Numerical Solutions to Partial Differential Equations

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A Framework for Interpolation Error Estimation of Affine Equivalent FEs

1. The polynomial quotient spaces of a Sobolev space and their equivalent quotient norms ((2) in the example);
2. The relations between the semi-norms of Sobolev spaces defined on affine-equivalent open sets ((3), (4) in the exmp.);
3. To estimate the constants appeared in the relations of the Sobolev semi-norms by means of the geometric parameters of the corresponding affine-equivalent open sets.
4. The abstract error estimates for the polynomial invariant operators ((1) in the example);

The Semi-norm $|v|_{k+1,p,\Omega}$ Is an Equivalent Norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

1. The quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$ is a Banach space.
2. The quotient norm of a function $\dot{v}$ is defined by
$$\dot{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega) \to \|\dot{v}\|_{k+1,p,\Omega} := \inf_{w \in \mathbb{P}_k(\Omega)} \|v+w\|_{k+1,p,\Omega}.$$
3. $\dot{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega) \to |\dot{v}|_{k+1,p,\Omega} = |v|_{k+1,p,\Omega}$ is a semi-norm, and obviously $|\dot{v}|_{k+1,p,\Omega} \leq \|\dot{v}\|_{k+1,p,\Omega}$.

Theorem

There exists a constant $C(\Omega)$ such that
$$\|\dot{v}\|_{k+1,p,\Omega} \leq C(\Omega)|\dot{v}|_{k+1,p,\Omega}, \qquad \forall \dot{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega).$$

Relations of Semi-norms on Open Sets Related by $F(\hat{x}) = B\hat{x} + b$

Theorem

Let Ω and $\hat{\Omega}$ be two affine equivalent open sets in $\mathbb{R}^n$. Let $v \in \mathbb{W}^{m,p}(\Omega)$ for some $p \in [1, \infty]$ and nonnegative integer m. Then, $\hat{v} = v \circ F \in \mathbb{W}^{m,p}(\hat{\Omega})$, and there exists a constant $C = C(m, n)$ such that

$$|\hat{v}|_{m,p,\hat{\Omega}} \leq C\|B\|^m \,|\det(B)|^{-1/p}\, |v|_{m,p,\Omega},$$

where B is the matrix in the affine mapping F, $\|\cdot\|$ represents the operator norms induced from the Euclidian norm of $\mathbb{R}^n$. Similarly, we also have

$$|v|_{m,p,\Omega} \leq C\|B^{-1}\|^m \,|\det(B)|^{1/p}\, |\hat{v}|_{m,p,\hat{\Omega}}.$$

Bound $\|B\|$ and $\|B^{-1}\|$ by the Interior and Exterior Diameters

1. Denote the exterior and interior of a region Ω as

$$\begin{cases} h_\Omega := \operatorname{diam}(\Omega), \\ \rho_\Omega := \sup\{\operatorname{diam}(S) : S \subset \Omega \text{ is a } n\text{-dimensional ball}\}. \end{cases}$$

Theorem

Let Ω and $\hat{\Omega}$ be two affine-equivalent open sets in $\mathbb{R}^n$, let $F(\hat{x}) = B\hat{x} + b$ be the invertible affine mapping, and $\Omega = F(\hat{\Omega})$. Then,

$$\|B\| \leq \frac{h}{\hat{\rho}}, \quad \textit{and} \quad \|B^{-1}\| \leq \frac{\hat{h}}{\rho},$$

where $h = h_\Omega$, $\hat{h} = h_{\hat{\Omega}}$, $\rho = \rho_\Omega$, $\hat{\rho} = \rho_{\hat{\Omega}}$.

Proof of $\|B\| \le \frac{h}{\hat{\rho}}$ and the Geometric Meaning of $\det(B)$

1. By the definition of $\|B\|$, we have
$$\|B\| = \frac{1}{\hat{\rho}} \sup_{\|\xi\|=\hat{\rho}} \|B\xi\|.$$
2. Let the vectors $\hat{x},\ \hat{y} \in \overline{\hat{\Omega}}$ be such that $\|\hat{y} - \hat{x}\| = \hat{\rho}$, then, we have $x = F(\hat{x}) \in \bar{\Omega}$, $y = F(\hat{y}) \in \bar{\Omega}$.
3. Therefore, $\|B(\hat{y} - \hat{x})\| = \|F(\hat{y}) - F(\hat{x})\| \le h \ \Rightarrow \|B\| \le \frac{h}{\hat{\rho}}$. ■

The determinant $\det(B)$ also has an obvious geometric meaning:
$$|\det(B)| = \frac{\mathrm{meas}(\Omega)}{\mathrm{meas}(\hat{\Omega})} \quad \text{and} \quad |\det(B^{-1})| = \frac{\mathrm{meas}(\hat{\Omega})}{\mathrm{meas}(\Omega)}.$$

A Result on Affine Equivalent Family of Polynomial Invariant Operators

Theorem

Let nonnegative integers k, m and p, $q \in [1, \infty]$ be such that $\mathbb{W}^{k+1,p}(\hat{\Omega}) \hookrightarrow \mathbb{W}^{m,q}(\hat{\Omega})$. Let the bounded linear operator $\hat{\Pi} \in \mathfrak{L}(\mathbb{W}^{k+1,p}(\hat{\Omega});\ \mathbb{W}^{m,q}(\hat{\Omega}))$ be $\mathbb{P}_k(\hat{\Omega})$ invariant, meaning

$$\hat{\Pi}\hat{w} = \hat{w}, \qquad \forall\, \hat{w} \in \mathbb{P}_k(\hat{\Omega}).$$

Let $\Omega = F(\hat{\Omega})$ be an arbitrary open set which is affine equivalent to $\hat{\Omega}$, where $F(\hat{x}) = B\hat{x} + b$. Let the linear operator $\Pi_\Omega \in \mathfrak{L}(\mathbb{W}^{k+1,p}(\Omega);\ \mathbb{W}^{m,q}(\Omega))$ be defined by

$$\Pi_\Omega\, v = \Big(\hat{\Pi}\,(v \circ F)\Big) \circ F^{-1}.$$

Then there exists a constant $C = C(\hat{\Pi}, \hat{\Omega}, k, m, n)$ independent of Ω, such that, for all $v \in \mathbb{W}^{k+1,p}(\Omega)$,

$$|v - \Pi_\Omega v|_{m,q,\Omega} \le C\,(\mathrm{meas}(\Omega))^{(1/q-1/p)}\ \frac{h_\Omega^{k+1}}{\rho_\Omega^{m}}|v|_{k+1,p,\Omega}.$$

Proof of Error Estimate for Affine-Family of Polynomial Invariant Operators

1. By the polynomial $\mathbb{P}_k(\hat{\Omega})$ invariant of $\hat{\Pi}$, we have
$\hat{v} - \hat{\Pi}\hat{v} = (I - \hat{\Pi})(\hat{v} + \hat{w}), \qquad \forall \hat{v} \in \mathbb{W}^{k+1,p}(\hat{\Omega}), \ \forall \hat{w} \in \mathbb{P}_k(\hat{\Omega}).$

2. By $\mathbb{W}^{k+1,p}(\hat{\Omega}) \hookrightarrow \mathbb{W}^{m,q}(\hat{\Omega})$, $\hat{\Pi} \in \mathfrak{L}(\mathbb{W}^{k+1,p}(\hat{\Omega});\ \mathbb{W}^{m,q}(\hat{\Omega}))$, and the semi-norm $|\cdot|_{k+1,p,\hat{\Omega}}$ is an equivalent quotient norm:

$$|\hat{v} - \hat{\Pi}\hat{v}|_{m,q,\hat{\Omega}} \le \|I - \hat{\Pi}\| \inf_{\hat{w} \in \mathbb{P}_k(\hat{\Omega})} \|\hat{v} + \hat{w}\|_{k+1,p,\hat{\Omega}} \le C(\hat{\Pi}, \hat{\Omega}) |\hat{v}|_{k+1,p,\hat{\Omega}}.$$

3. Since $v - \Pi_\Omega v = (\hat{v} - \hat{\Pi}\hat{v}) \circ F^{-1}$, by Theorem 7.3, we have

$$|v - \Pi v|_{m,q,\Omega} \le C(m,n) \|B^{-1}\|^m \det(B)^{1/q} |\hat{v} - \hat{\Pi}\hat{v}|_{m,q,\hat{\Omega}},$$

$$|\hat{v}|_{k+1,p,\hat{\Omega}} \le C(k+1,n) \|B\|^{k+1} \det(B)^{-1/p} |v|_{k+1,p,\Omega}.$$

4. The estimate follows as a consequence of the above three inequalities and Theorem 7.4. ■

Error Estimates of Affine Family Finite Element Interpolations

Theorem

Let $(\hat{K}, \hat{P}, \widehat{\Sigma})$ be a finite element, let s be the highest order of the partial derivatives appeared in the set of the degrees of freedom $\widehat{\Sigma}$. Let nonnegative integers k and m, and $p, q \in [1, \infty]$ be such that

$$\mathbb{W}^{k+1,p}(\hat{K}) \hookrightarrow \mathbb{C}^{s}(\hat{K}),$$
$$\mathbb{W}^{k+1,p}(\hat{K}) \hookrightarrow \mathbb{W}^{m,q}(\hat{K}),$$
$$\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{W}^{m,q}(\hat{K}).$$

Then, there exists a constant $C(\hat{K}, \hat{P}, \widehat{\Sigma})$ such that, for all finite elements (K, P, Σ) which are affine equivalent to $(\hat{K}, \hat{P}, \widehat{\Sigma})$, and for all $v \in \mathbb{W}^{k+1,p}(K)$,

$$|v - \Pi_K v|_{m,q,K} \leq C(\hat{K}, \hat{P}, \widehat{\Sigma})\,(\mathrm{meas}(K))^{(1/q-1/p)}\ \frac{h_K^{k+1}}{\rho_K^m}|v|_{k+1,p,K}.$$

Proof of Error Estimates for Affine-Family of Finite Element Interpolations

1. By the error estimates for affine-family of polynomial invariant operators (see Theorem 7.5), we only need to verify that the corresponding finite element interpolation operators $\hat{\Pi} = \hat{\Pi}_{\hat{K}} \in \mathfrak{L}(\mathbb{W}^{k+1,p}(\hat{K}); \mathbb{W}^{m,q}(\hat{K}))$.

2. Let $\{\hat{w}_i\}_{i=1}^N$ be a set of basis of $\hat{P}$, and $\{\hat{\varphi}_i\}_{i=1}^N \subset \widehat{\Sigma}$ be the corresponding dual basis.

3. $\{\hat{\varphi}_i\}_{i=1}^N$ are also bounded linear functionals on $\mathbb{C}^s(\hat{K})$, since s is the highest order of partial derivatives in $\widehat{\Sigma}$.

Proof of Error Estimates for Affine-Family of Finite Element Interpolations

4. By $\hat{\Pi}\hat{v} = \sum_{i=1}^{N} \hat{\varphi}_i(\hat{v})\hat{w}_i \in \hat{P} \subset \mathbb{W}^{m,q}(\hat{K})$, and $\mathbb{W}^{k+1,p}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K})$, we have, for all $\hat{v} \in \mathbb{W}^{k+1,p}(\hat{K})$,

$$\|\hat{\Pi}\hat{v}\|_{m,q,\hat{K}} \leq \sum_{i=1}^{N} |\hat{\varphi}_i(\hat{v})| \, \|\hat{w}_i\|_{m,q,\hat{K}} \leq$$

$$C \left(\sum_{i=1}^{N} \|\hat{w}_i\|_{m,q,\hat{K}} \right) \|\hat{v}\|_{s,\infty,\hat{K}} \leq C_1 \|\hat{v}\|_{k+1,p,\hat{K}}. \quad \blacksquare$$

Relations Between Geometric Parameters of a Finite Element K

- Denote $\sigma_n = \mathrm{meas}\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$, we have
$$\sigma_n \rho_K^n \leq \mathrm{meas}\,(K) \leq h_K^n.$$
- In the finite element error analysis, the convergence speed can be characterized by the powers of the element K's diameter h_K, if the finite elements satisfy certain geometric conditions, for example, the regularity condition given below.

Regular Family of Finite Element Triangulations

Definition

$\{\mathfrak{T}_h(\Omega)\}_{h>0}$ is said to be a regular family of finite element triangulations of Ω, if

(i) there exists a constant σ independent of h such that

$$h_K \leq \sigma\, \rho_K, \qquad \forall K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega);$$

(ii) 0 is the accumulation point of the parameter h.

Regular Affine Equivalent Family of Finite Element Triangulations

- Let $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ be a regular family of finite element triangulations of Ω.

- Let $\{(K, P_K, \Sigma_K)\}_{K\in\bigcup_{h>0}\mathfrak{T}_h(\Omega)}$ be a family of finite elements, each of which is affine equivalent to the reference finite element $(\hat{K},\ \hat{P},\ \hat{\Sigma})$.

Then, the finite elements $\{(K, P_K, \Sigma_K)\}_{K\in\bigcup_{h>0}\mathfrak{T}_h(\Omega)}$ are called a regular affine equivalent family of finite elements.

Error Estimates of Regular Affine Family of Finite Element Interpolations

Theorem

Let $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ *be a regular family of finite element triangulations of* Ω*, let* $\{(K, P_K, \Sigma_K)\}_{K\in\bigcup_{h>0}\mathfrak{T}_h(\Omega)}$ *be a regular affine equivalent family of finite elements with respect to the reference finite element* $(\hat{K}, \hat{P}, \hat{\Sigma})$*, which satisfies the conditions (7.2.11)–(7.2.13) of Theorem 7.6 (Them. on interpolation error of affine equivalent FEs). Then, there exists a constant* $C = C(\hat{K}, \hat{P}, \hat{\Sigma})$ *such that*

$$\|v - \Pi_K v\|_{m,q,K} \leq C\,(\mathrm{meas}\,(K))^{(1/q-1/p)}\,\sigma^m\,h_K^{k+1-m}\,|v|_{k+1,p,K},$$

$$\forall v \in \mathbb{W}^{k+1,p}(K),\ \ \forall K \in \bigcup_{h>0}\mathfrak{T}_h(\Omega),$$

where σ *is the constant in the definition of regularity.*

Proof of Error Estimates for Regular Affine Family of Finite Elements

1. By the error estimates for affine equivalent family of finite element interpolation operators (Theorem 7.6), we have that, for all $v \in \mathbb{W}^{k+1,p}(K)$,

$$|v - \Pi_K v|_{m,q,K} \leq C(\hat{K},\, \hat{P},\, \widehat{\Sigma})\, (\mathrm{meas}(K))^{(1/q-1/p)}\; \frac{h_K^{k+1}}{\rho_K^m} |v|_{k+1,p,K}.$$

2. By the regularity of the triangulation, we have

$$\rho_K^{-1} \leq \sigma h_K^{-1}, \quad \forall K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega).$$

3. Thus, the conclusion of the theorem follows. ■

The Sobolev Embedding Theorem

Theorem

Let Ω be a bounded connected domain with a Lipschitz continuous boundary $\partial\Omega$, then

$$
\begin{aligned}
&\mathbb{W}^{m+k,p}(\Omega) \hookrightarrow \mathbb{W}^{k,q}(\Omega),\ \forall\, 1 \le q \le \frac{np}{n-mp},\ k \ge 0, && \text{if } m < n/p;\\
&\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega),\ \forall\, 1 \le q < \frac{np}{n-mp},\ k \ge 0, && \text{if } m < n/p;\\
&\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega),\ \forall\, 1 \le q < \infty,\ k \ge 0, && \text{if } m = n/p;\\
&\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^{k}(\overline{\Omega}),\quad \forall\, k \ge 0, && \text{if } m > n/p.
\end{aligned}
$$

The Sobolev Embedding Theorem

Remark:

By the theorem (see Theorem 5.5), a "lower" order norm of a Sobolev function can be bounded by its "higher" order norms.

Generally speaking, the reverse inequalities do not hold. However, we can prove the inverse inequalities on certain finite element function spaces.

Family of Quasi-Uniform Finite Element Triangulations

Recall, $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ is a regular family of FE triangulations of Ω, if

(i) there exists a constant σ independent of h such that

$$h_K \leq \sigma \, \rho_K, \qquad \forall K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega);$$

(ii) 0 is the accumulation point of the parameter h.

Definition

Let $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ be a regular family of finite element triangulations of Ω. If there exists a constant γ such that

$$\max_{K' \in \mathfrak{T}_h(\Omega)} h_{K'} \leq \gamma \, h_K, \qquad \forall K \in \mathfrak{T}_h(\Omega), \ \ \forall h > 0,$$

then, $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ is called a family of quasi-uniform finite element triangulations of Ω.

The Inverse Inequalities of Finite Element Functions

Theorem

Let $\{\mathfrak{T}_h(\Omega)\}_{h>0}$ *be a family of quasi-uniform finite element triangulations of a bounded open set* Ω *in* $\mathbb{R}^n$, $\mathbb{V}_h(\Omega)$ *be the FE function spaces on* $\mathfrak{T}_h(\Omega)$, *where the FEs* (K, P_K, Σ_K) *are affine equivalent to the reference FE* $(\hat{K}, \hat{P}, \hat{\Sigma})$. *Let integers* $1 \leq l \leq m$ *and* p, $q \in [1, \infty]$ *satisfy* $\mathbb{P}_{l-1}(\hat{K}) \subset \hat{P} \subset \mathbb{W}^{l,p}(\hat{K}) \cap \mathbb{W}^{m,q}(\hat{K})$.

Then, there exists a constant $C(\sigma, \gamma, l, m)$, *where* σ *and* γ *are the regularity and quasi-uniform constants, such that, for all* $q < \infty$,

$$\Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|_{m,q,K}^q\Big)^{1/q} \leq C(\sigma, \gamma, l, m)\, h^{l-m-s}\Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|_{l,p,K}^p\Big)^{1/p}, \quad \forall v \in \mathbb{V}_h(\Omega),$$

where $s = \max\{0, n\,(1/p - 1/q)\}$; *and for* $q = \infty$,

$$\max_{K \in \mathfrak{T}_h(\Omega)} \{|v|_{m,\infty,K}\} \leq C(\sigma, \gamma, l, m)\, h^{l-m-n/p}\Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|_{l,p,K}^p\Big)^{1/p}, \quad \forall v \in \mathbb{V}_h(\Omega).$$

Remark: The conclusion of the Theorem can be extended to cover the case of $l = 0$ (with the proof modified slightly here and there).

Proof of the Inverse Inequalities of Finite Element Functions

1. It follows from the relations of semi-norms on affine equivalent open sets (see Theorem 7.3, and 7.4 as well) and the regularity of the triangulation that, there exists a constant C_0, which depends only on σ, γ, l, m, such that

$$\begin{aligned} |\hat{v}_K|_{l,p,\hat{K}} &\le C_0\, h_K^{l-n/p} |v|_{l,p,K}, \\ |v|_{m,q,K} &\le C_0\, h_K^{-m+n/q} |\hat{v}_K|_{m,q,\hat{K}}, \end{aligned}$$

where $\hat{v}_K = v \circ F_K$, $F_K : K \to \hat{K}$ are the corresponding affine mappings.

Proof of the Inverse Inequalities of Finite Element Functions

2. It follows from the equivalent norm on the polynomial quotient space (see Theorem 7.2) that, there exists a constant $C_1 = C(\hat{K})$ such that the quotient norm $\|\dot{w}\|_{l,p,\hat{K}}$ of the quotient space $\hat{P}/\mathbb{P}_{l-1}(\hat{K})$ satisfies
$$\|\dot{w}\|_{l,p,\hat{K}} \leq C_1 \, |w|_{l,p,\hat{K}}, \quad \forall w \in \hat{P}, \qquad \text{(holds for } l = 0 \text{ directly)}$$
where $\dot{w}$ is the equivalent class of w in the quotient space $\hat{P}/\mathbb{P}_{l-1}(\hat{K})$. (In fact, we have $\hat{P} = \hat{P}/\mathbb{P}_{l-1}(\hat{K}) \oplus \mathbb{P}_{l-1}(\hat{K})$).
3. On the other hand, $|w|_{m,q,\hat{K}} = 0$, $\forall w \in \mathbb{P}_{l-1}(\hat{K})$, since $l \leq m$.
(for $l = 0$, $|w|_{m,q,\hat{K}} = 0$, $\forall w \in \mathbb{P}_0(\hat{K})$, $\forall m \geq 1$)
4. Let $\{\hat{w}_i\}_{i=1}^{M} \subset \hat{P}$ be a basis of $\hat{P}$ with $\{\hat{w}_i\}_{i=1}^{L} \subset \mathbb{P}_{l-1}(\hat{K})$ being a basis of $\mathbb{P}_{l-1}(\hat{K})$. (for $l = 0$, let $\hat{w}_0 = 1 \subset \mathbb{P}_0(\hat{K})$)

Proof of the Inverse Inequalities of Finite Element Functions

5. Let $\{\varphi_i\}_{i=1}^M \subset \hat{\Sigma}$ be the corresponding dual basis of $\{\hat{w}_i\}_{i=1}^M$, meaning $\varphi_i(\hat{w}_j) = \delta_{ij}$, $i, j = 1, \dots, M$.
6. $\|\dot{w}\|_{m,q,\hat{K}} := |\dot{w}|_{m,q,\hat{K}} + \sum_{i=L+1}^M |\varphi_i(w)|$ defines a norm on $\hat{P}/\mathbb{P}_{l-1}(\hat{K})$, since $w \in \hat{P}$, $\|\dot{w}\|_{m,q,\hat{K}} = 0 \;\; \Rightarrow \;\; \varphi_i(w) = 0$, $i = L+1, \dots, M \;\; \Rightarrow \;\; w \in \mathbb{P}_{l-1}(\hat{K})$. (for $l = 0$, $L = 1$, replace $\mathbb{P}_{l-1}$ by $\mathbb{P}_0$)
7. Since any two norms on a finite dimensional space are equivalent, there exists a constant $C_2 = C(l, m)$ such that

$$|w|_{m,q,\hat{K}} = |\dot{w}|_{m,q,\hat{K}} \leq \|\dot{w}\|_{m,q,\hat{K}} \leq C_2 \|\dot{w}\|_{l,p,\hat{K}} \leq C_1 C_2 |w|_{l,p,\hat{K}}, \;\; \forall w \in \hat{P}.$$

8. By ①, ⑦ (see (7.21), (7.22) and (7.24)), and the quasi-uniformness of the triangulation, we have

$$|v|_{m,q,K} \leq C\, h^{l-m-n(1/p-1/q)} |v|_{l,p,K}, \;\; \forall v \in P_K, \; \forall K \in \mathfrak{T}_h(\Omega).$$

Proof of the Inverse Inequalities of Finite Element Functions

9. Hence, the conclusion of the theorem for $q = \infty$ follows.
10. For $p \leq q < \infty$, the conclusion of the theorem follows as a consequence of ⑧ and the Jensen's inequality
$$\Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|_{l,p,K}^q \Big)^{1/q} \leq \Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|_{l,p,K}^p \Big)^{1/p}.$$
11. If $q < p < \infty$, it follows from the Hölder's inequality that
$$\Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|_{l,p,K}^q \Big)^{1/q} \leq C(\mathfrak{T}_h(\Omega))^{(1/q-1/p)} \Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|_{l,p,K}^p \Big)^{1/p},$$
where $C(\mathfrak{T}_h(\Omega))$ is the $\#\mathfrak{T}_h(\Omega)$.

Proof of the Inverse Inequalities of Finite Element Functions

12. Since the triangulations are quasi-uniform, there exists a constant $C_3 = C(\sigma, \gamma)$ such that $C(\mathfrak{T}_h(\Omega)) \le C_3\, h^{-n}$.

13. Hence, the conclusion of the theorem for $q < p < \infty$ follows from 8 and 11 (see (7.25) and (7.26)).

14. For $q < p = \infty$, the conclusion of the theorem follows as a consequence of 8 (see (7.25)) and

$$\Big(\sum_{K \in \mathfrak{T}_h(\Omega)} |v|_{l,\infty,K}^q \Big)^{1/q} \le C(\mathfrak{T}_h(\Omega))^{1/q} \max_{K \in \mathfrak{T}_h(\Omega)} |v|_{l,\infty,K}. \quad \blacksquare$$

We Restrict Ourselves to the Simplest Ideal Situations

1. The second order linear elliptic problems defined on polygonal domains in $\mathbb{R}^n$.
2. The conditions in the Céa lemma are satisfied.
3. The domain Ω is exactly triangulated into polygonal finite elements.
4. The Dirichlet boundary $\partial\Omega_0$ consists exactly of some $(n-1)$-dimensional faces of the corresponding triangulation.
5. For further simplification, consider only the homogeneous Dirichlet boundary value problems, so that the function space of the corresponding variational problem is $\mathbb{V} = \mathbb{H}_0^1(\Omega)$.
6. Use only class $\mathbb{C}^0$ conforming finite elements, so that the finite element function spaces satisfy $\mathbb{V}_h \subset \mathbb{V}$.

Interpolation Error Estimates on Finite Element Function Spaces

Theorem

Let $\{(K, P_K, \Sigma_K)\}_{K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega)}$ *be a family of regular affine equivalent finite elements with* $(\hat{K},\, \hat{P},\, \hat{\Sigma})$ *being the reference finite element. Assume that there exist nonnegative integers* k, l *such that* $\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{H}^l(\hat{K}), \qquad \mathbb{H}^{k+1}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K}),$ *where* s *is the highest order of the partial derivatives appeared in the set of degrees of freedom* $\hat{\Sigma}$*. Then, there exists a constant* C *independent of* h *such that, for all* $v \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{V}$*,*

$$\|v - \Pi_h v\|_{m,\Omega} \le C\, h^{k+1-m}\, |v|_{k+1,\Omega}, \quad 0 \le m \le \min\{1,\, l\},$$

$$\Big(\sum_{K \in \mathfrak{T}_h(\Omega)} \|v - \Pi_h v\|^2_{m,K}\Big)^{1/2} \le C h^{k+1-m}\, |v|_{k+1,\Omega},\ 2 \le m \le \min\{k+1, l\},$$

where Π_h *is the* $\mathbb{V}_h$ *interpolation operator.*

Proof of Interpolation Error Estimates on Finite Element Function Spaces

1. For $p = q = 2$, $\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{H}^l(\hat{K})$, $\mathbb{H}^{k+1}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K}) \Rightarrow$ the conditions (7.2.11)–(7.2.13) of the theorem on error estimates of affine family finite element interpolations (see Theorem 7.6) hold for $m \leq \min\{k+1, l\}$.

2. Thus, by the theorem on error estimates of regular affine family finite element interpolations (see Theorem 7.7), we have

$$\|v - \Pi_K v\|_{m,K} \leq C\, h_K^{k+1-m}\, |v|_{k+1,K}, \quad 0 \leq m \leq \min\{k+1, l\}.$$

Proof of Interpolation Error Estimates on Finite Element Function Spaces

3. Since, by definition, $(\Pi_h v)|_K = \Pi_K(v|_K)$, $h_K \le h$, $\forall K \in \mathfrak{T}_h(\Omega)$, this leads to

$$\Big(\sum_{K\in\mathfrak{T}_h(\Omega)} \|v-\Pi_h v\|_{m,K}^2 \Big)^{1/2} \le C h^{k+1-m} |v|_{k+1,\Omega},\ 0 \le m \le \min\{k+1, l\},$$

4. For class $\mathbb{C}^0$ finite elements, and $0 \le m \le \min\{1, l\}$, we have

$$\|v - \Pi_h v\|_{m,\Omega} = \Big(\sum_{K\in\mathfrak{T}_h(\Omega)} \|v - \Pi_h v\|_{m,K}^2 \Big)^{1/2}. \qquad \blacksquare$$

Error Estimates for Regular Affine Family Finite Element Solutions

Theorem

Let $\{(K, P_K, \Sigma_K)\}_{K\in\bigcup_{h>0}\mathfrak{T}_h(\Omega)}$ *be a family of regular affine equivalent finite elements with* $(\hat{K}, \hat{P}, \hat{\Sigma})$ *being the reference finite element. Assume that there exists an integer* $k \geq 1$ *such that*

$$\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{H}^1(\hat{K}),$$
$$\mathbb{H}^{k+1}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K}),$$

where s *is the highest order of the partial derivatives appeared in the set of the degrees of freedom* $\widehat{\Sigma}$*. Assume that the solution* u *of the variational problem is in* $\mathbb{H}^{k+1}(\Omega) \cap \mathbb{V}$*. Then, there exists a constant* C *independent of* h *such that*

$$\|u - u_h\|_{1,\Omega} \leq C\, h^k\, |u|_{k+1,\Omega},$$

where $u_h \in \mathbb{V}_h \subset \mathbb{V}$ *is a finite element solution of the variational problem.*

Proof of Error Estimates for Regular Affine Family Finite Element Solutions

1. By the Céa lemma (see Theorem 7.1), we have

$$\|u - u_h\|_{1,\Omega} \leq C \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|_{1,\Omega} \leq C \|u - \Pi_h u\|_{1,\Omega}.$$

2. By the error estimates for regular affine family finite element interpolations (see Theorem 7.9. in particular (7.3.5)), since $u \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{V}$, we have

$$\|u - \Pi_h u\|_{1,\Omega} \leq C\, h^k\, |u|_{k+1,\Omega}.$$

3. Thus, the conclusion of the theorem follows. ∎

Proof of Error Estimates for Regular Affine Family Finite Element Solutions

Remark:

If the solution u of the variational problem has higher regularity, *i.e.* smoother, then, we can obtain finite element solutions with higher order of accuracy by selecting finite elements containing higher order polynomial function spaces.

Application of the General Result to $u \in \mathbb{H}^2(\Omega) \cap \mathbb{V}$ and $n \leq 3$

1. By the Sobolve embedding theorem, $\mathbb{W}^{m+s,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^s(\overline{\Omega})$, $\forall\, s \geq 0$, if $m > n/p$.
2. Here, we have $m = 2$, $n \leq 3$ and $p = 2$, so $2 > 3/2$ implies $\mathbb{H}^2(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^0(\overline{\Omega})$.
3. For class $\mathbb{C}^0$ type (1) Lagrange finite elements, we have $s = 0$, $k = 1$ and $\hat{P} = \mathbb{P}_1(\hat{K})$.
4. Thus, the conditions of Theorem 7.10: $\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{H}^1(\hat{K})$ and $\mathbb{H}^{k+1}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K})$ are satisfied for $k = 1$, $s = 0$.
5. As a consequence, we have $\|u - u_h\|_{1,\Omega} \leq C\, h\, |u|_{2,\Omega}$.
6. In this case, the error estimate for the finite element solution is optimal, meaning that the error estimate of the finite element solution is of the same order as that of the interpolation of the real solution in the finite element function space.

Application of the General Result to the Second Order Problem ($n \leq 3$)

1. For class $\mathbb{C}^0$ affine equivalent finite elements or finite elements which embed into an affine family, for example, the complete or incomplete type (k) n-simplex, type (k) n-rectangle, etc., by taking $l = 1$, we are lead to the error estimates

$$\|v - \Pi_h v\|_{m,\Omega} \leq C\, h^{k+1-m}\, |v|_{k+1,\Omega}, \quad m = 0, 1, \ \forall v \in \mathbb{H}^{k+1}(\Omega) \cap \mathbb{V}.$$

2. In particular, for type (1) Lagrange finite elements, we have

$$\|v - \Pi_h v\|_{m,\Omega} \leq C\, h^{2-m}\, |v|_{2,\Omega}, \qquad m = 0, 1, \ \forall v \in \mathbb{H}^{2}(\Omega) \cap \mathbb{V}.$$

3. In general, the finite element interpolation error in the $\mathbb{L}^2(\Omega)$ norm is an order higher than that in the $\mathbb{H}^1(\Omega)$ norm.

Finite Element Solution u Is Not in $\mathbb{H}^2(\Omega) \cap \mathbb{V}$ in General

1. For general variational problems, the condition $u \in \mathbb{H}^2(\Omega)$ is not always satisfied. For example, the Poisson equation on a $\heartsuit$ shaped region.

2. Under the minimal assumption that the solution $u \in \mathbb{H}^1(\Omega) \cap \mathbb{V}$, we can still prove the convergence of the finite element solutions, even though the order of the approximation accuracy is no longer available.

Convergence of Finite Element Solutions When u Is Only in $\mathbb{H}^1(\Omega) \cap \mathbb{V}$

Theorem

Let $\{(K, P_K, \Sigma_K)\}_{K \in \bigcup_{h>0} \mathfrak{T}_h(\Omega)}$ *be a family of regular class* $\mathbb{C}^0$ *affine equivalent finite elements with the reference finite element* $(\hat{K},\, \hat{P},\, \hat{\Sigma})$ *satisfying:* $\mathbb{P}_1(\hat{K}) \subset \hat{P} \subset \mathbb{H}^1(\hat{K})$, *and there is no partial derivatives of order greater than or equal to* 2 *in* $\hat{\Sigma}$. *Then,*

$$\lim_{h \to 0} \|u - u_h\|_{1,\Omega} = 0.$$

Proof of $\lim_{h\to 0}\|u-u_h\|_{1,\Omega}=0$ When u Is Only in $\mathbb{H}^1(\Omega)\cap\mathbb{V}$

1. Take $k=1$, $m=1$, $q=2$, $p=\infty$, $s=0$ or 1 accordingly. We have (see conditions (7.2.11)–(7.2.13) for Theorem 7.6, 7.7)

$$
\begin{aligned}
&\mathbb{W}^{2,\infty}(\hat{K}) \hookrightarrow \mathbb{C}^s(\hat{K}), \quad s=0,1\\
&\mathbb{W}^{2,\infty}(\hat{K}) \hookrightarrow \mathbb{H}^1(\hat{K}),\\
&\mathbb{P}_k(\hat{K}) \subset \hat{P} \subset \mathbb{H}^1(\hat{K}).
\end{aligned}
$$

2. Thus, by the interpolation error estimates on regular affine equivalent family of finite elements (see Theorem 7.7), for all $v\in\mathbb{W}^{2,\infty}(\Omega)\cap\mathbb{V}$, we have

$$
\|v-\Pi_h v\|_{1,\Omega}=\Big\{\sum_{K\in\mathfrak{T}(\Omega)}\|v-\Pi_K v\|_{1,K}^2\Big\}^{1/2}\le Ch(\mathrm{meas}\,(\Omega))^{1/2}|v|_{2,\infty,\Omega}.
$$

This implies $$\lim_{h\to 0}\|v-\Pi_h v\|_{1,\Omega}=0,\quad \forall v\in\mathbb{W}^{2,\infty}(\Omega)\cap\mathbb{V}.$$

Proof of $\lim_{h\to 0}\|u-u_h\|_{1,\Omega}=0$ When u Is Only in $\mathbb{H}^1(\Omega)\cap\mathbb{V}$

3. Since $u\in\mathbb{H}^1(\Omega)\cap\mathbb{V}$ and $\mathbb{W}^{2,\infty}(\Omega)\cap\mathbb{V}$ is dense in $\mathbb{H}^1(\Omega)\cap\mathbb{V}$, for any given $\varepsilon>0$, there exists a $v_\varepsilon\in\mathbb{W}^{2,\infty}(\Omega)\cap\mathbb{V}$ such that
$$\|u-v_\varepsilon\|_{1,\Omega}<\varepsilon/2.$$
4. For a fixed v_ε, by ②, there exists an $h(\varepsilon)>0$ such that $\|v_\varepsilon-\Pi_h v_\varepsilon\|_{1,\Omega}<\varepsilon/2$, if $0<h<h(\varepsilon)$.
5. Consequently
$$\|u-\Pi_h v_\varepsilon\|_{1,\Omega}\le\|u-v_\varepsilon\|_{1,\Omega}+\|v_\varepsilon-\Pi_h v_\varepsilon\|_{1,\Omega}<\varepsilon,\quad\forall h\in(0,\ h(\varepsilon)).$$
6. Therefore, we conclude $\displaystyle\lim_{h\to 0}\inf_{v_h\in\mathbb{V}_h}\|u-v_h\|_{1,\Omega}=0.$
7. The conclusion of the theorem follows now from the Céa lemma. ■

习题 7：5，6，8.

Thank You!