J_\Omega$ such that $D_{L_h}(\mathbf{j}) \subset \bar{\Omega}$;
4. Regular interior set $\mathring{J}_\Omega$: the set of all regular interior nodes;
5. Irregular interior set: $\tilde{J}_\Omega = J_\Omega \setminus \mathring{J}_\Omega$;
6. Irregular interior nodes (with respect to L_h): $\mathbf{j} \in \tilde{J}_\Omega$.

The control volume, grid functions and norms

1. Control volume of the node $\mathbf{j} \in J$:

$$\omega_{\mathbf{j}} = \{x \in \Omega : \ (j_i - \frac{1}{2})h_i \le x_i < (j_i + \frac{1}{2})h_i, \ 1 \le i \le n\},$$

   and denote $V_{\mathbf{j}} = \text{meas}(\omega_{\mathbf{j}})$;

2. Grid function $U(x)$: extend $U_{\mathbf{j}}$ to a piecewise constant function defined on Ω

$$U(x) = U_{\mathbf{j}}, \qquad \forall\, x \in \omega_{\mathbf{j}}.$$

3. $\mathbb{L}^p(\Omega)$ $(1 \le p \le \infty)$ norms of $U(x)$:

$$\|U\|_p = \Big\{ \sum_{j \in J} V_{\mathbf{j}} |U_{\mathbf{j}}|^p \Big\}^{1/p}, \qquad \|U\|_\infty = \max_{\mathbf{j} \in J} |U_{\mathbf{j}}|.$$

Basic Difference Operators

1. 1^{st}-order forward: $\triangle_{+x} v(x, x') := v(x + \triangle x, x') - v(x, x')$;
2. 1^{st}-order backward: $\triangle_{-x} v(x, x') := v(x, x') - v(x - \triangle x, x')$;
3. 1^{st}-order central: on one grid step
$$\delta_x v(x, x') := v(x + \frac{1}{2}\triangle x, x') - v(x - \frac{1}{2}\triangle x, x'),$$
and on two grid steps
$$\begin{aligned} \triangle_{0x} v(x, x') &:= \frac{1}{2}(\triangle_{+x} + \triangle_{-x})\, v(x, x') \\ &= \frac{1}{2}\left[v(x + \triangle x, x') - v(x - \triangle x, x')\right] \end{aligned}$$
4. 2nd order central: $\delta_x^2 v(x, x') = \delta_x(\delta_x v(x, x')) =$
$v(x + \triangle x, x') - 2v(x, x') + v(x - \triangle x, x')$.

A FD Scheme for the Steady State Convection-Diffusion Equation

$$-\nabla\cdot(a(x,y)\nabla u(x,y))+\nabla\cdot(u(x,y)\,\mathbf{v}(x,y)) = f(x,y),\ \forall(x,y)\in\Omega,$$

Substitute the differential operators by difference operators:

1. $(au_x)_x|_{i,j} \;\sim\; \delta_x(a_{i,j}\delta_x u_{i,j})/(\triangle x)^2$: where
   $$\delta_x(a_{i,j}\delta_x u_{i,j}) = a_{i+\frac{1}{2},j}(u_{i+1,j}-u_{i,j}) - a_{i-\frac{1}{2},j}(u_{i,j}-u_{i-1,j});$$
2. $(au_y)_y|_{i,j} \;\sim\; \delta_y(a_{i,j}\delta_y u_{i,j})/(\triangle y)^2$: where
   $$\delta_y(a_{i,j}\delta_y u_{i,j}) = a_{i,j+\frac{1}{2}}(u_{i,j+1}-u_{i,j}) - a_{i,j-\frac{1}{2}}(u_{i,j}-u_{i,j-1});$$
3. $(uv^1)_x|_{i,j}2\triangle x \;\sim\; \triangle_{0x}(uv^1)_{i,j} = (uv^1)_{i+1,j} - (uv^1)_{i-1,j};$
4. $(uv^2)_y|_{i,j}2\triangle y \;\sim\; \triangle_{0y}(uv^2)_{i,j} = (uv^2)_{i,j+1} - (uv^2)_{i,j-1};$

A FD Scheme for the Steady State Convection-Diffusion Equation

we are lead to the following finite difference scheme for the steady state convection-diffusion equation:

$$
\begin{aligned}
&-\frac{a_{i+\frac{1}{2},j}(U_{i+1,j}-U_{i,j})-a_{i-\frac{1}{2},j}(U_{i,j}-U_{i-1,j})}{(\triangle x)^2}\\
&\qquad -\frac{a_{i,j+\frac{1}{2}}(U_{i,j+1}-U_{i,j})-a_{i,j-\frac{1}{2}}(U_{i,j}-U_{i,j-1})}{(\triangle y)^2}\\
&\qquad\quad +\frac{(Uv^1)_{i+1,j}-(Uv^1)_{i-1,j}}{2\triangle x}+\frac{(Uv^2)_{i,j+1}-(Uv^2)_{i,j-1}}{2\triangle y}=f_{i,j}.
\end{aligned}
$$

A Finite Volume Scheme for the Steady State Convection-Diffusion Equation in Conservation Form

$$\int_{\partial\omega} (a(x,y)\nabla u(x,y) - u(x,y)\mathbf{v}(x,y))\cdot\nu(x,y)\,ds + \int_{\omega} f(x,y)\,dxdy = 0.$$

Take a proper control volume ω and substitute the differential operators by appropriate difference operators, and integrals by appropriate numerical quadratures:

1. for the index $(i,j) \in J_\Omega$, taking the control volume $\omega_{i,j} = \left\{(x,y) \in \Omega \cap \{[(i-\frac{1}{2})h_x, (i+\frac{1}{2})h_x) \times [(j-\frac{1}{2})h_y, (j+\frac{1}{2})h_y)\}\right\}$;
2. Applying the middle point quadrature on $\omega_{i,j}$ as well as on its four edges;
3. $\partial_\nu u(x_{i+\frac{1}{2}}, y_j) \;\sim\; (u_{i+1,j} - u_{i,j})/h_x$, etc.;

A Finite Volume Scheme for the Steady State Convection-Diffusion Equation in Conservation Form

we are lead to the following finite volume scheme for the steady state convection-diffusion equation:

$$
\begin{aligned}
&-\frac{a_{i+\frac{1}{2},j}(U_{i+1,j}-U_{i,j})-a_{i-\frac{1}{2},j}(U_{i,j}-U_{i-1,j})}{(\triangle x)^2}\\
&\quad-\frac{a_{i,j+\frac{1}{2}}(U_{i,j+1}-U_{i,j})-a_{i,j-\frac{1}{2}}(U_{i,j}-U_{i,j-1})}{(\triangle y)^2}\\
&\qquad+\frac{(U_{i+1,j}+U_{i,j})v^1_{i+\frac{1}{2},j}-(U_{i,j}+U_{i-1,j})v^1_{i-\frac{1}{2},j}}{2\triangle x}\\
&\qquad\quad+\frac{(U_{i,j+1}+U_{i,j})v^2_{i,j+\frac{1}{2}}-(U_{i,j}+U_{i,j-1})v^2_{i,j-\frac{1}{2}}}{2\triangle y}=f_{i,j},
\end{aligned}
$$

which is also called a conservative finite difference scheme.

A Finite Volume Scheme for Partial Differential Equations in Conservation Form

Finite volume methods:

1. control volume;
2. numerical flux;
3. conservative form.

More General Finite Difference Schemes

In more general case, say for triangular grid, hexagon grid, nonuniform grid, unstructured grid, and even grid less situations, in principle, we could still establish a finite difference scheme by

1. Taking proper neighboring nodes $J(P)$;
2. Approximating $Lu(P)$ by $L_h U_P := \sum_{i \in J(P)} c_i(P) U(Q_i)$;
3. Determining the weights $c_i(P)$ according to certain requirements, say the order of the local truncation error, local conservative property, discrete maximum principle, etc..

习题 1：1，3

Thank You!