Numerical Solutions to Partial Differential Equations

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Variational Problems of the Dirichlet BVP of the Poisson Equation

1. For the homogeneous Dirichlet BVP of the Poisson equation

$$\begin{cases} -\triangle u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

2. The weak form w.r.t. the virtual work principle:

$$\begin{cases} \text{Find } \ u \in \mathbb{H}_0^1(\Omega), \ \text{ such that} \\ a(u, v) = (f, v), \quad \forall v \in \mathbb{H}_0^1(\Omega), \end{cases}$$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, $(f, v) = \int_\Omega fv \, dx$.

3. The weak form w.r.t. the minimum potential energy principle:

$$\begin{cases} \text{Find } \ u \in \mathbb{H}_0^1(\Omega), \ \text{ such that} \\ J(u) = \min_{v \in \mathbb{H}_0^1(\Omega)} J(v), \end{cases}$$

where $J(v) = \frac{1}{2} a(v, v) - (f, v)$.

Use Finite Dimensional Trial, Test and Admissible Function Spaces

1. Replace the trial and test function spaces by appropriate finite dimensional subspaces, say $\mathbb{V}_h(0) \subset \mathbb{H}_0^1(\Omega)$, we are led to the discrete problem:
$$\begin{cases} \text{Find } \ u_h \in \mathbb{V}_h(0) \ \text{ such that} \\ a(u_h, v_h) = (f, v_h), \quad \forall v_h \in \mathbb{V}_h(0), \end{cases}$$
Such an approach is called the Galerkin method.
2. Replace the admissible function space by an appropriate finite dimensional subspace, say $\mathbb{V}_h(0) \subset \mathbb{H}_0^1(\Omega)$, we are led to the discrete problem:
$$\begin{cases} \text{Find } \ u_h \in \mathbb{V}_h(0) \ \text{ such that} \\ J(u_h) = \min_{v_h \in \mathbb{V}_h(0)} J(v_h). \end{cases}$$
Such an approach is called the Ritz method.
3. The two methods lead to an equivalent system of linear algebraic equations.

Derivation of Algebraic Equations of the Galerkin Method

Let $\{\varphi_i\}_{i=1}^{N_h}$ be a set of basis functions of $\mathbb{V}_h(0)$, let

$$u_h = \sum_{j=1}^{N_h} u_j \varphi_j, \quad v_h = \sum_{i=1}^{N_h} v_i \varphi_i,$$

then, the Galerkin method leads to

$$\begin{cases} \text{Find } \ \mathbf{u}_h = (u_1, \ldots, u_{N_h})^T \in \mathbb{R}^{N_h} \ \text{ such that} \\ \sum_{i,j=1}^{N_h} a(\varphi_j, \varphi_i) u_j v_i = \sum_{i=1}^{N_h} (f, \varphi_i) v_i, \ \ \forall \mathbf{v}_h = (v_1, \ldots, v_{N_h})^T \in \mathbb{R}^{N_h}, \end{cases}$$

which is equivalent to $\sum_{j=1}^{N_h} a(\varphi_j,\ \varphi_i) u_j = (f,\ \varphi_i)$, $i = 1, 2, \cdots, N_h$.

- The stiffness matrix: $K = (k_{ij}) = (a(\varphi_j,\ \varphi_i))$; the external load vector: $\mathbf{f}_h = (f_i) = ((f,\ \varphi_i))$; the displacement vector: $\mathbf{u}_h$; the linear algebraic equation: $K\, \mathbf{u}_h = \mathbf{f}_h$.

Derivation of Algebraic Equations of the Ritz Method

1. The Ritz method leads to a finite dimensional minimization problem, whose stationary points satisfy the equation given by the Galerkin method, and vice versa.

2. So, the Ritz method also leads to $K\,\mathbf{u}_h = \mathbf{f}_h$.

3. It follows from the symmetry of $a(\cdot,\cdot)$ and the Poincaré-Friedrichs inequality (see Theorem 5.4) that stiffness matrix K is a symmetric positive definite matrix, and thus the linear system has a unique solution, which is the minima of the discrete minimization problem.

The Key Is to Construct Finite Dimensional Subspaces

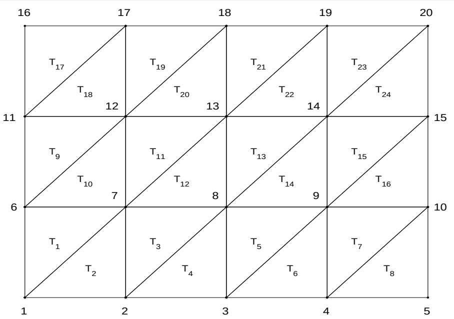
There are many ways to construct finite dimensional subspaces for the Galerkin method and Ritz method. For example

1. For $\Omega = (0, 1) \times (0, 1)$, the functions
$$\varphi_{mn}(x, y) = \sin(m\pi x)\sin(n\pi y), \quad m, n \geq 1,$$
which are the complete family of the eigenfunctions $\{\varphi_i\}_{i=1}^{\infty}$ of the corresponding eigenvalue problem
$$\begin{cases} -\triangle u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$
and form a set of basis of $\mathbb{H}_0^1(\Omega)$.
2. Define $\mathbb{V}_N = \text{span}\{\varphi_{mn} : m \leq N, n \leq N\}$, the corresponding numerical method is called the spectral method.
3. Finite element method is a systematic way to construct subspaces for more general domains.

Construction of a Finite Element Function Space for $\mathbb{H}_0^1([0,1]^2)$

1. The Dirichlet boundary value problem of the Poisson equation
$$-\triangle u = f, \ \ \forall x \in \Omega = (0,1)^2, \qquad u = 0, \ \ \forall x \in \partial\Omega.$$
2. We need to construct a finite element subspace of $\mathbb{H}_0^1((0,1)^2)$.
3. Firstly, introduce a triangulation $\mathfrak{T}_h(\Omega)$ on the domain $\overline{\Omega}$:

Closed triangular elements $\{T_i\}_{i=1}^M$ with $\overline{\Omega} = \cup_{i=1}^M T_i$;
$\mathring{T}_i \cap \mathring{T}_j = \emptyset$, $1 \le i \ne j \le M$;
If $T_i \cap T_j \ne \emptyset$: it must be a common edge or vertex;
$h = \max_i \operatorname{diam}(T_i)$;
Nodes $\{A_i\}_{i=1}^N$, which is globally numbered.



Construction of a Finite Element Function Space for $\mathbb{H}_0^1((0,1)^2)$

4. Secondly, define a finite element function space, which is a subspace of $\mathbb{H}^1((0,1)^2)$, on the triangulation $\mathfrak{T}_h(\Omega)$:
$$\mathbb{V}_h = \{u \in \mathbb{C}(\overline{\Omega}) : u|_{T_i} \in \mathbb{P}_1(T_i),\ \forall T_i \in \mathfrak{T}_h(\Omega)\}.$$
5. Then, define finite element trial and test function spaces, which are subspaces of $\mathbb{H}_0^1((0,1)^2)$:
$$\mathbb{V}_h(0) = \{u \in \mathbb{V}_h : u(A_i) = 0,\ \forall A_i \in \partial\Omega\}.$$
6. A function $u \in \mathbb{V}_h$ is uniquely determined by $\{u(A_i)\}_{i=1}^N$.
7. Basis $\{\varphi_i\}_{i=1}^N$ of $\mathbb{V}_h$: $\varphi_i(A_j) = \delta_{ij}$, $i = 1, 2, \ldots, N$.
8. $k_{ij} = a(\varphi_j, \varphi_i) \neq 0$, iff $A_i \cup A_j \subset T_e$ for some $1 \leq e \leq M$.
9. $\mathrm{supp}(\varphi_i)$ is small $\Rightarrow$ the stiffness matrix K is sparse.

Assemble the Global Stiffness Matrix K from the Element One K^e

1. Denote $a^e(u,v) = \int_{T_e} \nabla u \cdot \nabla v \, dx$, by the definition, then, $k_{ij} = a(\varphi_j, \varphi_i) = \sum_{e=1}^{M} a^e(\varphi_j, \varphi_i) = \sum_{e=1}^{M} k_{ij}^e$.
2. $k_{ij}^e = a^e(\varphi_j, \varphi_i) \neq 0$, iff $A_i \cup A_j \subset T_e$. For most e, $k_{ij}^e = 0$.
3. It is inefficient to calculate k_{ij} by scanning i, j node by node.
4. Element T_e with nodes $\{A_\alpha^e\}_{\alpha=1}^3 \Leftrightarrow$ the global nodes $A_{en(\alpha,e)}$.
5. Area coordinates $\lambda^e(A) = (\lambda_1^e(A), \lambda_2^e(A), \lambda_3^e(A))^T$ for $A \in T_e$, $\lambda_\alpha^e(A) = |\triangle A A_\beta^e A_\gamma^e| / |\triangle A_\alpha^e A_\beta^e A_\gamma^e| \in \mathbb{P}_1(T_e)$, $\lambda_\alpha^e(A_\beta^e) = \delta_{\alpha\beta}$.
6. $\varphi_{en(\alpha,e)}|_{T_e}(A) = \lambda_\alpha^e(A)$, $\forall A \in T_e$.

The Algorithm for Assembling Global K and $\mathbf{f}_h$

7. Define the element stiffness matrix

$$K^e = (k^e_{\alpha\beta}), \quad k^e_{\alpha\beta} \triangleq a^e(\lambda^e_\alpha, \lambda^e_\beta) = \int_{T_e} \nabla\lambda^e_\alpha \cdot \nabla\lambda^e_\beta \, dx,$$

8. Then, $k_{ij} = \sum\limits_{\substack{en(\alpha,e)=i\in T_e \\ en(\beta,e)=j\in T_e}} k^e_{\alpha\beta}$ can be assembled element wise.

9. The external load vector $\mathbf{f}_h = (f_i)$ can also be assembled by scanning through elements

$$f_i = \sum_{en(\alpha,e)=i\in T_e} \int_{T_e} f\,\lambda^e_\alpha \, dx = \sum_{en(\alpha,e)=i\in T_e} f^e_\alpha .$$

The Algorithm for Assembling Global K and $\mathbf{f}_h$

Algorithm 6.1: $K = (k(i,j)) := 0$; $\mathbf{f} = (f(i)) := 0$;

for $e = 1 : M$

$K^e = (k^e(\alpha,\ \beta))$; *% calculate the element stiffness matrix*

$\mathbf{f}^e = (f^e(\alpha))$; *% calculate the element external load vector*

$k(en(\alpha,\ e), en(\beta,\ e)) := k(en(\alpha,\ e), en(\beta,\ e)) + k^e(\alpha,\ \beta)$;

$f(en(\alpha,\ e)) := f(en(\alpha,\ e)) + f^e(\alpha)$;

end

Calculations of K^e and $\mathbf{f}^e$ Are Carried Out on a Reference Element

1. The standard reference triangle $T_s = \{\hat{x} = (\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2 : \hat{x}_1 \geq 0,\ \hat{x}_2 \geq 0 \text{ and } \hat{x}_1 + \hat{x}_2 \leq 1\}$, with $A_1^s = (0, 0)^T$, $A_2^s = (1, 0)^T$ and $A_3^s = (0, 1)^T$.
2. For T_e with $A_1^e = (x_1^1, x_2^1)^T$, $A_2^e = (x_1^2, x_2^2)^T$, $A_3^e = (x_1^3, x_2^3)^T$, define $A_e = (A_2^e - A_1^e, A_3^e - A_1^e)$, $a_e = A_1^e$.
3. $x = L_e(\hat{x}) := A_e\hat{x} + a_e : \ T_s \to T_e$ is an affine map.
4. The area coordinates of T_e: $\lambda_\alpha^e(x) = \lambda_\alpha^s(L_e^{-1}(x))$, since it is an affine function of x, and $\lambda_\alpha^s(L_e^{-1}(A_\beta^e)) = \lambda_\alpha^s(A_\beta^s) = \delta_{\alpha\beta}$.
5. $\nabla\lambda^e(x) = \nabla\lambda^s(\hat{x})\nabla L_e^{-1}(x) = \nabla\lambda^s(\hat{x})A_e^{-1}$.

Calculations of K^e and $\mathbf{f}^e$ Are Carried Out on a Reference Element

6. Change of integral variable $\hat{x} = L_e^{-1}(x) := A_e^{-1}x - A_e^{-1}A_1^e$,

$$K^e = \int_{T_e} \nabla\lambda^e(x)\,(\nabla\lambda^e(x))^T dx = \int_{T_s} \nabla\lambda^s(\hat{x})A_e^{-1}(\nabla\lambda^s(\hat{x})A_e^{-1})^T \det A_e d\hat{x},$$

$$\mathbf{f}^e = \int_{T_e} f(x)\lambda^e(x)\,dx = \det A_e \int_{T_s} f(L_e(\hat{x}))\lambda^s(\hat{x})\,d\hat{x}.$$

7. $\lambda_1^s(\hat{x}_1, \hat{x}_2) = 1 - \hat{x}_1 - \hat{x}_2$, $\lambda_2^s(\hat{x}_1, \hat{x}_2) = \hat{x}_1$, $\lambda_3^s(\hat{x}_1, \hat{x}_2) = \hat{x}_2$, so

$$\nabla\lambda^s(\hat{x}) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_e^{-1} = \frac{1}{\det A_e}\begin{pmatrix} x_2^3 - x_2^1 & x_1^1 - x_1^3 \\ x_2^1 - x_2^2 & x_1^2 - x_1^1 \end{pmatrix}.$$

Calculations of K^e and $\mathbf{f}^e$ in Terms of λ^s and A^e

8. The area of T_s is $1/2$, hence, the element stiffness matrix is

$$K^e = \frac{1}{2\det A_e}\begin{pmatrix} x_2^2 - x_2^3 & x_1^3 - x_1^2 \\ x_2^3 - x_2^1 & x_1^1 - x_1^3 \\ x_2^1 - x_2^2 & x_1^2 - x_1^1 \end{pmatrix}\begin{pmatrix} x_2^2 - x_2^3 & x_2^3 - x_2^1 & x_2^1 - x_2^2 \\ x_1^3 - x_1^2 & x_1^1 - x_1^3 & x_1^2 - x_1^1 \end{pmatrix}.$$

9. In general, it is necessary to apply a numerical quadrature to the calculation of the element external load vector $\mathbf{f}^e$.

10. If f is a constant on T_e, then

$$\mathbf{f}^e = \frac{1}{6} f(T_e) \det A_e \,(1,\, 1,\, 1)^T = \frac{1}{3} f(T_e)\,|T_e|\,(1,\, 1,\, 1)^T.$$

Extension of the Example to More General Boundary Conditions

- For a Dirichlet boundary condition $u(x) = u_0(x) \neq 0$, on $\partial\Omega$, FE trial function space $\mathbb{V}_h(0)$ should be replaced by

$$\mathbb{V}_h(u_0) = \{u \in \mathbb{V}_h : u(A_i) = u_0(A_i),\ \forall A_i \in \partial\Omega\}.$$

- For a more general mixed type boundary condition

$$\begin{cases} u(x) = u_0(x), & \forall x \in \partial\Omega_0, \\ \dfrac{\partial u}{\partial \nu} + bu = g, & \forall x \in \partial\Omega_1, \end{cases}$$

We need to

1. add contributions of $\int_{\partial\Omega_1} buv\,dx$ and $\int_{\partial\Omega_1} gv\,dx$ to K and $\mathbf{f}$ by scanning through edges on $\partial\Omega_1$;

Extension of the Example to More General Boundary Conditions

2. Set finite element trial function space:

$$\mathbb{V}_h(u_0; \partial\Omega_0) = \{u \in \mathbb{V}_h : u(A_i) = u_0(A_i),\ \forall A_i \in \partial\Omega_0\},$$

if $\partial\Omega_0 \neq \emptyset$ (mixed boundary condition);

$\mathbb{V}_h$, if $\partial\Omega_0 = \emptyset$ but $b > 0$ (the 3rd type boundary condition);

$$\mathbb{V}_h(0; A_i) = \{u \in \mathbb{V}_h : u(A_i) = 0, \text{ on a specified node } A_i \in \overline{\Omega}\},$$

if $\partial\Omega_0 = \emptyset$ and $b = 0$ (pure Neumann boundary condition).

Note: In the case of pure Neumann boundary condition, the solution is unique up to an additive constant. $\mathbb{V}_h(0; A_i)$ removes such uncertainty, so the solution in $\mathbb{V}_h(0; A_i)$ is unique. Likewise, let l be a non-zero linear functional on $\mathbb{V}_h$, then we may as well take $\mathbb{V}_h(0; l) = \{u \in \mathbb{V}_h : l(u) = 0\}$.

Summary of the Typical Example on FEM

1. introduce a finite element partition (triangulation) $\mathcal{T}_h$ to the region $\overline{\Omega}$, such as the triangular partition shown above.
2. Establish finite element trial and test function spaces on $\mathcal{T}_h(\Omega)$, such as continuous piecewise affine function spaces satisfy appropriate boundary conditions shown above.
3. Select a set of basis functions, known as the shape functions, for example, the area coordinates on the triangular element.
4. Calculate the element stiffness matrixes K^e and element external load vector $\mathbf{f}_h^e$, and form the global stiffness matrix K and external load vector $\mathbf{f}_h$.

Some General Remarks on the Implementation of FEM

Arrays used in the algorithm:

1. $en(\alpha, e)$: assigns a global node number to a node with the local node number α on the eth element.
2. $edg0(\alpha, edg)$: assigns a global node number to a node with the local node number α on the edgth edge on $\partial\Omega_0$. $edg1(\alpha, edg)$, $edg2(\alpha, edg)$ are similar arrays with respect to Neumann and Robin type boundaries.
3. $cd(i, nd)$: assigns the ith component of the spatial coordinates to a node with the global node number nd.

Some General Remarks on the Implementation of FEM

Arrays used in the algorithm:

4. In iterative methods for solving $K\mathbf{u}_h = \mathbf{f}_h$, it is not necessary to form the global stiffness matrix K, since it always appears in the form $K\mathbf{v}_h = \sum_{e \in \mathcal{T}_h} K^e \mathbf{v}_h^e$. In such cases, we may need:

5. $et(i, \tau)$: assigns the global element number to the τth local element of the ith global node. And $edgrt(i, \tau)$, etc.

Three Basic Ingredients in a Finite Element Function Space

(FEM 1) Introduce a finite element triangulation $\mathcal{T}_h$ on the region $\overline{\Omega}$, which divides the region $\overline{\Omega}$ into finite numbers of subsets K, generally called finite element, such that

- ($\mathcal{T}_h$1) $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$;
- ($\mathcal{T}_h$2) each finite element $K \in \mathcal{T}_h$ is a closed set with a nonempty interior set $\overset{\circ}{K}$;
- ($\mathcal{T}_h$3) $\overset{\circ}{K}_1 \cap \overset{\circ}{K}_2 = \emptyset$, for any two different finite elements $K_1,\ K_2 \in \mathcal{T}_h$;
- ($\mathcal{T}_h$4) every finite element $K \in \mathcal{T}_h$ has a Lipschitz continuous boundary.

Three Basic Ingredients in a Finite Element Function Space

(FEM 2) Introduce on each finite element $K \in \mathcal{T}_h$ a function space P_K which consists of some polynomials or other functions having certain approximation properties and at the same time easily manipulated analytically and numerically;

(FEM 3) The finite element function space $\mathbb{V}_h$ has a set of "normalized" basis functions which are easily computed, and each basis function has a "small" support.

Generally speaking, a finite element is not just a subset K, it includes also the finite dimensional function space P_K defined on K and the corresponding "normalized" basis functions.

General Abstract Definition of a Finite Element

Definition

A triple (K, P_K, Σ_K) is called a finite element, if

1. $K \subset \mathbb{R}^n$, called an element, is a closed set with non-empty interior and a Lipschitz continuous boundary;
2. $P_K : K \to \mathbb{R}$ is a finite dimensional function space consisting of sufficiently smooth functions defined on the element K;
3. Σ_K is a set of linearly independent linear functionals $\{\varphi_i\}_{i=1}^N$ defined on $\mathbb{C}^\infty(K)$, which are called the degrees of freedom of the finite element and form a dual basis corresponding to a "normalized" basis of P_K, meaning that there exists a unique basis $\{p_i\}_{i=1}^N$ of P_K such that $\varphi_i(p_j) = \delta_{ij}$.

An Additional Requirement on the Partition

In applications, an element K is usually taken to be

1. a triangle in $\mathbb{R}^2$; a tetrahedron in $\mathbb{R}^3$; a n simplex in $\mathbb{R}^n$;
2. a rectangle or parallelogram in $\mathbb{R}^2$; a cuboid or a parallelepiped or more generally a convex hexahedron in $\mathbb{R}^3$; a parallelepiped or more generally a convex $2n$ polyhedron in $\mathbb{R}^n$;
3. a triangle with curved edges or a tetrahedron with curved faces, etc..

An Additional Requirement on the Partition

When a region $\overline{\Omega}$ is partitioned into a finite element triangulation $\mathcal{T}_h$ with such elements, to ensure that (FEM 3) holds, the adjacent elements are required to satisfy the following compatibility condition:

($\mathcal{T}_h$5) For any pair of K_1, $K_2 \in \mathcal{T}_h$, if $K_1 \bigcap K_2 \neq \emptyset$, then, there must exists an $0 \leq i \leq n-1$, such that $K_1 \bigcap K_2$ is exactly a common i dimensional face of K_1 and K_2.

The Function Space P_K Usually Consists of Polynomials

1. The finite element of the n-simplex of type (k): K is a n-simplex, $P_K = \mathbb{P}_k(K)$, which is the space of all polynomials of degree no greater than k defined on K.

   For example, the piecewise affine triangular element (2-simplex of type (1), or type (1) 2-simplex, or type(1) triangle).

2. The finite element of n-rectangle of type (k) (abbreviated as the n-k element): K is a n-rectangle, $P_K = \mathbb{Q}_k(K)$, which is the space of all polynomials of degree no greater than k with respect to each one of the n variables.

   For example, the bilinear element (the 2-rectangle of type (1), or type (1) 2-rectangle, or 2-1 rectangle); etc..

The Nodal Degrees of Freedom Σ_K

The degrees of freedom in the nodal form:

$$\begin{cases} \varphi_i^0 : \quad p \to p\,(a_i^0), & \text{Lagrange FE, if contains point values only} \\ \varphi_{ij}^1 : \quad p \to \partial_{\nu_{ij}^1} p\,(a_i^1), & \text{Hermite FE, if contains at least} \\ \varphi_{ijk}^2 : \quad p \to \partial^2_{\nu_{ij}^2 \nu_{ik}^2} p\,(a_i^2), & \text{one of the derivatives} \end{cases}$$

where the points $a_i^s \in K$, $s = 0, 1, 2$ are called nodes, $\nu_{ij}^s \in \mathbb{R}^n$, $s = 1, 2$ are specified nonzero vectors.

The Nodal and Integral Degrees of Freedom Σ_K

The degrees of freedom in the integral form:

$$\psi_i^s : p \to \frac{1}{\mathrm{meas}_s(K_i^s)} \int_{K_i^s} p(x)\, dx,$$

where K_i^s, $s = 0, 1, \ldots, n$ are s-dimensional faces of the element K, and $\mathrm{meas}_s(K_i^s)$ is the s-dimensional Lebesgue measure of K_i^s.

For example, if $s = n$, then the corresponding degree of freedom is the average of the element integral.

P_K Interpolation for a Given Finite Element (K, P_K, Σ_K)

Definition

Let (K, P_K, Σ_K) be a finite element, and let $\{\varphi_i\}_{i=1}^N$ be its degrees of freedom with $\{p_i\}_{i=1}^N \in P_K$ being the corresponding dual basis, meaning $\varphi_i(p_j) = \delta_{ij}$. Define the P_K interpolation operator $\Pi_K : \mathbb{C}^\infty(K) \to P_K$ by

$$\Pi_K(v) = \sum_{i=1}^{N} \varphi_i(v)\, p_i, \quad \forall v \in \mathbb{C}^\infty(K),$$

and define $\Pi_K(v)$ as the P_K interpolation function of v.

In applications, it is often necessary to extend the domain of the definition of the P_K interpolation operator, for example, to extend the domain of the definition of a Lagrange finite element to $\mathbb{C}(K)$.

The P_K Interpolation Operator Is Independent of the Choice of Basis

Definition

Let two finite elements (K, P_K, Σ_K) and (L, P_L, Σ_L) satisfy

$$K = L, \quad P_K = P_L, \text{ and } \Pi_K = \Pi_L,$$

where Π_K and Π_L are respectively P_K and P_L interpolation operators, then the two finite elements are said to be equivalent.

Compatibility Conditions for P_K and Σ_K on Adjacent Elements

1. $\mathcal{T}_h$: a finite element triangulation of Ω; $\{(K, P_K, \Sigma_K)\}_{K \in \mathcal{T}_h}$: a given set of corresponding finite elements.
2. $\mathbb{V}_h = \{v : \bigcup_{K \in \mathcal{T}_h} K \to \mathbb{R} : v|_K \in P_K\}$: FE function space.
3. Compatibility conditions are required to assure $\mathbb{V}_h$ satisfies (FEM 3), as well as a subspace of $\mathbb{V}$.

For example, for polyhedron elements and nodal degrees of freedom, if $K_1 \bigcap K_2 \neq \emptyset$, then, we require that a point $a_i^s \in K_1 \bigcap K_2$ is a node of K_1, if and only if it is also the same type of node of K_2.

$\mathbb{V}_h$ Interpolation Operator and $\mathbb{V}_h$ Interpolation

Denote $\Sigma_h = \bigcup_{K \in \mathcal{T}_h} \Sigma_K$ as the degrees of freedom of the finite element function space $\mathbb{V}_h$.

Definition

Define the $\mathbb{V}_h$ interpolation operator $\Pi_h : \mathbb{C}^\infty(\overline{\Omega}) \to \mathbb{V}_h$ by

$$\Pi_h(v)|_K = \Pi_K(v|_K), \quad \forall v \in \mathbb{C}^\infty(\overline{\Omega}),$$

and define $\Pi_h(v)$ as the $\mathbb{V}_h$ interpolant of v.

In applications, similar as for the P_K interpolation operator, the domain of definition of the $\mathbb{V}_h$ interpolation operator is often extended to meet certain requirements.

Reference Finite Element $(\hat{K}, \hat{P}, \hat{\Sigma})$ and Its Isoparametric Equivalent Family

Definition

Let $\hat{K}$, $K \in \mathbb{R}^n$, $(\hat{K},\ \hat{P},\ \hat{\Sigma})$ and $(K,\ P_K,\ \Sigma_K)$ be two finite elements. Suppose that there exists a sufficiently smooth invertible map $F_K : \hat{K} \to K$, such that

$$\begin{cases} F_K(\hat{K}) = K; \\ p_i = \hat{p}_i \circ F_K^{-1}, \quad i = 1, \dots, N; \\ \varphi_i(p) = \hat{\varphi}_i(p \circ F_K), \quad \forall p \in P_K,\ \ i = 1, \dots, N, \end{cases}$$

where $\{\hat{\varphi}_i\}_{i=1}^N$ and $\{\varphi_i\}_{i=1}^N$ are the basis of the degrees of freedom spaces $\hat{\Sigma}$ and Σ_K respectively, $\{\hat{p}_i\}_{i=1}^N$ and $\{p_i\}_{i=1}^N$ are the corresponding dual basis of $\hat{P}$ and P_K respectively. Then, the two finite elements are said to be isoparametrically equivalent. In particular, if F_K is an affine mapping, the two finite elements are said to be affine-equivalent.

An Isoparametric (Affine) Family of Finite Elements

If all finite elements in a family are isoparametrically (affine-) equivalent to a given reference finite element, then we call the family an isoparametric (affine) family.

For example, the finite elements with triangular elements and piecewise linear function space used in the previous subsection, *i.e.* finite elements of 2-simplex of type (1), are an affine family.

习题 6：5，6，7

Thank You!