Numerical Solutions to Partial Differential Equations

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The Relations Between the Errors in $\mathbb{L}^2$ and $\mathbb{H}^1$ norms

1. It follows from the interpolation error estimates on regular affine family of finite element function spaces (see Theorem 7.7) that

$$\|v - \Pi_h v\|_{m,\Omega} \leq C\, h^{k+1-m}\, |v|_{k+1,\Omega}, \quad m = 0, 1.$$

2. In other words, under the same conditions, the error of the finite element interpolation in the $\mathbb{L}^2(\Omega)$-norm is one order higher than that in the $\mathbb{H}^1(\Omega)$-norm.

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3. By the Céa lemma, the error of the finite element solution u_h in $\mathbb{H}^1(\Omega)$-norm is optimal. However, the error in $\mathbb{L}^2(\Omega)$-norm thus obtained

$$\|u - u_h\|_{0,\Omega} \leq \|u - u_h\|_{1,\Omega} \leq C\|u - \Pi_h u\|_{1,\Omega},$$

is obviously not optimal.

4. Under certain additional conditions, optimal $\mathbb{L}^2(\Omega)$-norm error estimate for FE solutions can be obtained by applying the Aubin-Nische technique based on the dual variational problem.

Dual Variational Problem and Relations of Errors in $\mathbb{L}^2$ and $\mathbb{H}^1$ norms

1. Consider the variational problem
$$\begin{cases} \text{Find } \ u \in \mathbb{V} \text{ such that} \\ a(u, v) = f(v), \quad \forall v \in \mathbb{V}, \end{cases}$$
where $\mathbb{V} \subset \mathbb{H}^1(\Omega)$, the bilinear form $a(\cdot, \ \cdot)$ and the linear form $f(\cdot)$ satisfy the conditions of the Lax-Milgram lemma.
2. Let $\mathbb{V}_h$ be a closed linear subspace of $\mathbb{V}$, and $u_h \in \mathbb{V}_h$ satisfy the equation $$a(u_h, v_h) = f(v_h), \quad \forall v_h \in \mathbb{V}_h.$$
3. Define the dual variational problem:
$$\begin{cases} \text{Find } \ \varphi \in \mathbb{V} \text{ such that} \\ a(v, \varphi) = (u - u_h, \ v), \quad \forall v \in \mathbb{V}, \end{cases}$$
where $(\cdot, \ \cdot)$ is the $\mathbb{L}^2(\Omega)$ inner product.

Dual Variational Problem and Relations of Errors in $\mathbb{L}^2$ and $\mathbb{H}^1$ norms

Lemma

Let $\varphi \in \mathbb{V}$ be the solution of the dual variational problem, and let $\varphi_h \in \mathbb{V}_h$ satisfy the equation

$$a(v_h, \varphi_h) = (u - u_h, \ v_h), \quad \forall v_h \in \mathbb{V}_h.$$

Then, we have

$$\|u - u_h\|_{0,\Omega}^2 \leq M \|u - u_h\|_{1,\Omega} \|\varphi - \varphi_h\|_{1,\Omega}.$$

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proof:

Take $v = u - u_h$ in the dual variational equation, and by the facts that $a(u - u_h, v_h) = 0$, $\forall v_h \in \mathbb{V}_h$ and $a(\cdot, \cdot)$ is bounded, we are lead to

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 = a(u - u_h, \varphi) &= a(u - u_h, \varphi - \varphi_h) \\ &\leq M \|u - u_h\|_{1,\Omega} \|\varphi - \varphi_h\|_{1,\Omega}. \quad \blacksquare \end{aligned}$$

An Optimal Error Estimate in $\mathbb{L}^2$-Norm

Theorem

Let the space dimension $n \le 3$. *Assume that the solution* φ *of the dual variational problem (7.3.10) is in* $\mathbb{H}^2(\Omega) \cap \mathbb{V}$, *and satisfies*

$$\|\varphi\|_{2,\Omega} \le C \, \|u - u_h\|_{0,\Omega}.$$

Let $\{(K, P_K, \Sigma_K)\}_{K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega)}$ *be a family of regular class* $\mathbb{C}^0$ *type* (1) *Lagrange affine equivalent finite elements. Then, the* $\mathbb{L}^2(\Omega)$*-norm error of the finite element solutions of the variational problem (7.1.1) satisfy*

$$\|u - u_h\|_{0,\Omega} \le C \, h \, \|u - u_h\|_{1,\Omega}.$$

Furthermore, if the solution u *of the variational problem (7.1.1) is in* $\mathbb{H}^2(\Omega) \cap \mathbb{V}$, *then*

$$\|u - u_h\|_{0,\Omega} \le C \, h^2 \, |u|_{2,\Omega},$$

Here C *in the three inequalities represent generally different constants which are independent of* h.

Proof of the Optimal Error Estimate in $\mathbb{L}^2$-Norm

1. By the Sobolve embedding theorem, $\mathbb{W}^{m+s,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^s(\overline{\Omega})$, $\forall\ s \geq 0$, if $m > n/p$. In particular, $\mathbb{H}^2(\Omega) \hookrightarrow \mathbb{C}(\bar{\Omega})$, if $n \leq 3$.

2. Thus, by applying the error estimates for finite element solutions in $\mathbb{H}^1(\Omega)$ norm (see Theorem 7.10 with $k = 1$ and $s = 0$) to the dual variational problem (7.3.10), we obtain

$$\|\varphi - \varphi_h\|_{1,\Omega} \leq Ch\,|\varphi|_{2,\Omega}.$$

Proof of the Optimal Error Estimate in $\mathbb{L}^2$-Norm

3. Therefore, by the lemma on the dual problem and $\|\varphi\|_{2,\Omega} \leq C\,\|u - u_h\|_{0,\Omega}$, we have

$$\|u - u_h\|_{0,\Omega} \leq C\,h\,\|u - u_h\|_{1,\Omega}.$$

4. Applying again Theorem 7.10 with $k = 1$ and $s = 0$ to $\|u - u_h\|_{1,\Omega}$, we are lead to

$$\|u - u_h\|_{0,\Omega} \leq C\,h^2\,|u|_{2,\Omega}. \qquad \blacksquare$$

Remarks on the Optimal Error Estimate in $\mathbb{L}^2$-Norm

1. The key to increase the $\mathbb{L}^2$-norm error estimate by an order is $\|\varphi\|_{2,\Omega} \leq C\,\|u - u_h\|_{0,\Omega}$, which does hold, if the coefficients of the second order elliptic operator are sufficiently smooth, and Ω is a convex polygonal region or a region with sufficiently smooth boundary.
2. In the general case, if we have $\|\varphi - \varphi_h\|_{1,\Omega} \propto h^\alpha \|u - u_h\|_{0,\Omega}$ and $\|u - u_h\|_{1,\Omega} \propto h^\alpha$, then,
$$\|u - u_h\|_{0,\Omega} \propto h^{2\alpha}.$$
3. Generally, we expect the convergence rate of finite element solutions in the $\mathbb{L}^2$-norm is twice of that in the $\mathbb{H}^1$-norm.

Nonconformity and Consistency Error

- The conformity of finite element methods is often broken, so it is necessary to extend abstract error estimates accordingly.
- Numerical quadratures break the conformity and introduce consistency error.

First Strang Lemma — Abstract Error Estimate Including Consistency Error

Theorem

Let $\mathbb{V}_h \subset \mathbb{V}$, and let the bilinear form $a_h(\cdot,\cdot)$ defined on $\mathbb{V}_h \times \mathbb{V}_h$ be uniform $\mathbb{V}_h$-elliptic, i.e. *there exists a constant $\hat{\alpha} > 0$ independent of h such that*

$$a_h(v_h, v_h) \geq \hat{\alpha}\|v_h\|^2, \quad \forall v_h \in \mathbb{V}_h.$$

Then, there exists a constant C independent of h such that

$$\|u - u_h\| \leq C\Big(\inf_{v_h \in \mathbb{V}_h} \Big\{ \|u - v_h\| + \sup_{w_h \in \mathbb{V}_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|} \Big\} + \sup_{w_h \in \mathbb{V}_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|} \Big).$$

Proof of the First Strang Lemma

1. Since $\mathbb{V}_h \subset \mathbb{V}$ and $a(u, v) = f(v)$, $\forall v \in \mathbb{V}$, we have
$$a(u - v_h, u_h - v_h) + a(v_h, u_h - v_h) - f(u_h - v_h) = 0, \quad \forall v_h \in \mathbb{V}_h.$$

2. Since $a_h(u_h, v_h) = f_h(v_h)$, $\forall v_h \in \mathbb{V}_h$, we have
$$a_h(u_h - v_h, u_h - v_h) = f_h(u_h - v_h) - a_h(v_h, u_h - v_h), \quad \forall v_h \in \mathbb{V}_h.$$

3. Therefore, by the uniform $\mathbb{V}_h$-ellipticity of $a_h(\cdot, \cdot)$ on $\mathbb{V}_h$, we have
$$\begin{aligned} & \hat{\alpha}\|v_h - u_h\|^2 \leq a_h(u_h - v_h, u_h - v_h) \\ = \; & a(u - v_h, u_h - v_h) + \{a(v_h, u_h - v_h) - a_h(v_h, u_h - v_h)\} \\ & + \{f_h(u_h - v_h) - f(u_h - v_h)\}. \end{aligned}$$

Proof of the First Strang Lemma

4. Hence, by the boundedness of the bilinear form $a(\cdot,\cdot)$, and $|f_h(u_h - v_h) - f(u_h - v_h)| \le \sup_{w_h \in \mathbb{V}_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|} \|u_h - v_h\|$

$$|a(v_h, u_h - v_h) - a_h(v_h, u_h - v_h)| \le \sup_{w_h \in \mathbb{V}_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|} \|u_h - v_h\|,$$

we are lead to

$$\hat{\alpha}\|u_h - v_h\| \le M\|u - v_h\| + \sup_{w_h \in \mathbb{V}_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|} + \sup_{w_h \in \mathbb{V}_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|}.$$

5. Since $\|u - u_h\| \le \|u - v_h\| + \|u_h - v_h\|$, the conclusion of the theorem follows for $C = \max\{\hat{\alpha}^{-1}, 1 + \hat{\alpha}^{-1}M\}$. ■

Use of Non-Conforming Finite Element Function Spaces

1. The conformity will be broken, if a non-conforming finite element is used to construct the finite element function spaces.
2. In such a case, $\mathbb{V}_h \not\subset \mathbb{V}$, therefore, $\|\cdot\|$, $f(\cdot)$ and $a(\cdot,\cdot)$ must be extended to $\|\cdot\|_h$, $f_h(\cdot)$ and $a_h(\cdot,\cdot)$ defined on $\mathbb{V}+\mathbb{V}_h$.
3. For example, if $\mathbb{V}=\mathbb{H}_0^1(\Omega)$ and $a(u,v)=\int_\Omega \nabla u\cdot\nabla v\,dx$, we may define

$$v_h \mapsto \|v_h\|_h := \Big(\sum_{K\in\mathfrak{T}_h(\Omega)} |v_h|_{1,K}^2\Big)^{1/2},$$

$$(u_h, v_h) \mapsto a_h(u_h, v_h) := \sum_{K\in\mathfrak{T}_h(\Omega)} \int_K \nabla u_h\cdot\nabla v_h\,dx.$$

Error Bound of Non-Conforming Finite Element solution

The following abstract error estimate again bounds the error of the finite element solution in the non-conforming finite element function spaces by

- the approximation error of the finite element function space;
- and the consistency error of the approximation functionals $a_h(\cdot,\cdot)$ and $f_h(\cdot)$.

Second Strang Lemma — Abstract Error Estimate for Non-Conforming FE

Theorem

Let the bilinear form $a_h(\cdot, \cdot)$ *be uniformly bounded on* $(\mathbb{V} + \mathbb{V}_h) \times (\mathbb{V} + \mathbb{V}_h)$*, and be uniformly* $\mathbb{V}_h$*-elliptic,* i.e. *there exist constants* $\hat{M}$ *and* $\hat{\alpha} > 0$ *independent of* h *such that*

$$|a_h(u_h, v_h)| \leq \hat{M} \|u_h\|_h \|v_h\|_h, \quad \forall u_h, v_h \in \mathbb{V} + \mathbb{V}_h,$$

$$a_h(v_h, v_h) \geq \hat{\alpha} \|v_h\|_h^2, \quad \forall v_h \in \mathbb{V}_h.$$

Then, the error of the solution u_h *of the corresponding approximation variational problem with respect to the solution* u *of the original variational problem satisfies*

$$\|u - u_h\|_h \cong \Big(\inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_h + \sup_{w_h \in \mathbb{V}_h} \frac{|a_h(u, w_h) - f_h(w_h)|}{\|w_h\|_h} \Big).$$

Here $A_h(u) \cong B_h(u)$ *means that there exist positive constants* C_1 *and* C_2 *independent of* u *and* h *s.t.* $C_1 B_h(u) \leq A_h(u) \leq C_2 B_h(u)$*, for all* $h > 0$ *sufficiently small.*

Proof of the Second Strang Lemma

1. Since $a_h(u_h, v_h) = f_h(v_h)$, $\forall v_h \in \mathbb{V}_h$, we have

$$a_h(u_h - v_h,\, u_h - v_h) = f_h(u_h - v_h) - a_h(v_h,\, u_h - v_h), \quad \forall v_h \in \mathbb{V}_h.$$

2. Therefore, by the uniform $\mathbb{V}_h$-ellipticity of $a_h(\cdot,\cdot)$ on $\mathbb{V}_h$, we have

$$\begin{aligned} \hat{\alpha}\|v_h - u_h\|^2 &\le a_h(u_h - v_h,\, u_h - v_h) \\ &= a_h(u - v_h, u_h - v_h) + \{f_h(u_h - v_h) - a_h(u,\, u_h - v_h)\}. \end{aligned}$$

3. Thus, the uniform boundedness of $a_h(\cdot,\,\cdot)$ and

$$\|u - u_h\|_h \le \|u - v_h\|_h + \|u_h - v_h\|_h$$

led to "$\le$" part of the theorem for $C_2 = \max\{\hat{\alpha}^{-1}, 1 + \hat{\alpha}^{-1}\hat{M}\}$.

Proof of the Second Strang Lemma

4. On the other hand, it follows from the uniform boundedness of $a_h(\cdot, \cdot)$ that

$$a_h(u, w_h) - f_h(w_h) = a_h(u - u_h, w_h) \le \hat{M} \|u - u_h\|_h \|w_h\|_h, \quad \forall w_h \in \mathbb{V}_h.$$

5. Thus, by the arbitrariness of w_h, we have

$$\|u - u_h\|_h \ge \hat{M}^{-1} \sup_{w_h \in \mathbb{V}_h} \frac{|a_h(u, w_h) - f_h(w_h)|}{\|w_h\|_h}.$$

6. This together with $\|u - u_h\|_h \ge \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_h$ yield the "$\ge$" part of theorem with $C_1 = \frac{1}{2} \min\{\hat{M}^{-1}, 1\}$. ∎

In General $\mathbb{V}_h \not\subseteq \mathbb{V}$ for Non-Polygon Ω

1. If Ω is not a polygonal region, the region Ω_h covered by a finite element triangulation is generally not equal to Ω, this will also lead to nonconformity $\mathbb{V}_h \not\subseteq \mathbb{V}$.
2. For a general case when there is nonconformity, it follows from the first and second Strang lemmas that, to obtain the **error estimates** for finite element solutions, in addition to the **interpolation error** estimates, the **consistency errors** of the approximate bilinear forms $a_h(\cdot, \cdot)$ and linear forms $f_h(\cdot)$ must also be properly estimated.
3. It is usually required that the **approximate operators are uniformly continuous and stable** (*i.e.* the approximate bilinear forms $a_h(\cdot, \cdot)$ are uniformly bounded and uniformly $\mathbb{V}_h$-elliptic).

The Basic Tools for the Analysis of Consistency Error

1. The basic tools for the analysis are still the equivalent quotient norms in the polynomial quotient spaces and relations between the Sobolev semi-norms on affine equivalent open sets.

2. Error estimates on polynomial invariant operators play a very important role in the interpolation error estimates. The following two lemmas are the counterparts in the consistency error estimates for polynomial vanishing linear and bilinear forms.

The Bramble-Hilbert Lemma — — An Abstract Estimate on Polynomial Vanishing Linear Forms

Theorem

Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. For some $p \in [1, \infty]$ and some integer $k \geq 0$, let the bounded linear form f defined on $\mathbb{W}^{k+1,p}(\Omega)$ be such that

$$f(w) = 0, \qquad \forall w \in \mathbb{P}_k(\Omega).$$

Then, there exists a constant $C(\Omega)$ such that

$$|f(v)| \leq C(\Omega) \, \|f\|^*_{k+1,p,\Omega} |v|_{k+1,p,\Omega},$$

where $\| \cdot \|^*_{k+1,p,\Omega}$ is the norm on the dual space of $\mathbb{W}^{k+1,p}(\Omega)$.

Proof of the Bramble-Hilbert Lemma

1. For any $v \in \mathbb{W}^{k+1,p}(\Omega)$, it follows from $f(w) = 0$, $\forall w \in \mathbb{P}_k(\Omega)$, that
$$|f(v)| = |f(v+w)| \le \|f\|^*_{k+1,p,\Omega} \|v+w\|_{k+1,p,\Omega}, \quad \forall w \in \mathbb{P}_k(\Omega),$$
2. Thus,
$$|f(v)| \le \|f\|^*_{k+1,p,\Omega} \inf_{w \in \mathbb{P}_k} \|v + w\|_{k+1,p,\Omega}.$$
3. Since $|\cdot|_{k+1,p,\Omega}$ is an equivalent quotient norm in the polynomial quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$ (see Theorem 7.2), the conclusion of the theorem follows. ∎

The Bilinear Lemma —

— An Abstract Estimate on Polynomial Vanishing Bilinear Forms

Theorem

Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz continuous boundary. For some $p,\ q \in [1,\ \infty]$, some integers $k,\ l \geq 0$ and a subspace $\mathbb{W}$ which satisfies the inclusion relation $\mathbb{P}_l(\Omega) \subset \mathbb{W} \subset \mathbb{W}^{l+1,q}(\Omega)$ and is endowed with the norm $\|\cdot\|_{l+1,q,\Omega}$, let the bounded bilinear form b defined on $\mathbb{W}^{k+1,p}(\Omega) \times \mathbb{W}$ be such that

$$\begin{aligned} b(r,\ w) &= 0, \qquad \forall r \in \mathbb{P}_k(\Omega),\ \ \forall w \in \mathbb{W}, \\ b(v,\ r) &= 0, \qquad \forall v \in \mathbb{W}^{k+1,p}(\Omega),\ \ \forall r \in \mathbb{P}_l(\Omega). \end{aligned}$$

Then, there exists a constant $C(\Omega)$ such that

$$|b(v,w)| \leq C(\Omega)\|b\||v|_{k+1,p,\Omega}|w|_{l+1,q,\Omega},\ \ \forall v \in \mathbb{W}^{k+1,p}(\Omega),\ \forall w \in \mathbb{W},$$

where $\|b\|$ is the norm of the bilinear form b on $\mathbb{W}^{k+1,p}(\Omega) \times \mathbb{W}$.

Proof of the Bilinear Lemma

1. For any given $w \in \mathbb{W}$, $b(\cdot,\ w)$, regarded as a bounded linear form defined on $\mathbb{W}^{k+1,p}(\Omega)$ satisfies the conditions of the Bramble-Hilbert lemma.

2. Hence, there exists a constant $C_1(\Omega)$ such that

$$|b(v,\ w)| \leq C_1(\Omega)\, \|b(\cdot,\ w)\|^*_{k+1,p,\Omega}\, |v|_{k+1,p,\Omega},\ \ \forall v \in \mathbb{W}^{k+1,p}(\Omega).$$

3. On the other hand, since for any $v \in \mathbb{W}^{k+1,p}(\Omega)$, $b(v,\ r) = 0$, $\forall r \in \mathbb{P}_l(\Omega)$, we have

$$|b(v, w)| = |b(v, w+r)| \leq \|b\| \|v\|_{k+1,p,\Omega} \|w+r\|_{l+1,q,\Omega},\ \ \forall r \in \mathbb{P}_l.$$

Proof of the Bilinear Lemma

4. Since $|\cdot|_{l+1,p,\Omega}$ is an equivalent quotient norm in the polynomial quotient space $\mathbb{W}^{l+1,p}(\Omega)/\mathbb{P}_l(\Omega)$, $\exists$ const. $C_2(\Omega)$ s.t. $\inf_{r\in\mathbb{P}_l}\|w+r\|_{l+1,q,\Omega} \leq C_2(\Omega)|w|_{l+1,q,\Omega}$.
5. Therefore, for any $v \in \mathbb{W}^{k+1,p}(\Omega)$, we have
$$|b(v,w)| \leq C_2(\Omega)\|b\|\|v\|_{k+1,p,\Omega}|w|_{l+1,q,\Omega}, \quad \forall w \in \mathbb{W}.$$
6. This implies
$$\|b(\cdot,w)\|^*_{k+1,p,\Omega} = \sup_{v\in\mathbb{W}^{k+1,p,\Omega}} \frac{|b(v,\ w)|}{\|v\|_{k+1,p,\Omega}} \leq C_2(\Omega)\|b\||w|_{l+1,q,\Omega}.$$
7. Combining this with ② (see (7.4.11)) leads to the conclusion of the theorem. ■

习题 7：9，10

Thank You!