Numerical Solutions to Partial Differential Equations

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Céa Lemma — an Abstract Error Estimate Theorem

1. Consider the variational problem of the form

$$\begin{cases} \text{Find } \ u \in \mathbb{V} \text{ such that} \\ a(u, v) = f(v), \quad \forall v \in \mathbb{V}. \end{cases}$$

2. Consider the conforming finite element method of the form

$$\begin{cases} \text{Find } \ u_h \in \mathbb{V}_h \subset \mathbb{V} \text{ such that} \\ a(u_h, v_h) = f(v_h), \quad \forall v_h \in \mathbb{V}_h. \end{cases}$$

3. The problem: how to estimate the error $\|u - u_h\|$?
4. The method used for FDM is not an ideal framework for FEM.
5. The standard approach for the error estimations of a finite element solution is to use an abstract error estimate to reduce the problem to a function approximation problem.

Céa Lemma — an Abstract Error Estimate Theorem

Theorem

Let $\mathbb{V}$ be a Hilbert space, $\mathbb{V}_h$ be a linear subspace of $\mathbb{V}$. Let the bilinear form $a(\cdot, \cdot)$ and the linear form $f(\cdot)$ satisfy the conditions of the Lax-Milgram lemma (see Theorem 5.1). *Let $u \in \mathbb{V}$ be the solution to the variational problem, and $u_h \in \mathbb{V}_h$ satisfy the equation*

$$a(u_h, v_h) = f(v_h), \quad \forall v_h \in \mathbb{V}_h.$$

Then, there exist a constant C independent of $\mathbb{V}_h$, such that

$$\|u - u_h\| \leq C \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|,$$

where $\|\cdot\|$ is the norm of $\mathbb{V}$.

Proof of the Céa Lemma

1. Since u and u_h satisfy the equations, and $\mathbb{V}_h \subset \mathbb{V}$, we have
$$a(u-u_h,\, w_h) = a(u,\, w_h) - a(u_h,\, w_h) = f(w_h) - f(w_h) = 0, \quad \forall w_h \in \mathbb{V}_h.$$
2. In particular, taking $w_h = u_h - v_h$ leads to
$$a(u - u_h,\, u_h - v_h) = 0.$$
3. The $\mathbb{V}$-ellipticity $\Rightarrow$ $\alpha\|u - u_h\|^2 \le a(u - u_h,\, u - u_h)$.
4. The boundedness $\Rightarrow a(u - u_h,\, u - v_h) \le M\|u - u_h\|\|u - v_h\|$.
5. Hence, $\alpha\|u - u_h\|^2 \le a(u - u_h,\, u - v_h) \le M\|u - u_h\|\|u - v_h\|$.
6. Take $C = M/\alpha$, we have
$$\|u - u_h\| \le C\|u - v_h\|, \quad \forall v_h \in \mathbb{V}_h.$$
7. The conclusion of the theorem follows. ■

Remarks on the Céa Lemma

1. The Céa lemma reduces the error estimation problem of $\|u - u_h\|$ to the optimal approximation problem of $\inf_{v_h \in \mathbb{V}_h} \|u - v_h\|$.
2. Error of the finite element solution $\|u - u_h\|$ is of the same order as the optimal approximation error $\inf_{v_h \in \mathbb{V}_h} \|u - v_h\|$.
3. Suppose the $\mathbb{V}_h$-interpolation function $\Pi_h u$ of u is well defined in the finite element function space $\mathbb{V}_h$, then,
$$\|u - u_h\| \leq C \inf_{v_h \in \mathbb{V}_h} \|u - v_h\| \leq C \|u - \Pi_h u\|.$$
4. Therefore, the error estimation problem of $\|u - u_h\|$ can be further reduced to the error estimation problem for the $\mathbb{V}_h$-interpolation error $\|u - \Pi_h u\|$.

For Symmetric $a(\cdot,\cdot)$, u_h Is a Orthogonal Projection of u on $\mathbb{V}_h$

1. If the $\mathbb{V}$-elliptic bounded bilinear form $a(\cdot,\ \cdot)$ is symmetric, then, $a(\cdot,\ \cdot)$ defines an inner product on $\mathbb{V}$, with the induced norm equivalent to the $\mathbb{V}$-norm.
2. Denote $\mathbf{P}_h : \mathbb{V} \to \mathbb{V}_h$ as the orthogonal projection operator induced by the inner product $a(\cdot,\ \cdot)$. Then,
$$a(u - \mathbf{P}_h u, v_h) = 0, \quad \forall v_h \in \mathbb{V}_h.$$
3. Therefore, the finite element solution $u_h = \mathbf{P}_h u$, *i.e.* it is the orthogonal projection of u on $\mathbb{V}_h$ with respect to the inner product $a(\cdot,\ \cdot)$.

Céa Lemma for Symmetric $a(\cdot, \cdot)$

Corollary

Under the conditions of the Céa Lemma, if the bilinear form $a(\cdot,\ \cdot)$ is in addition symmetric, then, the solution u_h is the orthogonal projection, which is induced by the inner product $a(\cdot,\ \cdot)$, of the solution u on the subspace $\mathbb{V}_h$, meaning $u_h = \mathbf{P}_h u$.

Furthermore, we have

$$a(u - u_h, u - u_h) = \inf_{v_h \in \mathbb{V}_h} a(u - v_h, u - v_h).$$

The proof follows the same lines as the proof of the Céa lemma. The only difference here is that $\alpha = M = 1$.

Céa Lemma in the Form of Orthogonal Projection Error Estimate

Denote $\tilde{P}_h : \mathbb{V} \to \mathbb{V}_h$ as the orthogonal projection operator induced by the inner product $(\cdot,\cdot)_{\mathbb{V}}$ of $\mathbb{V}$, then,

$$\|u - \tilde{P}_h u\| = \|(I - \tilde{P}_h)u\| = \inf_{v_h \in \mathbb{V}_h} \|u - v_h\|.$$

Therefore, as a corollary of the Céa lemma, we have

Corollary

Let $\mathbb{V}$ be a Hilbert space, and $\mathbb{V}_h$ be a linear subspace of $\mathbb{V}$. Let $a(\cdot,\ \cdot)$ be a symmetric bilinear form on $\mathbb{V}$ satisfying the conditions of the Lax-Milgram lemma. Let P_h and $\tilde{P}_h$ be the orthogonal projection operators from $\mathbb{V}$ to $\mathbb{V}_h$ induced by the inner products $a(\cdot,\ \cdot)$ and $(\cdot,\cdot)_{\mathbb{V}}$ respectively. Then, we have

$$\|I - \tilde{P}_h\| \leq \|I - P_h\| \leq \frac{M}{\alpha}\|I - \tilde{P}_h\|.$$

1-D Example on Linear Interpolation Error Estimation for $\mathbb{H}^2$ Functions

1. $\hat{\Omega} = (0,\ 1)$, $\Omega = (b,\ b+h)$, $h > 0$.
2. $F : \hat{x} \in [0,\ 1] \to [b,\ b+h]$, $F(\hat{x}) = h\hat{x} + b$: an invertible affine mapping from $\overline{\hat{\Omega}}$ to $\overline{\Omega}$.
3. $\hat{\Pi} : \mathbb{C}([0,\ 1]) \to \mathbb{P}_1([0,\ 1])$: the interpolation operator with $\hat{\Pi}\hat{v}(0) = \hat{v}(0)$, $\hat{\Pi}\hat{v}(1) = \hat{v}(1)$.
4. $\Pi : \mathbb{C}([b,\ b+h]) \to \mathbb{P}_1([b,\ b+h])$: the interpolation operator with $\Pi v(b) = v(b)$, $\Pi v(b+h) = v(b+h)$.

1-D Example on Linear Interpolation Error Estimation for $\mathbb{H}^2$ Functions

5. Let $u \in \mathbb{H}^2(\Omega)$, denote $\hat{u}(\hat{x}) = u \circ F(\hat{x}) = u(h\hat{x} + b)$, then, it can be shown $\hat{u} \in \mathbb{H}^2(\hat{\Omega})$, thus, $\hat{u} \in \mathbb{C}([0,\ 1])$.

6. $\hat{\Pi}$ **is** $\mathbb{P}_1([0,1])$ **invariant**: $\hat{\Pi}\hat{w} = \hat{w}$, $\forall \hat{w} \in \mathbb{P}_1([0,1])$, thus,

$$\|(I - \hat{\Pi})\hat{u}\|_{0,\hat{\Omega}} = \|(I - \hat{\Pi})(\hat{u} + \hat{w})\|_{0,\hat{\Omega}} \le \|I - \hat{\Pi}\|\, \|\hat{u} + \hat{w}\|_{2,\hat{\Omega}},$$

where $\|I - \hat{\Pi}\|$ is the norm of $I - \hat{\Pi} : \mathbb{H}^2(\hat{\Omega}) \to \mathbb{L}^2(\hat{\Omega})$.

★ This shows that $I - \hat{\Pi} \in \mathfrak{L}(\mathbb{H}^2(0,1)/\mathbb{P}_1([0,1]); \mathbb{L}^2(0,1))$, and

$$\|\hat{u} - \hat{\Pi}\hat{u}\|_{0,\hat{\Omega}} \le \|I - \hat{\Pi}\| \inf_{\hat{w} \in \mathbb{P}_1(\hat{\Omega})} \|\hat{u} + \hat{w}\|_{2,\hat{\Omega}}, \tag{1}$$

where $\inf_{\hat{w} \in \mathbb{P}_1(\hat{\Omega})} \|\hat{u} + \hat{w}\|_{2,\hat{\Omega}}$ is the norm of $\hat{u}$ in the quotient space $\mathbb{H}^2(0,1)/\mathbb{P}_1([0,1])$.

1-D Example on Linear Interpolation Error Estimation for $\mathbb{H}^2$ Functions

★ It can be shown that, $\exists$ const. $C(\hat{\Omega}) > 0$ s.t.

$$(2)\quad |\hat{u}|_{2,\hat{\Omega}} \leq \inf_{\hat{w}\in\mathbb{P}_1(\hat{\Omega})} \|\hat{u}+\hat{w}\|_{2,\hat{\Omega}} \leq C(\hat{\Omega})|\hat{u}|_{2,\hat{\Omega}}.$$

★ It follows from the chain rule that $\hat{u}''(\hat{x}) = h^2 u''(x)$.

★ By a change of the integral variable, and $dx = hd\hat{x}$, we obtain

$$(3)\quad \hat{u} \in \mathbb{H}^2(\hat{\Omega}), \text{ and } |\hat{u}|^2_{2,\hat{\Omega}} = h^3 |u|^2_{2,\Omega};$$

$$(4)\quad \|u - \Pi u\|^2_{0,\Omega} = h\|\hat{u} - \hat{\Pi}\hat{u}\|^2_{0,\hat{\Omega}}.$$

1-D Example on Linear Interpolation Error Estimation for $\mathbb{H}^2$ Functions

- The conclusion (1) says that the $\mathbb{L}^2$ norm of the error of a $\mathbb{P}_1$ invariant interpolation can be bounded by the quotient norm of the function in $\mathbb{H}^2(0,1)/\mathbb{P}_1([0,1])$.

- The conclusion (2) says that the semi norm $|\cdot|_{2,(0,1)}$ is an equivalent norm of the quotient space $\mathbb{H}^2(0,1)/\mathbb{P}_1([0,1])$.

- The conclusions (3) and (4) present the relations between the semi-norms of Sobolev spaces defined on affine-equivalent open sets.

1-D Example on Linear Interpolation Error Estimation for $\mathbb{H}^2$ Functions

★ The combination of (4) and (1) yields

$$\|u - \Pi u\|_{0,\Omega} \leq h^{\frac{1}{2}} \|I - \hat{\Pi}\| \inf_{\hat{w} \in \mathbb{P}_1(\hat{\Omega})} \|\hat{u} + \hat{w}\|_{2,\hat{\Omega}}$$

★ This together with (2) and (3) lead to the expected interpolation error estimate:

$$\|u - \Pi u\|_{0,\Omega} \leq \|I - \hat{\Pi}\| C(\hat{\Omega}) |u|_{2,\Omega} h^2, \quad \forall u \in \mathbb{H}^2(\Omega).$$

A Framework for Interpolation Error Estimation of Affine Equivalent FEs

1. The polynomial quotient spaces of a Sobolev space and their equivalent quotient norms ((2) in the example);
2. The relations between the semi-norms of Sobolev spaces defined on affine-equivalent open sets ((3), (4) in the exmample);
3. The abstract error estimates for the polynomial invariant operators ((1) in the example);
4. To estimate the constants appeared in the relations of the Sobolev semi-norms by means of the geometric parameters of the corresponding affine-equivalent open sets.

A Framework for Interpolation Error Estimation of Affine Equivalent FEs

- the change of integral variable will introduce the Jacobi determinant $\det\left(\frac{\partial F(\hat{x})}{\partial \hat{x}}\right)$;
- in high dimensions, the Jacobi determinant represents the ratio of the volumes $|\Omega|/|\hat{\Omega}|$;
- the chain rule for the mth derivative will produce h^m.
- h actually represents the ratio of the lengths in the directions of corresponding directional derivatives of the regions $\Omega = F(\hat{\Omega})$ and $\hat{\Omega}$.

The related technique is often referred to as the scaling technique.

Polynomial Quotient Spaces

1. The quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$, in which a function $\dot{v}$ is the equivalent class of $v \in \mathbb{W}^{k+1,p}(\Omega)$ in the sense that

$$\dot{v} = \{w \in \mathbb{W}^{k+1,p}(\Omega) : (w - v) \in \mathbb{P}_k(\Omega)\}.$$

2. The quotient norm of a function $\dot{v}$ is defined by

$$\dot{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega) \to \|\dot{v}\|_{k+1,p,\Omega} := \inf_{w \in \mathbb{P}_k(\Omega)} \|v+w\|_{k+1,p,\Omega}.$$

Polynomial Quotient Spaces

3. The quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$ is a Banach space.

4. $\dot{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega) \to |\dot{v}|_{k+1,p,\Omega} = |v|_{k+1,p,\Omega}$ is a semi-norm of the quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$, and obviously $|\dot{v}|_{k+1,p,\Omega} \leq \|\dot{v}\|_{k+1,p,\Omega}$.

5. In fact, $|\dot{v}|_{k+1,p,\Omega} = |v|_{k+1,p,\Omega}$ is an equivalent norm of the quotient space $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$.

The Semi-norm $|v|_{k+1,p,\Omega}$ Is an equivalent Norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

Theorem

There exists a constant $C(\Omega)$ such that

$$\|\dot{v}\|_{k+1,p,\Omega} \le C(\Omega)|\dot{v}|_{k+1,p,\Omega}, \qquad \forall \dot{v} \in \mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega).$$

Proof:

1. Let $\{p_i\}_{i=1}^N$ be a basis of $\mathbb{P}_k(\Omega)$, and let f_i, $i = 1, \dots, N$, be the corresponding dual basis, meaning $f_i(p_j) = \delta_{ij}$.
2. For any $w \in \mathbb{P}_k(\Omega)$, $f_i(w) = 0$, $i = 1, \dots, N \quad \Leftrightarrow \quad w = 0$.
3. Extend f_i, $i = 1, \dots, N$, to a set of bounded linear functionals defined on $\mathbb{W}^{k+1,p}(\Omega)$.

The Semi-norm $|v|_{k+1,p,\Omega}$ Is an equivalent Norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

4. We claim that there exists a constant $C(\Omega)$ such that $\|v\|_{k+1,p,\Omega} \leq C(\Omega)(|v|_{k+1,p,\Omega} + \sum_{i=1}^{N} |f_i(v)|)$, $\forall v \in \mathbb{W}^{k+1,p}(\Omega)$.

5. For $v \in \mathbb{W}^{k+1,p}(\Omega)$, define $\tilde{w} = -\sum_{j=1}^{N} f_j(v) p_j$, then, $f_i(v + \tilde{w}) = 0$, $i = 1, \ldots, N$, consequently, $\inf_{w \in \mathbb{P}_k(\Omega)} \|v + w\|_{k+1,p,\Omega} \leq \|v + \tilde{w}\|_{k+1,p,\Omega} \leq C(\Omega)|v|_{k+1,p,\Omega}$.

What remains to show is ④. Suppose ④ doesn't hold.

6. Then, there exists a sequence $\{v_j\}_{j=1}^{\infty}$ in $\mathbb{W}^{k+1,p}(\Omega)$ such that

$$\|v_j\|_{k+1,p,\Omega} = 1,\ \forall j \geq 1 \text{ and } \lim_{j\to\infty}(|v_j|_{k+1,p,\Omega} + \sum_{i=1}^{N} |f_i(v_j)|) = 0.$$

7. $\mathbb{W}^{k+1,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,p}(\Omega)$, $1 \leq p < \infty$; $\mathbb{W}^{k+1,\infty}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^k(\bar{\Omega})$.

The Semi-norm $|v|_{k+1,p,\Omega}$ Is an equivalent Norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

8. So, there exist a subsequence of $\{v_j\}_{j=1}^{\infty}$, denoted again as $\{v_j\}_{j=1}^{\infty}$, and a function $v \in \mathbb{W}^{k,p}(\Omega)$, such that

$$\lim_{j\to\infty} \|v_j - v\|_{k,p,\Omega} = 0.$$

9. ⑥ and ⑧ imply $\{v_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathbb{W}^{k+1,p}(\Omega)$.

10. Therefore, v in ⑧ is actually a function in $\mathbb{W}^{k+1,p}(\Omega)$.

11. Thus, it follows from ⑥ that

$$|\partial^\alpha v|_{0,p,\Omega} = \lim_{j\to\infty} |\partial^\alpha v_j|_{0,p,\Omega} = 0, \quad \forall \alpha, \ \ |\alpha| = k+1,$$

The Semi-norm $|v|_{k+1,p,\Omega}$ Is an equivalent Norm of $\mathbb{W}^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega)$

12. By Theorem 5.2, (11) implies $v \in \mathbb{P}_k(\Omega)$.

13. On the other hand, it follows from (6) that
$$f_i(v) = \lim_{j\to\infty} f_i(v_j) = 0, \quad i = 1, \dots, N,$$

14. Therefore, by (2), we have $v = 0$.

15. On the other hand, since v_j converges to v in $\mathbb{W}^{k+1,p}(\Omega)$, by (6), we have $\|v\|_{k+1,p,\Omega} = \lim_{j\to\infty} \|v_j\|_{k+1,p,\Omega} = 1$.

16. The contradiction of (14) and (15) completes the proof. ■

Relations of Semi-norms on Open Sets Related by $F(\hat{x}) = h\hat{x} + b \in \mathbb{R}^n$

1. Let $F : \hat{x} \in \mathbb{R}^n \to F(\hat{x}) = h\hat{x} + b \in \mathbb{R}^n$, and $\Omega = F(\hat{\Omega})$,
   $\Rightarrow \quad \mathrm{diam}(\Omega)/\mathrm{diam}(\hat{\Omega}) = h$ and $\|\frac{\partial F(\hat{x})}{\partial \hat{x}}\| = h$.

2. Then, $\partial^\alpha v(x) = h^{-|\alpha|}\partial^\alpha \hat{v}(\hat{x})$, and $dx = |\det(B)|\, d\hat{x} = h^n\, d\hat{x}$, (where $B = \frac{\partial F(\hat{x})}{\partial \hat{x}} = h\, I_{n\times n}$).

3. Therefore, by a change of integral variable, we have
   $$|v|_{m,p,\Omega} = \|B^{-1}\|^m |\det(B)|^{1/p} |\hat{v}|_{m,p,\hat{\Omega}} = h^{-m+n/p} |\hat{v}|_{m,p,\hat{\Omega}}.$$

4. $|v|_{m,p,\Omega}/|\hat{v}|_{m,p,\hat{\Omega}} \propto \|B^{-1}\|^m |\det(B)|^{1/p} = h^{-m+n/p}$.

Affine Equivalent Open Sets Related by $F(\hat{x}) = B\hat{x} + b \in \mathbb{R}^n$

Let $\Omega = F(\hat{\Omega})$ be an affine equivalent open set in $\mathbb{R}^n$ with

$$F : \hat{x} \in \mathbb{R}^n \to F(\hat{x}) = B\hat{x} + b \in \mathbb{R}^n,$$

For $v \in \mathbb{W}^{m,p}(\Omega)$ and $\hat{v}(\hat{x}) = v(F(\hat{x}))$, the Sobolev semi-norms $|v|_{m,p,\Omega}$ and $|\hat{v}|_{m,p,\hat{\Omega}}$ have a similar relation for general B, *i.e.*

$$|v|_{m,p,\Omega} / |\hat{v}|_{m,p,\hat{\Omega}} \propto \|B^{-1}\|^m |\det(B)|^{1/p} \propto h^{-m+n/p},$$

provided $F(\hat{x})$ satisfies certain regularity conditions.

Relations of Semi-norms on Open Sets Related by $F(\hat{x}) = B\hat{x} + b$

Theorem

Let Ω and $\hat{\Omega}$ be two affine equivalent open sets in $\mathbb{R}^n$. Let $v \in \mathbb{W}^{m,p}(\Omega)$ for some $p \in [1, \infty]$ and nonnegative integer m. Then, $\hat{v} = v \circ F \in \mathbb{W}^{m,p}(\hat{\Omega})$, and there exists a constant $C = C(m, n)$ such that

$$|\hat{v}|_{m,p,\hat{\Omega}} \leq C\|B\|^m \, |\det(B)|^{-1/p} \, |v|_{m,p,\Omega},$$

where B is the matrix in the affine mapping F, $\|\cdot\|$ represents the operator norms induced from the Euclidian norm of $\mathbb{R}^n$. Similarly, we also have

$$|v|_{m,p,\Omega} \leq C\|B^{-1}\|^m \, |\det(B)|^{1/p} \, |\hat{v}|_{m,p,\hat{\Omega}}.$$

Proof of $|\hat{v}|_{m,p,\hat{\Omega}} \le C(n,m)\|B\|^m\,|\det(B)|^{-1/p}\,|v|_{m,p,\Omega}$

1. Let $\xi_i = (\xi_{i1}, \ldots, \xi_{in})^T \in \mathbb{R}^n$, $i = 1, \cdots, m$, be unit vectors, $D = (\partial_1, \ldots, \partial_n)$, $D^m\hat{v}(\hat{x})(\xi_1, \ldots, \xi_m) = \left(\prod_{i=1}^m D\cdot\xi_i\right)\hat{v}(\hat{x})$.

2. Assume $v \in \mathbb{C}^m(\bar{\Omega})$, therefore, $\hat{v} \in \mathbb{C}^m(\bar{\hat{\Omega}})$ also. We have

$$|\partial^\alpha \hat{v}(\hat{x})| \le \|D^m\hat{v}(\hat{x})\| := \sup_{\substack{\|\xi_i\|=1 \\ 1\le i\le m}} |D^m\hat{v}(\hat{x})(\xi_1, \ldots, \xi_m)|, \quad \forall|\alpha| = m.$$

3. Let $C_1(m,n)$ be the cardinal number of α, then

$$|\hat{v}|_{m,p,\hat{\Omega}} = \left(\int_{\hat{\Omega}} \sum_{|\alpha|=m} |\partial^\alpha \hat{v}(\hat{x})|^p d\hat{x}\right)^{1/p} \le C_1(m,n)\left(\int_{\hat{\Omega}} \|D^m\hat{v}(\hat{x})\|^p d\hat{x}\right)^{1/p}.$$

Proof of $|\hat{v}|_{m,p,\hat{\Omega}} \leq C(n,m)\|B\|^m \,|\det(B)|^{-1/p}\, |v|_{m,p,\Omega}$

4. On the other hand, by the chain rule of differentiations for composition of functions,

$$(D\cdot\xi)\hat{v}(\hat{x}) = D(v\circ F(\hat{x}))\,\xi = Dv(x)\frac{\partial F(\hat{x})}{\partial \hat{x}}\,\xi = (D\cdot B\xi)v(x).$$

5. Therefore, $\left(\prod_{i=1}^m D\cdot\xi_i\right)\hat{v}(\hat{x}) = \left(\prod_{i=1}^m D\cdot B\xi_i\right)v(x)$, *i.e.*

$$D^m\hat{v}(\hat{x})(\xi_1,\ldots,\xi_m) = D^m v(x)(B\xi_1,\ldots,B\xi_m).$$

6. Consequently, $\|D^m\hat{v}(\hat{x})\| \leq \|B\|^m\,\|D^m v(x)\|$.

7. Thus, by a change of integral variable, we obtain

$$\int_{\hat{\Omega}} \|D^m\hat{v}(\hat{x})\|^p d\hat{x} \leq \|B\|^{mp}\,\left|\det\left(B^{-1}\right)\right| \int_{\Omega} \|D^m v(x)\|^p dx.$$

Proof of $|\hat{v}|_{m,p,\hat{\Omega}} \le C(n,m)\|B\|^m \,|\det(B)|^{-1/p}\, |v|_{m,p,\Omega}$

8. For any given $\eta_i \in \mathbb{R}^n$ with $\|\eta_i\| = 1$, $1 \le i \le m$, we have

$$D^m v(x)(\eta_1, \ldots, \eta_m) = \left[\prod_{i=1}^{m}\sum_{j=1}^{n}\eta_{ij}\partial_j\right] v(x) = \sum_{j_1,\cdots,j_m=1}^{n}\left[\prod_{i=1}^{m}\eta_{ij_i}\partial_{j_i}\right] v(x).$$

9. Since, $|\eta_{ij}| \le 1$, $1 \le i \le m$, $1 \le j \le n$, we are lead to

$$\|D^m v(x)\| \le n^m \max_{|\alpha|=m} |\partial^\alpha v(x)| \le n^m \Big(\sum_{|\alpha|=m} |\partial^\alpha v(x)|^p\Big)^{1/p}.$$

Proof of $|\hat{v}|_{m,p,\hat{\Omega}} \leq C(n,m)\|B\|^m \,|\det(B)|^{-1/p}\, |v|_{m,p,\Omega}$

10. By ③, ⑦ and ⑨, the inequality hold for $v \in \mathbb{C}^m(\bar{\Omega})$.

11. For $1 \leq p < \infty$, $\mathbb{C}^m(\bar{\Omega})$ is dense in $\mathbb{W}^{m,p}(\Omega)$, so the inequality also holds for all $v \in \mathbb{W}^{m,p}(\Omega)$.

12. If $p = \infty$, since the inequality holds uniformly for $1 \leq q < \infty$, and for the bounded domain Ω

$$\|w\|_{0,\infty,\Omega} = \lim_{q\to\infty} \|w\|_{0,q,\Omega}, \qquad \forall w \in \mathbb{L}^\infty(\Omega),$$

therefore, the inequality holds also for $v \in \mathbb{W}^{m,\infty}(\Omega)$. ■

Bound $\|B\|$ and $\|B^{-1}\|$ by the Interior and Exterior Diameters

1. Denote the exterior and interior diameters of a region Ω as

$$\begin{cases} h_\Omega := \text{diam}\,(\Omega), \\ \rho_\Omega := \sup\{\text{diam}\,(S) : S \subset \Omega \ \text{ is a } n\text{-dimensional ball}\}. \end{cases}$$

Theorem

Let Ω and $\hat{\Omega}$ be two affine-equivalent open sets in $\mathbb{R}^n$, let $F(\hat{x}) = B\hat{x} + b$ be the invertible affine mapping, and $\Omega = F(\hat{\Omega})$. Then,

$$\|B\| \le \frac{h}{\hat{\rho}}, \quad \text{and} \quad \|B^{-1}\| \le \frac{\hat{h}}{\rho},$$

where $h = h_\Omega$, $\hat{h} = h_{\hat{\Omega}}$, $\rho = \rho_\Omega$, $\hat{\rho} = \rho_{\hat{\Omega}}$.

Proof of $\|B\| \leq \frac{h}{\hat{\rho}}$ and the Geometric Meaning of $\det(B)$

1. By the definition of $\|B\|$, we have
$$\|B\| = \frac{1}{\hat{\rho}} \sup_{\|\xi\|=\hat{\rho}} \|B\xi\|.$$
2. Let the vectors $\hat{x}$, $\hat{y} \in \overline{\hat{\Omega}}$ be such that $\|\hat{y} - \hat{x}\| = \hat{\rho}$, then, we have $x = F(\hat{x}) \in \bar{\Omega}$, $y = F(\hat{y}) \in \bar{\Omega}$.
3. Therefore, $\|B(\hat{y} - \hat{x})\| = \|F(\hat{y}) - F(\hat{x})\| \leq h \;\Rightarrow \|B\| \leq \frac{h}{\hat{\rho}}$. ■

The determinant $\det(B)$ also has an obvious geometric meaning:

$$|\det(B)| = \frac{\mathrm{meas}(\Omega)}{\mathrm{meas}(\hat{\Omega})} \quad \text{and} \quad |\det(B^{-1})| = \frac{\mathrm{meas}(\hat{\Omega})}{\mathrm{meas}(\Omega)}.$$

习题 7：1，3，4

Thank You!