Numerical Solutions to Partial Differential Equations

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General Abstract Definition of a Finite Element

Definition

A triple (K, P_K, Σ_K) is called a finite element, if

1. $K \subset \mathbb{R}^n$, called an element, is a closed set with non-empty interior and a Lipschitz continuous boundary;
2. $P_K : K \to \mathbb{R}$ is a finite dimensional function space consisting of sufficiently smooth functions defined on the element K;
3. Σ_K is a set of linearly independent linear functionals $\{\varphi_i\}_{i=1}^N$ defined on $\mathbb{C}^\infty(K)$, which are called the degrees of freedom of the finite element and form a dual basis corresponding to a "normalized" basis of P_K, meaning that there exists a unique basis $\{p_i\}_{i=1}^N$ of P_K such that $\varphi_i(p_j) = \delta_{ij}$.

Type (k) n-Simplex — The Simplest Class of Lagrange Finite Elements

1. $K^n = \{\mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathbf{a}_i : 0 \le \lambda_i \le 1, 1 \le i \le n+1, \sum_{i=1}^{n+1} \lambda_i = 1\}$ is the convex hull of vertices $\mathbf{a}_j = (a_{ij})_{i=1}^n$, $j = 1, \ldots, n+1$, with
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n+1} \\ a_{21} & a_{22} & \cdots & a_{2,n+1} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,n+1} \\ 1 & 1 & \cdots & 1 \end{pmatrix} = (\tilde{\mathbf{a}}_1, \cdots, \tilde{\mathbf{a}}_{n+1}) \quad \text{non-singular.}$$
2. Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})^T$, $\tilde{\mathbf{x}} = (x_1, x_2, \ldots, x_n, 1)^T$, if $A\boldsymbol{\lambda} = \tilde{\mathbf{x}}$, then $\boldsymbol{\lambda}(\mathbf{x}) = A^{-1}\tilde{\mathbf{x}}$ is called barycentric coordinates.
3. By $\boldsymbol{\lambda}(\mathbf{a}_j) = A^{-1}\tilde{\mathbf{a}}_j$, and $\tilde{\mathbf{a}}_j$ is the jth column of A, we have $\lambda_i(\mathbf{a}_j) = \delta_{ij}$, $1 \le i, j \le n+1$.

Type (k) n-Simplex — $P_K = \mathbb{P}_k(K)$, $\Sigma_K = K_k^n$, the Principal Lattice

1. $P_K = \mathbb{P}_k(K)$: polynomials of degree no greater than k for the n variables $x_1, x_2, \ldots, x_n$ defined on K. $\dim \mathbb{P}_k(K) = C_{n+k}^n$.

2. For $k = 1$, $\dim \mathbb{P}_1(K) = n + 1$. Since $\lambda_i(\mathbf{x}) \in \mathbb{P}_1(K)$ and $\lambda_i(\mathbf{a}_j) = \delta_{ij}$, if we take $\Sigma_K = \{p(\mathbf{a}_i), 1 \leq i \leq n+1\}$, then, the barycentric coordinates $\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}), \ldots, \lambda_{n+1}(\mathbf{x})$ form the normalized dual basis of $\mathbb{P}_1(K)$ with respect to Σ_K.

3. In general, for $k \geq 1$, the principal lattice

$$K_k^n = \left\{ \mathbf{x} \in \sum_{i=1}^{n+1} \lambda_i \mathbf{a}_i : \sum_{i=1}^{n+1} \lambda_i = 1,\ \lambda_i \in \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\right\},\ 1 \leq i \leq n+1 \right\},$$

form a dual basis of $\mathbb{P}_k(K^n)$.

$\Sigma_K = K_k^n$, the Principal Lattice, Form a Dual Basis of $P_K = \mathbb{P}_k(K^n)$

Theorem

For $k = 0$, denote $K_0^n = \left\{ \frac{1}{n+1} \sum_{i=1}^{n+1} \mathbf{a}_i \right\}$; for $k \geq 1$, denote

$$K_k^n = \left\{ \mathbf{x} \in \sum_{i=1}^{n+1} \lambda_i \mathbf{a}_i : \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \in \left\{ 0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1 \right\}, 1 \leq i \leq n+1 \right\},$$

and call them the kth order principal lattice of the n-simplex $K^n \subset \mathbb{R}^n$. Then, the degrees of freedom given by $\Sigma_k^n = \{p(x) : x \in K_k^n\}$ form a dual basis of $\mathbb{P}_k(K^n)$, and are called the kth order principal degrees of freedom of the n-simplex K^n.

Proof of the Principal Lattice Form a Dual Basis of $P_K = \mathbb{P}_k(K^n)$

The key points to the proof:

- There are exactly $\dim \mathbb{P}_k(K^n) = C_{n+k}^n$ points in K_k^n.
- If $p \in \mathbb{P}_k(K^n)$ satisfies $p(\mathbf{x}) = 0$ on K_k^n, then, $p(\mathbf{x}) \equiv 0$.

Proof:

1. $\alpha_i = k\lambda_i$, $i = 1, 2, \ldots, n$ is 1–1 to the multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i \leq k$. $\sharp K_k^n = \dim \mathbb{P}_k(K^n)$.
2. For $n = 1$, the conclusion of the theorem obviously holds for all $k \geq 0$. We will prove by the principle of induction.
3. Let $n \geq 2$, assume that, for all space dimensions less than n, the conclusion of the theorem holds for all $k \geq 0$.

Proof of the Principal Lattice Form a Dual Basis of $P_K = \mathbb{P}_k(K^n)$

4. Since $\tilde{\mathbf{x}} = A\boldsymbol{\lambda}$, $p \in \mathbb{P}_k(K^n)$ can be written as $p(\mathbf{x}) = \sum_{|\alpha| \leq k} a_\alpha \lambda_1^{\alpha_1}(\mathbf{x}) \cdots \lambda_{n+1}^{\alpha_{n+1}}(\mathbf{x})$, and in particular, written as

$$p(x) = \sum_{i=0}^{k} \left[p_{k-i}(\lambda_1(x), \ldots, \lambda_n(x)) \prod_{j=1}^{i} \left(\lambda_{n+1}(x) - \frac{j-1}{k} \right) \right],$$

where $p_{k-i}(\lambda_1, \ldots, \lambda_n)$ is a polynomial of $\lambda_1, \ldots, \lambda_n$ of degree no greater than $k - i$.

Proof of the Principal Lattice Form a Dual Basis of $P_K = \mathbb{P}_k(K^n)$

5. Let $\hat{\lambda}_j = \frac{k}{k-i}\lambda_j$, $\hat{\mathbf{a}}_j = \frac{k-i}{k}\mathbf{a}_j + \frac{i}{k}\mathbf{a}_{n+1}$, $j = 1, 2, \ldots, n$, then

$$\tilde{K}^{n-1}_{k-i} = \left\{ \mathbf{x} \in \sum_{j=1}^{n} \lambda_j \mathbf{a}_j + \frac{i}{k}\mathbf{a}_{n+1} : \sum_{j=1}^{n} \lambda_j = 1 - \frac{i}{k}, \lambda_j \in \left\{ 0, \frac{1}{k}, \ldots, \frac{k-i}{k} \right\}, 1 \le j \le n \right\},$$

$$\Leftrightarrow \quad \tilde{K}^{n-1}_{k-i} = \left\{ \mathbf{x} \in \sum_{j=1}^{n} \hat{\lambda}_j \hat{\mathbf{a}}_j : \sum_{j=1}^{n} \hat{\lambda}_j = 1, \hat{\lambda}_j \in \left\{ 0, \frac{1}{k-i}, \ldots, 1 \right\}, 1 \le j \le n \right\}.$$

6. $\tilde{K}^{n-1}_{k-i}$ is the $(k-i)$th order principal lattice of the $(n-1)$-simplex $K^{n-1}_{i,k} = \{\mathbf{x} \in K^n : \lambda_{n+1}(\mathbf{x}) = \frac{i}{k}\}$.

Proof of $\Sigma_K = K_k^n$ Form a Dual Basis of $P_K = \mathbb{P}_k(K^n)$ — continue

7. $p(x) = 0$ on $K_k^n \Rightarrow p(x) = 0$ on $\tilde{K}_{k-i}^{n-1}$.

8. $p(x) = p_k(\lambda_1(x), \ldots, \lambda_n(x)) \in \mathbb{P}_k(K_{0,k}^{n-1})$, $p(x) = 0$ on $\tilde{K}_k^{n-1}$, by the induction assumption $\Rightarrow p_k(\lambda_1(x), \ldots, \lambda_n(x)) \equiv 0 \Rightarrow$

9. $p(x) = \lambda_{n+1}(x) p_{k-1}(\lambda_1(x), \ldots, \lambda_n(x)) \in \mathbb{P}_k(K_{1,k}^{n-1})$, thus, $p_{k-1} = 0$ on $\tilde{K}_{k-1}^{n-1} \Rightarrow p_{k-1}(\lambda_1(x), \ldots, \lambda_n(x)) \equiv 0$. Similarly,

10. $p_{k-i}(\lambda_1(x), \ldots, \lambda_n(x)) \equiv 0$, $2 \le i \le k$, $\Rightarrow p(x) \equiv 0$. ■

Type (k) n-Simplex Finite Elements

1. A finite element (K, P_K, Σ_K) is called a type (k) n-simplex, if K is a n-simplex, $P_K = \mathbb{P}_k(K)$, and Σ_K is the kth order principal degrees of freedom Σ_k^n of K.
2. Type (k) n-simplex finite elements are an affine family.
3. The normalized dual basis of $\mathbb{P}_k(K)$ corresponding to Σ_k^n of the n-simplex K can be easily expressed in barycentric coordinates.

Type (k) n-Simplex Finite Elements

4. For example, for the type (2) n-simplex, the normalized dual basis of $\mathbb{P}_2(K)$ corresponding to Σ_2^n is given by

$$\lambda_i(x)(2\lambda_i(x)-1),\ i=1,2,\ldots,n+1;\ \ 4\lambda_i(x)\lambda_j(x),\ 1\leq i<j\leq n+1.$$

5. In fact, denoting $\mathbf{a}_{ij}=(\mathbf{a}_i+\mathbf{a}_j)/2$, we have

$$p(x)=\sum_{i=1}^{n+1}\lambda_i(x)(2\lambda_i(x)-1)p(\mathbf{a}_i)+\sum_{1\leq i<j\leq n+1}4\lambda_i(x)\lambda_j(x)p(\mathbf{a}_{ij}),\ \ \forall p\in\mathbb{P}_2(K).$$

Type (k) n-Rectangle — Another Class of Lagrange Finite Elements

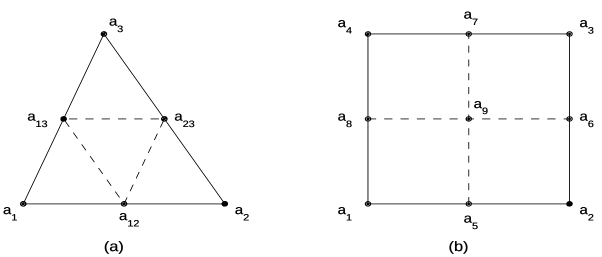
1. $K^n = [X_{11},\, X_{12}] \times [X_{21},\, X_{22}] \times \cdots \times [X_{n1},\, X_{n2}]$ is a n-rectangle.
2. $P_K \triangleq \mathbb{Q}_k(K^n) = \left\{ p\,(\mathbf{x}) : p\,(\mathbf{x}) = \sum_{\substack{\alpha_i \le k \\ 1 \le i \le n}} p_{\alpha_1 \cdots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \right\}$.
3. $\dim \mathbb{Q}_k(K^n) = (k+1)^n$.
4. Let $h_i = X_{i2} - X_{i1}$, $1 \le i \le n$, define the kth order principal lattice of the n-rectangle K^n:

$$\bar{K}_k^n = \left\{ \mathbf{x} = (X_{11} + \frac{i_1}{k} h_1, \cdots, X_{n1} + \frac{i_n}{k} h_n)^T \in \mathbb{R}^n : i_j \in \{0, 1, \ldots, k\},\ 1 \le j \le n \right\},$$

5. $\sharp \bar{K}_k^n = (k+1)^n$.
6. If $p \in \mathbb{Q}_k(K^n)$ and $p = 0$ on $\bar{K}_k^n$, then, $p \equiv 0$.
7. $\bar{\Sigma}_k^n = \left\{ p\,(\mathbf{x}) : \mathbf{x} \in \bar{K}_k^n \right\}$ form a dual basis of $\mathbb{Q}_k(K^n)$.

Type (k) n-Rectangle Lagrange Finite Elements

1. Type (k) n-rectangle finite element: K a n-rectangle K^n, $P_K = \mathbb{Q}_k(K^n)$, and $\Sigma_K = \bar{\Sigma}_k^n = \{p(\mathbf{x}) : \mathbf{x} \in \bar{K}_k^n\}$.
2. The type (k) n-rectangles form a particular subset of an affine family.
3. Figures (a): a type (2) triangle; (b): a type (2) rectangle.



Incomplete Type (k) n-Simplex and Type (k) n-Rectangle

1. Finite elements can be obtained by removing some of the principal degrees of freedom and the corresponding dual basis functions from a type (k) n-simplex or a type (k) n-rectangle.

2. For example, by removing the nodal degree of freedom a_9 and its corresponding basis function

$$16(h_1 h_2)^{-1}(x_1 - X_{11})(x_1 - X_{12})(x_2 - X_{21})(x_2 - X_{22})$$

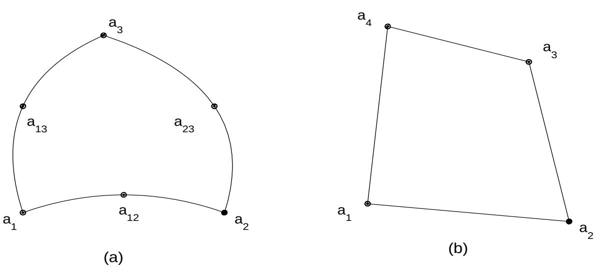
from the type (2) rectangle, we obtain a finite element called the type $(2)'$ rectangle, or incomplete biquadratic rectangle.

Isoparametric Family Given by a Type (k) Simplex or Rectangle

1. Isoparametric families of finite elements can be constructed by a complete or incomplete type (k) n-simplex or n-rectangle.
2. Let the reference finite element (K, P_K, Σ_K) be a complete or incomplete type (k) n-simplex or type (k) n-rectangle.
3. Let $\{\hat{p}_i\}_{i=1}^N$ be the dual basis of P_K corresponding to the kth order principal degrees of freedom of K.
4. Then, for any given invertible map $F : K \to F(K) \in \mathbb{R}^n$, the maps
$$\begin{cases} \mathbf{x} = F(\hat{\mathbf{x}}) := \sum_{i=1}^N \mathbf{a}_i \hat{p}_i(\hat{\mathbf{x}}), \\ u = \sum_{i=1}^N u_i \hat{p}_i(\hat{\mathbf{x}}), \end{cases} \qquad \hat{\mathbf{x}} \in K,$$
define a finite element $(F(K), P_{F(K)}, \Sigma_{F(K)})$, where $P_{F(K)} = \mathrm{span}\,\{\hat{p}_i \circ F^{-1},\ 1 \le i \le N\}$, $\Sigma_{F(K)} = \{\mathbf{a}_i,\ 1 \le i \le N\}$.
5. The finite elements so defined form an isoparametrically equivalent family with (K, P_K, Σ_K) as a reference FE.

Isoparametric Family Given by a Type (k) Simplex or Rectangle

6. Computations are on the reference finite element, it is unnecessary to calculate F^{-1}.
7. $u(\mathbf{x})$ is implicitly expressed by the same set of parameters.
8. Figures (a): type (2) curved triangle; (b): type (1) quadrilateral.



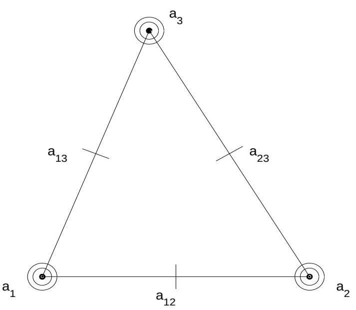
Finite Element of Class $\mathbb{C}^k$

1. A finite element is said to be of class $\mathbb{C}^k$, if all functions in a finite element function space, which is composed of such type of finite elements, are in $\mathbb{C}^k(\overline{\Omega})$.
2. The Lagrange finite elements introduce above are all of class $\mathbb{C}^0$, since the face value of a finite element function is completely determined by its nodal values on the face.
3. For 2nd order elliptic problems, finite elements of class $\mathbb{C}^0$ are sufficient, since the underlying function space is $H^1(\Omega)$.
4. For 4th order elliptic problems, we need finite elements of class $\mathbb{C}^1$ to construct a conforming finite element function space.
5. To construct finite element of class $\mathbb{C}^k$, $k \geq 1$, we need to use Hermite finite elements.

An Example of Hermite Finite Element — The Argyris Triangle

1. $K \subset \mathbb{R}^2$: a triangle with vertices $\mathbf{a}_i$, $i = 1, 2, 3$; $P_K = \mathbb{P}_5(K)$;
$$\Sigma_K = \{p(\mathbf{a}_i), \partial_j p(\mathbf{a}_i), \partial^2_{jk} p(\mathbf{a}_i), 1 \leq i \leq 3, 1 \leq j \leq k \leq 2; \\ \partial_\nu p(\mathbf{a}_{ij}), 1 \leq i < j \leq 3\}.$$
2. $\dim \mathbb{P}_5(K) = C_7^2 = 21$, and $\sharp\Sigma_K = 21$.

Need to show:
If $p \in \mathbb{P}_5(K)$,
$p = 0$ on Σ_K,
then, $p \equiv 0$.



Show $p \equiv 0$, if $p \in \mathbb{P}_5(K)$ and $p = 0$ on Σ_K for the Argyris triangle

1. Let t be the coordinate on the edge $K_{12}^1 = \{\mathbf{a}_1 + t(\mathbf{a}_2 - \mathbf{a}_1) : 0 \le t \le 1\} = \{\mathbf{x} \in K : \lambda_3(\mathbf{x}) = 0\}$.
2. $p \in \mathbb{P}_5(K)$, $p = 0$, $dp = 0$, $d^2p = 0$ on $\mathbf{a}_1$, $\mathbf{a}_2$ $\Rightarrow$ $q(t) = p|_{K_{12}^1} \in \mathbb{P}_5(K_{12}^1)$, $q = 0$, $q' = 0$, $q'' = 0$ on $\mathbf{a}_1$, $\mathbf{a}_2 \Rightarrow$ $q \equiv 0$.
3. Similarly, $\partial_\nu p \in \mathbb{P}_4(K)$, $\partial_\nu p = 0$, on $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_{12}$, and as a function of t, $(\partial_\nu p)'_t = 0$ on $\mathbf{a}_1$, $\mathbf{a}_2$ $\Rightarrow$ $\partial_\nu p|_{K_{12}^1} \equiv 0$.
4. $p|_{K_{12}^1} \equiv 0$, $\partial_\nu p|_{K_{12}^1} \equiv 0$ $\Rightarrow$ $\nabla p|_{K_{12}^1} \equiv 0$ $\Rightarrow$ $p|_{\lambda_3=0} \equiv 0$ and $\partial_{\lambda_3} p|_{\lambda_3=0} \equiv 0$ $\Rightarrow$ λ_3^2 must be a factor of p.
5. Similarly, λ_1^2 and λ_2^2 must also be factors of p.
6. Thus $p = r\lambda_1^2\lambda_2^2\lambda_3^2$ $\Rightarrow$ $r \equiv 0$, since $p \in \mathbb{P}_5(K)$. ■

Show $p \equiv 0$, if $p \in \mathbb{P}_5(K)$ and $p = 0$ on Σ_K for the Argyris triangle

Remark:

The Argyris triangle is a class $\mathbb{C}^1$ finite element, since the values of the function and its first order derivatives on an edge are completely determined by the nodal degrees of freedom there.

The Argyris triangles are not an affine family, since the normals are not affine invariant.

Another Type (5) Hermite Triangle — in an Affine Equivalent Family

1. $K \subset \mathbb{R}^2$: a triangle with vertices $\mathbf{a}_i$, $i = 1, 2, 3$; $P_K = \mathbb{P}_5(K)$;

$$\Sigma'_K = \Big\{ p(\mathbf{a}_i),\ \partial_{\xi_{ij}} p(\mathbf{a}_i),\ \partial^2_{\xi_{ij}\xi_{ik}} p(\mathbf{a}_i), 1 \le i \le 3,\ 1 \le j \le k \le 3,$$
$$i \notin \{j, k\}; \partial_{\eta_{ijk}} p(\mathbf{a}_{ij}),\ 1 \le i < j \le 3,\ k \notin \{i, j\} \Big\},$$

   where $\xi_{ij} = \mathbf{a}_j - \mathbf{a}_i$, $\eta_{ijk} = \mathbf{a}_{ij} - \mathbf{a}_k$.
2. $\dim \mathbb{P}_5(K) = C_7^2 = 21$, and $\sharp\Sigma_K = 21$.
3. Let Π_K and Π'_K be the $P_K = \mathbb{P}_5(K)$ interpolation operators defined by Σ_K and Σ'_K respectively, then,

$$\Pi_K v = \Pi'_K v, \quad \forall v \in \mathbb{P}_5(K), \quad \text{(or equivalently } \Pi'_K \Pi_K v = \Pi_K v, \ \ \forall K \in C^\infty(K)\text{)}.$$

Finite Elements Embedded into an Affine Equivalent Family

Definition

Let (K, P_K, Σ_K) and (K, P_K, Σ'_K) be finite elements, and the latter is in an affine family. The former is said to embed into the affine family of the latter, if the two finite elements satisfy

$$\Pi_K v = \Pi'_K v, \quad \forall v \in P_K, \quad \text{(or equivalently } \Pi'_K \Pi_K v = \Pi_K v, \ \ \forall K \in C^\infty(K)).$$

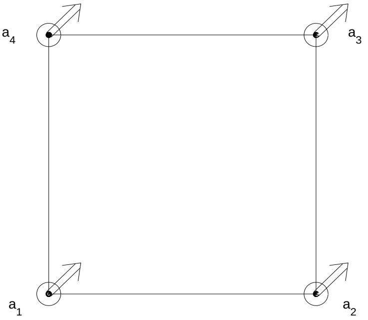
Remak 1: On the finite element function space consists of Argyris triangles, one can still compute the global stiffness matrix by working on reference finite element using the degrees of freedom Σ'_K and the corresponding dual basis functions expressed in barycentric coordinates.

Remak 2: Such an embedding property is useful in coding and error analysis of finite element solutions, when a finite element which is not in an affine equivalent family is used in constructing the finite element function space.

A Class $\mathbb{C}^1$ Type (3) Hermite FE — Bogner-Fox-Schmit Rectangle

1. $K \subset \mathbb{R}^2$: a rectangle with vertices $\{\mathbf{a}_i\}_{i=1}^4$; $P_K = \mathbb{Q}_3(K)$;
$$\Sigma_K = \left\{p(\mathbf{a}_i),\ \partial_j p(\mathbf{a}_i),\ \partial_{12}^2 p(\mathbf{a}_i),\ 1 \le i \le 4,\ j = 1, 2\right\},$$
2. $\dim \mathbb{Q}_3(K) = (3+1)^2 = 16$, and $\sharp\Sigma_K = 16$.

Easily verified:
If $p \in \mathbb{Q}_3(K)$,
$p = 0$ on Σ_K,
then, $p \equiv 0$.



An Example of Finite Element Equations of Elliptic Problems

1. The weak form w.r.t. the homogeneous Dirichlet boundary value problem of the Poisson equation:

$$\begin{cases} \text{Find} \quad u \in \mathbb{H}_0^1(\Omega), \quad \text{such that} \\ a(u,\ v) = (f,\ v), \quad \forall v \in \mathbb{H}_0^1(\Omega), \end{cases}$$

where $a(u,\ v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, $(f,\ v) = \int_\Omega fv \, dx$.

2. Let $\mathbb{V}_h(0) \subset \mathbb{H}_0^1(\Omega)$ be a finite element space, then

$$\begin{cases} \text{Find} \quad u_h \in \mathbb{V}_h(0) \quad \text{such that} \\ a(u_h,\ v_h) = (f,\ v_h), \quad \forall v_h \in \mathbb{V}_h(0), \end{cases}$$

is called the finite element problem of the original problem.

An Example of Finite Element Equations of Elliptic Problems

3. Let $\{\varphi_i\}_{i=1}^{N_h}$ be a set of basis functions of $\mathbb{V}_h(0)$. Denote
$$u_h = \sum_{j=1}^{N_h} u_j \varphi_j, \quad \mathbf{u}_h = (u_1, \ldots, u_{N_h})^T$$
4. $K\,\mathbf{u}_h = \mathbf{f}_h$ is called the finite element equation of the original problem, where $K = (k_{ij}) = (a(\varphi_j, \varphi_i))$ is the stiffness matrix, $\mathbf{f}_h = (f_i) = ((f, \varphi_i))$ is the external load vector.
5. In general, a finite element method discretizes a problem of partial differential equations to a finite dimensional algebraic problem.
6. In particular, a finite element equation derived from a linear problem is usually linear.

Some Numerical Methods for Solving Finite Element Equations

1. For the Dirichlet boundary value problem of the Poisson equation, the finite element equation is usually a symmetric positive definite linear algebraic equation.
2. Classical numerical methods include: Cholesky decomposition, the Gauss-Seidel iterative method, the successive over relaxation iterative method, the conjugate gradient method, the preconditioned conjugate gradient method, etc..
3. In solving large scale symmetric positive definite finite element equations, the preconditioned conjugate method with the incomplete Cholesky decomposition method serving as a preconditioner (ICCG) is a highly recommended method.

The Multigrid Method for Solving Finite Element Equations

1. Observation: for classical iterative methods, (a): the highest frequency modes of the initial error decay very fast; (b): the smaller the grid size, the slower the final convergence speed.
2. Observation: after a very limited number of iterations, the error $\delta u_h^{(k)} = u_h - u_h^{(k)}$ of the finite element solution and the residual $r_h^{(k)} = \sum_{i=1}^{N_h} r_i^{(k)} \varphi_i$, where $(r_1^{(k)}, \cdots, r_{N_h}^{(k)})^T = \mathbf{r}_h^{(k)} = \mathbf{f}_h - K\mathbf{u}_h^{(k)}$ will become very smooth.
3. To increase the efficiency of the computation, one could consider to restrict the residual error on a coarser grid.
4. A typical two-grid method consists of the following 5 parts: pre-smoothing, restriction, coarse grid correction, prolongation and post-smoothing.

Pre-smoothing Perform a few iterations using the Gauss-Seidel, SOR etc., to smooth out the residual and obtain an approximate solution $u_h^{(k)}$ on the fine grid;

Restriction Calculate the residual and restrict the information on to the coarse grid by, say, interpolation, projection or integral average, etc.;

Coarse grid correction Solve the error equation on the coarse grid;

Prolongation Inject the correction solution defined on the coarse grid to the fine grid by, say, interpolation, etc., and add it to $u_h^{(k)}$ to obtain a better approximation;

Post-smoothing Perform a few more smoothing iterations to diminishing the high frequency errors possibly introduced in the restriction and prolongation steps.

The Domain Decomposition Method for Solving PDEs

1. In numerically solving large scale partial differential equations, the domain decomposition method is a type of highly efficient iterative methods, which are particularly suitable for parallel computation.
2. Divide the domain Ω into subdomains Ω_i, $i = 1, 2, \ldots, M$, with or without overlapping.
3. Decompose the problem into subproblems defined on the subdomains Ω_i.
4. Improve the current approximate solution iteratively using the information exchanged between the subdomains.
5. The process could be coupled with some postprocessing

Mixed Finite Element Problem

Typical mixed finite element problem:

$$\begin{cases} \text{Find } \mathbf{p}_h \in \mathbb{X}_h,\ u_h \in \mathbb{Y}_h \text{ such that} \\ a(\mathbf{p}_h, \mathbf{q}_h) + b(\mathbf{q}_h, u_h) = G(\mathbf{q}_h), \quad \forall \mathbf{q}_h \in \mathbb{X}_h, \\ \qquad\qquad\qquad\quad b(\mathbf{p}_h, v_h) = F(v_h), \quad \forall v_h \in \mathbb{Y}_h. \end{cases}$$

Existence Theorem of Mixed Finite Element Problem

Theorem

(**Brezzi**) *Let* $a(\mathbf{p}, \mathbf{q})$ *and* $b(\mathbf{q}, u)$ *be bounded bilinear forms on* $\mathbb{X} \times \mathbb{X}$ *and* $\mathbb{X} \times \mathbb{Y}$ *respectively, let* $G(\mathbf{q})$ *and* $F(v)$ *be bounded linear forms on* $\mathbb{X}$ *and* $\mathbb{Y}$ *respectively. Denote* $\mathbb{V}_{h0} = \{\mathbf{p}_h \in \mathbb{X}_h : b(\mathbf{p}_h, v_h) = 0, \ \forall v_h \in \mathbb{Y}_h\}$. *Suppose*

$(1)'$ *there exists* $\alpha_h > 0$, *such that*

$$a(\mathbf{p}_h, \mathbf{p}_h) \geq \alpha_h \|\mathbf{p}_h\|_{\mathbb{X}}^2, \quad \forall \mathbf{p}_h \in \mathbb{V}_{h0},$$

$(2)'$ *there exists* $\beta_h > 0$, *such that*

$$\sup_{0 \neq \mathbf{p}_h \in \mathbb{X}_h} \frac{b(\mathbf{p}_h, v_h)}{\|\mathbf{p}_h\|_{\mathbb{X}}} \geq \beta_h \|v_h\|_{\mathbb{Y}}, \quad \forall v_h \in \mathbb{Y}_h.$$

Then, the mixed finite element problem has a unique solution.

Finite Element Function Spaces $\mathbb{X}_h$ and $\mathbb{Y}_h$ Must be Properly Coupled

1. Condition $(2)'$: Babuška-Brezzi condition or B-B condition.
2. To guarantee the convergence, the constants α_h and β_h are usually required to be independent of h.
3. The B-B condition imposes restrictions on the choice of finite element function spaces.
4. Let $\dim(\mathbb{X}_h) = N$, $\dim(\mathbb{Y}_h) = M$, and $\{\varphi_i\}_{i=1}^N$ and $\{\psi_j\}_{j=1}^M$ be the normalized bases of $\mathbb{X}_h$ and $\mathbb{Y}_h$ respectively.

Finite Element Function Spaces $\mathbb{X}_h$ and $\mathbb{Y}_h$ Must be Properly Coupled

5. A necessary condition for the finite element equation

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p}_h \\ u_h \end{pmatrix} = \begin{pmatrix} \mathbf{g}_h \\ \mathbf{f}_h \end{pmatrix}.$$

to have no more than one solution is $\mathrm{rank}(B) = M \leq N$.

6. If A is positive definite, then,
$\mathrm{rank}(B) = M \leq N \quad \Leftrightarrow \quad$ B-B condition holds.

Conforming: $\mathbb{V}_h \subset \mathbb{V}$; Non-Conforming: $\mathbb{V}_h \not\subset \mathbb{V}$

An example of the non-conforming finite element method.

1. Consider the variational problem on a polygon region $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \text{Find} \quad u \in \mathbb{H}_0^1(\Omega), \quad \text{such that} \\ \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx, \quad \forall v \in \mathbb{H}_0^1(\Omega). \end{cases}$$

2. K: triangle with vertices $\{\mathbf{a}_i\}_{i=1}^3$, $P_K = \mathbb{P}_1(K)$, $\Sigma_K = \{\mathbf{a}_{ij}\}$.

3. $\tilde{\mathbb{V}}_h = \{u|_{T_i} \in \mathbb{P}_1, \ \forall\, T_i \in \mathfrak{T}_h(\Omega)$, continuous on $Q_i \in \mathbb{Q}_h\}$, $\tilde{\mathbb{V}}_h(0) = \left\{u \in \tilde{\mathbb{V}}_h : u(Q_i) = 0, \ \forall Q_i \in \mathbb{Q}_h \cap \partial\Omega\right\}$, where $\mathbb{Q}_h = \Sigma_K = \{\mathbf{a}_{ij}\}$ is the set of middle points of edges in $\mathfrak{T}_h(\Omega)$.

Conforming: $\mathbb{V}_h \subset \mathbb{V}$; Non-Conforming: $\mathbb{V}_h \not\subset \mathbb{V}$

4. $a_h(u,\, v) = \sum_{T \in \mathfrak{T}_h(\Omega)} \int_T \nabla u \cdot \nabla v \, dx.$

5. The nonconforming finite element problem:

$$\begin{cases} \text{Find} \quad u_h \in \tilde{\mathbb{V}}_h(0) \ \text{ such} \\ a(u_h,\, v_h) = (f,\ v_h), \quad \forall v_h \in \tilde{\mathbb{V}}_h(0). \end{cases}$$

6. Provide a lot of convenience, accompanied by additional difficulties.

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Thank You!