Numerical Solutions to Partial Differential Equations

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Embedding Operator and Embedding Relation of Banach Spaces

1. $\mathbb{X}$, $\mathbb{Y}$: Banach spaces with norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$.
2. If $x \in \mathbb{X} \Rightarrow x \in \mathbb{Y}$, & $\exists$ const. $C > 0$ independent of x s.t. $\|x\|_{\mathbb{Y}} \leq C\|x\|_{\mathbb{X}}$, $\forall x \in \mathbb{X}$, then the identity map $I : \mathbb{X} \to \mathbb{Y}$, $I\,x = x$ is called an embedding operator, and the corresponding embedding relation is denoted by $\mathbb{X} \hookrightarrow \mathbb{Y}$.
3. The embedding operator $I : \mathbb{X} \to \mathbb{Y}$ is a bounded linear map.
4. If, in addition, I is happened to be a compact map, then, the corresponding embedding is called a compact embedding, and is denoted by $\mathbb{X} \overset{c}{\hookrightarrow} \mathbb{Y}$.

The Sobolev Embedding Theorem

Theorem

Let Ω be a bounded connected domain with a Lipschitz continuous boundary $\partial\Omega$, then

$$
\begin{aligned}
&\mathbb{W}^{m+k,p}(\Omega) \hookrightarrow \mathbb{W}^{k,q}(\Omega),\ \forall\, 1 \le q \le \frac{np}{n-mp},\ k \ge 0, && \text{if}\ \ m < n/p;\\
&\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega),\ \forall\, 1 \le q < \frac{np}{n-mp},\ k \ge 0, && \text{if}\ \ m < n/p;\\
&\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega),\ \forall\, 1 \le q < \infty,\ k \ge 0, && \text{if}\ \ m = n/p;\\
&\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^{k}(\overline{\Omega}),\quad \forall\, k \ge 0, && \text{if}\ \ m > n/p.
\end{aligned}
$$

Trace of a Function and Trace Operators

1. Since the n dimensional Lebesgue measure of a Lipschitz continuous boundary $\partial\Omega$ is zero, a function in $\mathbb{W}^{m,p}(\Omega)$ is generally not well defined on $\partial\Omega$.
2. $\mathbb{C}^\infty(\overline{\Omega})$ is dense in $\mathbb{W}^{m,p}(\Omega)$ for $1 \leq p < \infty$.
3. For $u \in \mathbb{W}^{m,p}(\Omega)$, let $\{u_k\} \subset \mathbb{C}^\infty(\overline{\Omega})$ be such that

$$\|u_k - u\|_{m,p,\Omega} \longrightarrow 0, \quad \text{as} \;\; k \to \infty,$$

4. If, for any such a sequence, $u_k|_{\partial\Omega} \to \nu(u)$ in $\mathbb{L}^q(\partial\Omega)$, then, we call $u|_{\partial\Omega} \triangleq \nu(u) \in \mathbb{L}^q(\partial\Omega)$ the trace of u on $\partial\Omega$, and call $\nu : \mathbb{W}^{m,p}(\Omega) \to \mathbb{L}^q(\partial\Omega)$, $\nu(u) = u|_{\partial\Omega}$ the trace operator.

Trace of a Function and Trace Operators

5. If ν is continuous, we say $\mathbb{W}^{m,p}(\Omega)$ embeds into $\mathbb{L}^q(\partial\Omega)$, and denote the embedding relation as $\mathbb{W}^{m,p}(\Omega) \hookrightarrow \mathbb{L}^q(\partial\Omega)$.

6. Trace operators, as well as corresponding embedding and compact embedding, into other Banach spaces defined on the whole or a part of $\partial\Omega$ can be defined similarly.

Obviously, under the conditions of the embedding theorem, $\mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^k(\partial\Omega)$, if $m > n/p$. In general, we have the following trace theorem.

The Trace Theorem

Theorem

If the boundary $\partial\Omega$ of a bounded connected open domain Ω is an order $m \geq 1$ continuously differentiable surface, then, we have

$$\mathbb{W}^{m,p}(\Omega) \hookrightarrow \mathbb{L}^q(\partial\Omega), \quad \textit{for } 1 \leq q \leq \frac{(n-1)p}{n-mp}, \quad \textit{if} \quad m < n/p;$$

$$\mathbb{W}^{m,p}(\Omega) \hookrightarrow \mathbb{L}^q(\partial\Omega), \quad \textit{for } 1 \leq q < \infty, \quad \textit{if} \quad m = n/p.$$

In addition, if $m = 1$ and $p = q = 2$, and if the boundary $\partial\Omega$ is a Lipschitz continuous surface, then, we have in particular

$$\mathbb{H}^1(\Omega) \hookrightarrow \mathbb{L}^2(\partial\Omega).$$

Remarks on $\mathbb{H}_0^1(\Omega)$ and $\mathbb{H}_0^2(\Omega)$

For a bounded connected open domain Ω with Lipschitz continuous boundary $\partial\Omega$,

1. by definition the Hilbert space $\mathbb{H}_0^m(\Omega)$ is the closure of $\mathbb{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_m = \|\cdot\|_{m,2} := \|\cdot\|_{m,2,\Omega}$;
2. in particular, $\mathbb{H}_0^1(\Omega) = \{u \in \mathbb{H}^1(\Omega) : u|_{\partial\Omega} = 0\}$;
3. $\mathbb{H}_0^2(\Omega) = \{u \in \mathbb{H}^2(\Omega) : u|_{\partial\Omega} = 0,\ \partial_\nu u|_{\partial\Omega} = 0\}$, where $\partial_\nu u|_{\partial\Omega}$ is the outer normal derivative of u in the sense of trace.

Derivation of a Variational Form

1. The Dirichlet boundary value problem of the Poisson equation

$$-\triangle u = f, \ \ \forall x \in \Omega, \qquad u = \bar{u}_0, \ \ \forall x \in \partial\Omega.$$

2. Assume the problem admits a classical solution $u \in \mathbb{C}^2(\overline{\Omega})$.

3. For any test function $v \in \mathbb{C}_0^\infty(\Omega)$, by the Green's formula,

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} v \, \partial_\nu u \, dx = \int_\Omega fv \, dx.$$

Derivation of a Variational Form

4. Let $a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$; $(\cdot, \cdot)$ the inner product of $\mathbb{L}^2(\Omega)$.

5. By the denseness of $\mathbb{C}_0^\infty(\Omega)$ in $\mathbb{H}_0^1(\Omega)$, we are lead to

$$a(u, v) = (f, v), \quad \forall v \in \mathbb{H}_0^1(\Omega).$$

6. u does not have to be in $\mathbb{C}^2(\overline{\Omega})$ to satisfy such a variational equation, $u \in \mathbb{H}^1(\Omega)$ makes sense.

A Variational Form of Dirichlet BVP of the Poisson Equation

Definition

If $u \in \mathbb{V}(\bar{u}_0;\Omega) = \{u \in \mathbb{H}^1(\Omega) : u|_{\partial\Omega} = \bar{u}_0\}$ satisfies the variational equation

$$a(u,\ v) = (f,\ v), \quad \forall v \in \mathbb{H}_0^1(\Omega),$$

then, u is called a weak solution of the Dirichlet boundary value problem of the Poisson equation; the corresponding variational problem is called a variational form, or weak form, of the Dirichlet boundary value problem of the Poisson equation; and the function spaces $\mathbb{V}(\bar{u}_0;\Omega)$ and $\mathbb{H}_0^1(\Omega)$ are called respectively the trial and test function spaces of the variational problem.

- Obviously, the classical solution, if exists, is a weak solution.
- Let $\tilde{u} \in \mathbb{H}^1(\Omega)$ and $\tilde{u}|_{\partial\Omega} = \bar{u}_0$, then $\mathbb{V}(\bar{u}_0;\Omega) = \tilde{u} + \mathbb{H}_0^1(\Omega)$.

The Relationship Between Weak and Classical Solutions

Theorem

Let $f \in \mathbb{C}(\overline{\Omega})$ and $\bar{u}_0 \in \mathbb{C}(\partial\Omega)$. If $u \in \mathbb{C}^2(\overline{\Omega})$ is a classical solution of the Dirichlet boundary value problem of the Poisson equation, then, it must also be a weak solution of the problem. On the other hand, if u is a weak solution of the Dirichlet boundary value problem of the Poisson equation, and in addition $u \in \mathbb{C}^2(\overline{\Omega})$, then it must also be a classical solution of the problem.

- The classical solution, if exists, is a weak solution, follows directly from the derivation of the variational form of the problem.
- We only need to show the second part of the theorem.

Proof of Weak Solution + $u \in \mathbb{C}^2(\overline{\Omega})$ ⇒ Classical Solution

1. Let u be a weak solution, and $u \in \mathbb{C}^2(\overline{\Omega})$.
2. Since u is a weak solution and $u \in \mathbb{C}^2(\overline{\Omega})$, by the Green's formula:
$$\int_\Omega (\triangle u + f)\, v\, dx = 0, \quad \forall v \in \mathbb{C}_0^\infty(\Omega).$$
3. $\triangle u + f$ is continuous $\Rightarrow -\triangle u = f$, $\forall x \in \Omega$.
4. By the definition of trace, $u|_{\partial\Omega} = \bar{u}_0$ also holds in the classical sense.
5. u is a classical solution of the Dirichlet BVP of the Poisson equation. ∎

Another Variational Form of Dirichlet BVP of the Poisson Equation

- The quadratic functional $J(v) = \frac{1}{2}\, a(v,\ v) - (f,\ v)$ on $\mathbb{H}^1(\Omega)$.
- Its Fréchet differential $J'(u)v = a(u,\ v) - (f,\ v)$.
- The weak form above is simply $J'(u)v = 0$, $\forall v \in \mathbb{H}_0^1(\Omega)$.

Definition

If $u \in \mathbb{V}(\bar{u}_0; \Omega)$ is a minima of the functional $J(\cdot)$ in $\mathbb{V}(\bar{u}_0,\ \Omega)$, meaning

$$J(u) = \min_{v \in \mathbb{V}(\bar{u}_0;\Omega)} J(v),$$

then, u is called a weak solution of the Dirichlet BVP of the Poisson equation. The corresponding functional minimization problem is called a variational form (or weak form) of the Dirichlet BVP of the Poisson Equation.

Equivalence of the Two Variational Forms

Theorem

The weak solutions of the two variational problems are equivalent.

Proof: Let $u \in \mathbb{V}(\bar{u}_0;\Omega)$ be a minima of J in $\mathbb{V}(\bar{u}_0;\Omega)$, then

$$J'(u)v = 0, \ \ \forall v \in \mathbb{H}_0^1(\Omega); \ \Rightarrow \ \ a(u,\, v) = (f,\ v), \ \ \forall v \in \mathbb{H}_0^1(\Omega).$$

Let $u \in \mathbb{V}(\bar{u}_0;\Omega)$ satisfy the above equation. Then, by the symmetry of the bilinear form $a(u,\, v)$, we have

$$J(v) - J(u) = a(u,\, v-u) - (f,\ v-u) + \frac{1}{2}\, a(v-u,\, v-u).$$

Since $v - u \in \mathbb{H}_0^1(\Omega)$, we are lead to

$$J(v) - J(u) = \frac{1}{2}\, a(v-u,\, v-u) \geq 0, \quad \forall v \in \mathbb{V}(\bar{u}_0;\Omega).$$

Therefore, $u \in \mathbb{V}(\bar{u}_0;\Omega)$ is a minima of J in $\mathbb{V}(\bar{u}_0;\Omega)$. ■

Existence and Uniqueness of Weak Solutions

Theorem

Let Ω be a bounded connected domain with Lipschitz continuous $\partial\Omega$. Let $f \in \mathbb{L}^2(\Omega)$. Suppose $\{u_0 \in \mathbb{H}^1(\Omega) : u_0|_{\partial\Omega} = \bar{u}_0\} \neq \emptyset$. Then, the Dirichlet BVP of the Poisson equation has a unique weak solution.

Proof of Existence and Uniqueness Theorem on Weak Solutions

1. Define $F(v) = (f,\ v) - a(u_0,\ v)$ on $\mathbb{V} = \mathbb{H}_0^1(\Omega)$.
2. By the Poincaré-Friedrichs inequality (see Theorem 5.4) that
$$\exists\ const.\ \alpha(\Omega) > 0, \quad s.t. \quad a(v,\ v) \geq \alpha \|v\|_{1,2,\Omega}^2, \quad \forall v \in \mathbb{V},$$
3. By the Lax-Milgram lemma (see Theorem 5.1), the variational problem
$$\begin{cases} \text{Find} \ \ u \in \mathbb{V} \text{ such that} \\ a(u, v) = F(v), \quad \forall v \in \mathbb{V}, \end{cases}$$
has a unique solution.
4. u solves the above problem $\Leftrightarrow$ $u + u_0$ is a weak solution. ■

Derivation of a Variational Form for Neumann BVP of the Poisson Eqn

1. The Neumann BVP of the Poisson equation

$$-\triangle u = f, \ \ \forall x \in \Omega, \qquad \partial_\nu u = g, \ \ \forall x \in \partial\Omega.$$

2. Assume the problem admits a classical solution $u \in \mathbb{C}^2(\overline{\Omega})$.

3. For any test function $v \in \mathbb{C}^\infty(\overline{\Omega})$, by the Green's formula,

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} v \, \partial_\nu u \, dx = \int_\Omega fv \, dx.$$

Derivation of a Variational Form for Neumann BVP of the Poisson Eqn

4. Let $(g,\ v)_{\partial\Omega} = \int_{\partial\Omega} g\, v\, ds$, $a(u,\ v)$ and $(f,\ v)$ as before.

5. By the denseness of $\mathbb{C}^{\infty}(\overline{\Omega})$ in $\mathbb{H}^1(\Omega)$, we are lead to

$$a(u,\ v) = (f,\ v) + (g,\ v)_{\partial\Omega}, \quad \forall v \in \mathbb{H}^1(\Omega).$$

6. u does not have to be in $\mathbb{C}^2(\overline{\Omega})$ to satisfy such a variational equation, $u \in \mathbb{H}^1(\Omega)$ makes sense.

A Variational Form of the Neumann BVP of the Poisson Equation

Definition

$u \in \mathbb{H}^1(\Omega)$ is said to be a weak solution of the Neumann BVP of the Poisson equation, if it satisfies

$$a(u,\ v) = (f,\ v) + (g,\ v)_{\partial\Omega}, \quad \forall v \in \mathbb{H}^1(\Omega),$$

which is called the variational form (or weak form) of the Neumann BVP of the Poisson equation.

1. Obviously, the classical solution, if exists, is a weak solution.
2. Here, both the trial and test function spaces are $\mathbb{H}^1(\Omega)$.
3. If u is a solution, then, $u + const.$ is also a solution.
4. Taking $v \equiv 1$ as a test function, we obtain a necessary condition for the existence of a solution
$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,ds = 0.$$

The Relationship Between Weak and Classical Solutions

Theorem

Let $f \in \mathbb{C}(\overline{\Omega})$ and $g \in \mathbb{C}(\partial\Omega)$. If $u \in \mathbb{C}^2(\overline{\Omega})$ is a classical solution of the Neumann boundary value problem of the Poisson equation, then, it must also be a weak solution of the problem.

On the other hand, if u is a weak solution of the Neumann boundary value problem of the Poisson equation, and in addition $u \in \mathbb{C}^2(\overline{\Omega})$, then it must also be a classical solution of the problem.

- The classical solution, if exists, is a weak solution, follows directly from the derivation of the variational form of the problem.
- We only need to show the second part of the theorem.

Proof of Weak Solution + $u \in \mathbb{C}^2(\overline{\Omega}) \Rightarrow$ Classical Solution

1. Let u be a weak solution, and $u \in \mathbb{C}^2(\overline{\Omega})$.
2. By the Green's formula,
$$\int_\Omega (\triangle u + f)\, v\, dx = 0, \quad \forall v \in \mathbb{C}_0^\infty(\Omega).$$
3. $\triangle u + f$ is continuous $\Rightarrow -\triangle u = f$, $\forall x \in \Omega$.
4. By this and by the Green's formula, we have
$$\int_{\partial\Omega} (\partial_\nu u - g)\, v\, ds = 0, \quad \forall v \in \mathbb{C}^\infty(\overline{\Omega}).$$
5. $(\partial_\nu u - g)$ is continuous $\Rightarrow \partial_\nu u = g$, $\forall x \in \partial\Omega$.
6. u is a classical solution of the Neumann BVP of the Poisson equation. ■

Existence of Weak Solutions for the Neumann BVP of Poisson Eqn.

Theorem

Let Ω be a bounded connected domain with Lipschitz continuous boundary $\partial\Omega$. Let $f \in \mathbb{L}^2(\Omega)$ and $g \in \mathbb{L}^2(\partial\Omega)$ satisfy the relation $\int_\Omega f\,dx + \int_{\partial\Omega} g\,ds = 0$. Let $\mathbb{V}_0 = \left\{u \in \mathbb{H}^1(\Omega) : \int_\Omega u\,dx = 0\right\}$, and $F : \mathbb{V}_0 \to \mathbb{R}$ be defined by $F(v) = (f,\ v) + (g,\ v)_{\partial\Omega}$. Then, the variational problem

$$\begin{cases} \text{Find } \ u \in \mathbb{V}_0 \text{ such that} \\ a(u,v) = F(v), \quad \forall v \in \mathbb{V}_0, \end{cases}$$

has a unique solution, which is a weak solution of the Neumann BVP of the Poisson equation. On the other hand, if u is a weak solution of the Neumann BVP of the Poisson equation, then $\tilde{u} \triangleq u - \frac{1}{\text{meas}\Omega}\int_\Omega u\,dx \in \mathbb{V}_0$ is a solution to the above variational problem.

- The second part of the theorem is left as an exercise.

Proof of the Existence Theorem for the Neumann BVP of Poisson Eqn.

To prove the first part of the theorem, we need to show

- $a(\cdot, \cdot)$ is a continuous, $\mathbb{V}_0$-elliptic bilinear form on $\mathbb{V}_0$.
- $F(v)$ is a continuous linear form on $\mathbb{V}_0$.
- If u is a solution of the variational problem, then, it is also a weak solution of the Neumann BVP of the Poisson equation.

■

The second and third claims above can be verified by definitions, and are left as exercises.

The key to the first claim is to show the $\mathbb{V}_0$-ellipticity of $a(\cdot, \cdot)$ on $\mathbb{V}_0 := \left\{u \in \mathbb{H}^1(\Omega) : \int_\Omega u\, dx = 0\right\}$, *i.e.* $|u|_{1,2,\Omega} \geq \gamma_0 \|u\|_{1,2,\Omega}$, for some constant $\gamma_0 > 0$. In fact, we have the following stronger result.

Poincaré-Friedrichs Inequality on $\mathbb{H}^1(\Omega)$

Theorem

Let Ω be a bounded connected domain with Lipschitz continuous boundary $\partial\Omega$. Then, there exist constants $\gamma_1 \geq \gamma_0 > 0$ such that

$$\gamma_0 \, \|u\|_{1,2,\Omega} \leq \left| \int_\Omega u \, dx \right| + |u|_{1,2,\Omega} \leq \gamma_1 \|u\|_{1,2,\Omega}, \quad \forall u \in \mathbb{H}^1(\Omega).$$

The inequality is also named as the Poincaré-Friedrichs inequality.

Poincaré-Friedrichs Inequality on $\mathbb{H}^1(\Omega)$

Remarks:

1. The Poincaré-Friedrichs inequality given in Theorem 5.4 is on $\mathbb{W}_0^{m,p}(\Omega)$.
2. Another form of the Poincaré-Friedrichs inequality, in which $\left|\int_\Omega u\, dx\right|$ is replaced by $\|u\|_{0,2,\partial\Omega_0}$, is given in Exercise 5.6.
3. The Poincaré-Friedrichs inequality in a more general form on $\mathbb{W}^{m,p}(\Omega)$ can also be given.

Proof of the Poincaré-Friedrichs Inequality on $\mathbb{H}^1(\Omega)$

1. The Schwarz inequality $\Rightarrow$ the second inequality.
2. Assume the first doesn't hold, *i.e.* $\exists \{u_k\} \subset \mathbb{H}^1(\Omega)$, $\|u_k\|_{1,2,\Omega} \equiv 1$, s.t. $\left|\int_\Omega u_k \, dx\right| + |u_k|_{1,2,\Omega} \to 0$ as $k \to 0$.
3. A bounded set in the Hilbert space $\mathbb{H}^1(\Omega)$ is sequentially weakly precompact, and $\mathbb{H}^1(\Omega)$ compactly embeds into $\mathbb{L}^2(\Omega)$.
4. $\exists$ a subsequence $\{u_k\}$, $u \in \mathbb{H}^1(\Omega)$ and $v \in \mathbb{L}^2(\Omega)$, such that
$$u_k \rightharpoonup u, \text{ in } \mathbb{H}^1(\Omega); \qquad u_k \to v, \text{ in } \mathbb{L}^2(\Omega).$$
5. $|u_k|_1 \to 0$ and $\|u_k - v\|_0 \to 0 \Rightarrow \{u_k\}$ is a Cauchy sequence in $\mathbb{H}^1(\Omega)$, therefore, $\|u_k - u\|_1 \to 0 \quad \Rrightarrow \nabla u = 0 \quad \Rightarrow u \equiv C$.
6. $\|u_k\|_1 \equiv 1$, $|u_k|_1 \to 0$, $\|u_k - u\|_0 \to 0 \Rightarrow \|u\|_0 = 1 \Rightarrow C \neq 0$.
7. $\left|\int_\Omega u_k \, dx\right| \to 0$ and $\|u_k - u\|_0 \to 0 \Rightarrow \int_\Omega u \, dx = 0 \Rightarrow$ $C \, \mathrm{meas}(\Omega) = \int_\Omega u \, dx = 0 \Rightarrow C = 0$, a contradiction. $\blacksquare$

Remarks on the Derivation of Variational Forms of a PDE Problem

1. Coercive (or essential) boundary conditions: those appear in the admissible function space of the variational problem.
2. Natural boundary conditions: those appear in the variational equation (or functional) of the variational problem.
3. The underlying function space: determined by the highest order derivatives of the trial function u in $a(\cdot,\cdot)$.
4. The trial function space: all functions in the underlying function space satisfying the coercive boundary condition.
5. The test function space: u in the underlying function space, with $u = 0$ on the coercive boundary.

Remarks on the Derivation of Variational Forms of a PDE Problem

6. The variational equation: obtained by using smooth test functions on the PDE, applying the Green's formula, and coupling the natural boundary condition.

7. Recall the BVPs of the Poisson equation.

8. $-\triangle u = f \quad \Rightarrow \quad \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} v \, \partial_\nu u \, dx = \int_\Omega fv \, dx.$

9. The underlying function space is $\mathbb{H}^1(\Omega)$.

10. $\mathbb{H}^1(\Omega) \hookrightarrow \mathbb{L}^2(\partial\Omega_0)$, $u|_{\partial\Omega_0}$ is well defined in $\mathbb{L}^2(\partial\Omega_0)$, however $u|_{\partial\Omega_0}$ does not appear in the boundary integral, therefore, the Dirichlet boundary condition on $\partial\Omega_0$ is coercive.

Remarks on the Derivation of Variational Forms of a PDE Problem

11. $\partial_\nu u|_{\partial\Omega_1}$, which appears in the boundary integral, is not well defined in $\mathbb{L}^2(\partial\Omega_1)$ in general, therefore the 2nd and 3rd type boundary conditions appear as natural boundaries.
12. The trial function space $\mathbb{V}(\bar{u}_0; \partial\Omega_0)$; the test one $\mathbb{V}(0; \partial\Omega_0)$.
13. The variational equation ($\partial_\nu u = g - \beta u$ on $\partial\Omega_1$):

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega_1} \beta uv \, dx = \int_\Omega fv \, dx + \int_{\partial\Omega_1} gv \, dx.$$

14. The variational form of the problem:

$$\begin{cases} \text{Find } \ u \in \mathbb{V}(\bar{u}_0; \partial\Omega_0) \text{ such that} \\ a(u, v) = F(v), \quad \forall v \in \mathbb{V}(0; \partial\Omega_0), \end{cases}$$

A Mixed Variational Form of the Dirichlet BVP of the Poisson Equation

1. The Poisson equation $-\triangle u = f$ can be transformed into an equivalent system of 1st order PDEs:

$$\begin{cases} p_i = \partial_i u, \quad i = 1, \dots, n, \\ -\sum_{i=1}^{n} \partial_i p_i = f, \end{cases} \qquad x \in \Omega.$$

2. Take test functions $\mathbf{q} = (q_1, \dots, q_n)$, $q_i \in \mathbb{C}^\infty(\overline{\Omega})$, $i = 1, \dots, n$, and $v \in \mathbb{C}^\infty(\overline{\Omega})$.

3. By the Green's formula (applying to the integral of $\nabla u \cdot \mathbf{q}$) and the boundary condition, we see that the underlying function spaces for $\mathbf{p}$ and u are $(\mathbb{H}^1(\Omega))^n$ and $\mathbb{L}^2(\Omega)$ respectively.

A Mixed Variational Form of the Dirichlet BVP of the Poisson Equation

4. let ν be the unit exterior normal, and

$$a(\mathbf{p},\mathbf{q}) = \int_\Omega \mathbf{p}\cdot\mathbf{q}\,dx = \int_\Omega \sum_{i=1}^n p_i\,q_i\,dx,$$

$$b(\mathbf{q},u) = \int_\Omega u\,\mathrm{div}\,(\mathbf{q})\,dx = \int_\Omega u \sum_{i=1}^n \partial_i q_i\,dx,$$

$$G(\mathbf{q}) = \int_{\partial\Omega} \bar{u}_0\,\mathbf{q}\cdot\nu\,ds = \int_{\partial\Omega} \bar{u}_0 \sum_{i=1}^n q_i\,\nu_i\,ds,$$

$$F(v) = -\int_\Omega f\,v\,dx.$$

A Mixed Variational Form of the Dirichlet BVP of the Poisson Equation

5. Since $u \in \mathbb{L}^2(\Omega)$, $u|_{\partial\Omega}$ doesn't make sense in general, the term $\int_{\partial\Omega} \bar{u}_0 \, \mathbf{q} \cdot \nu \, ds$ should be kept in the variational equation, *.i.e.* the Dirichlet boundary condition appears as a natural boundary condition in this case.

6. Thus, we are led to the following variational problem:

$$\begin{cases} \text{Find } \ \mathbf{p} \in (\mathbb{H}^1(\Omega))^n, \ u \in \mathbb{L}^2(\Omega) \text{ such that} \\ a(\mathbf{p}, \mathbf{q}) + b(\mathbf{q}, u) = G(\mathbf{q}), \quad \forall \mathbf{q} \in (\mathbb{H}^1(\Omega))^n, \\ \qquad\qquad b(\mathbf{p}, v) = F(v), \quad \forall\, v \in \mathbb{L}^2(\Omega). \end{cases}$$

Remark: Neumann boundary condition will appear as a coercive boundary condition. (Robin boundary condition does not apply here. Why?)

Another Mixed Variational Form of the Dirichlet BVP of the Poisson Equation

1. If we apply the Green's formula to transform $-\int_\Omega \mathrm{div}\mathbf{p}\, v\, dx$ into the form $-\int_{\partial\Omega} v\,\mathbf{p}\cdot\nu\, ds + \int_\Omega \mathbf{p}\cdot\nabla v\, dx$ instead, then, the underlying function spaces for $\mathbf{p}$ and u are $(\mathbb{L}^2(\Omega))^n$ and $\mathbb{H}^1(\Omega)$ respectively.

2. Since $u \in \mathbb{H}^1(\Omega)$, $u|_\Omega$ makes sense, while $\mathbf{p} \in (\mathbb{L}^2(\Omega))^n$, the term $-\int_{\partial\Omega} v\,\mathbf{p}\cdot\nu\, ds$ doesn't make sense. Therefore, the Dirichlet boundary condition appears as a coercive boundary condition, while the Neumann and Robin boundary conditions appear as natural boundary conditions here in this case.

Another Mixed Variational Form of the Dirichlet BVP of the Poisson Equation

3. We are lead to the variational problem

$$\begin{cases} \text{Find } \ \mathbf{p} \in (\mathbb{L}^2(\Omega))^n, \ u \in \mathbb{H}^1(\Omega), \ u|_{\partial\Omega} = \bar{u}_0 \text{ such that} \\ a(\mathbf{p}, \mathbf{q}) + b^*(\mathbf{q}, u) = 0, \qquad \forall \mathbf{q} \in (\mathbb{L}^2(\Omega))^n, \\ \qquad b^*(\mathbf{p}, v) = F(v), \qquad \forall\, v \in \mathbb{H}_0^1(\Omega), \end{cases}$$

where

$$b^*(\mathbf{q}, u) = -\int_\Omega \mathbf{q} \cdot \nabla u \, dx = -\int_\Omega \sum_{i=1}^n q_i \partial_i u \, dx.$$

Remarks on the Mixed Variational Forms of BVP of the Poisson Eqn.

1. The classical solution is also a solution to the mixed variational problem (named again as the weak solution).
2. Weak solution $+\ u \in \mathbb{C}^2(\bar{\Omega}),\ \mathbf{p} \in (\mathbb{C}^1(\bar{\Omega}))^n \quad \Rightarrow \quad u$ is a classical solution.
3. The weak mixed forms have their corresponding functional extremum problems.
4. Under the so called B-B conditions, the weak mixed variational problems can be shown to have a unique stable solution.

习题 5：7，8，12(3)

Thank You!