Numerical Solutions to Partial Differential Equations

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Finite Element Method — a Method Based on Variational Problems

Finite Difference Method:

1. Based on PDE problem.
2. Introduce a grid (or mesh) on $\overline{\Omega}$.
3. Define grid function spaces.
4. Approximate differential operators by difference operators.
5. PDE discretized into a finite algebraic equation.

Finite Element Method:

1. Based on variational problem, say $F(u) = \inf_{v \in \mathbb{X}} F(v)$.
2. Introduce a grid (or mesh) on $\overline{\Omega}$.
3. Establish finite dimensional subspaces $\mathbb{X}_h$ of $\mathbb{X}$.
4. Restrict the original problem on the subspaces, say $F(U_h) = \inf_{V_h \in \mathbb{X}_h} F(V_h)$.
5. PDE discretized into a finite algebraic equation.

An Abstract Variational Form of Energy Minimization Problem

Many physics problems, such as minimum potential energy principle in elasticity, lead to an abstract variational problem:

$$\begin{cases} \text{Find} \quad u \in \mathbb{U} \text{ such that} \\ J(u) = \inf\limits_{v \in \mathbb{U}} J(v), \end{cases}$$

where $\mathbb{U}$ is a nonempty closed subset of a Banach space $\mathbb{V}$, and $J : v \in \mathbb{U} \to \mathbb{R}$ is a functional. In many practical linear problems,

- $\mathbb{V}$ is a Hilbert space, $\mathbb{U}$ a closed linear subspace of $\mathbb{V}$;
- the functional J often has the form

$$J(v) = \frac{1}{2}\, a(v,\ v) - f(v),$$

- $a(\cdot,\ \cdot)$ and f are continuous bilinear and linear functionals.

Find Solutions to a Functional Minimization Problem

Method 1 — Direct method of calculus of variations:

1. Find a minimizing sequence, say, by gradient type methods;
2. Find a convergent subsequence of the minimizing sequence, say, by certain kind of compactness;
3. Show the limit is a minimizer, say, by lower semi-continuity of the functional.

Find Solutions to a Functional Minimization Problem

Method 2 — Solving the Euler-Lagrange equation:

1. Work out the corresponding Euler-Lagrange equation;
2. For smooth solutions, the Euler-Lagrange equation leads to classical partial differential equations;
3. In general, the Euler-Lagrange equation leads to another form of variational problems (weak form of classical partial differential equations).

Both methods involve the derivatives of the functional J.

Fréchet Derivatives of Maps on Banach Spaces

Let $\mathbb{X}$, $\mathbb{Y}$ be real normed linear spaces, Ω is an open set of $\mathbb{X}$. Let $F : \Omega \to \mathbb{Y}$ be a map, nonlinear in general.

Definition

F is said to be Fréchet differentiable at $x \in \Omega$, if there exists a linear map $A : \mathbb{X} \to \mathbb{Y}$ satisfying: for any $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$\|F(x+z) - F(x) - Az\| \leq \varepsilon \|z\|, \quad \forall z \in \mathbb{X} \text{ with } \|z\| \leq \delta.$$

The map A is called the Fréchet derivative of F at x, denoted as $F'(x) = A$, or $dF(x) = A$. $F'(x)z = Az$ is called the Fréchet differential of F at x, or the first order variation.

The Fréchet differential is an extension of total differential in the multidimensional calculus.

Higher Order Fréchet Derivatives

Definition

If for any $z \in \mathbb{X}$, $F'(x)z$ is Fréchet differentiable at $x \in \Omega$, F is said to be second order Fréchet differentiable at $x \in \Omega$.

The second order Fréchet derivative of F at x is a $\mathbb{X} \times \mathbb{X} \to \mathbb{Y}$ bilinear form, denoted as $F''(x)$ or $d^2F(x)$.

$F''(x)(z, y) = d^2F(x)(z, y) = (F'(x)z)'y$ is called the second order Fréchet differential of F at x, or the second order variation.

Recursively, we can define the mth order Fréchet derivative of F at x by $d^mF(x) \triangleq d(d^{m-1}F(x))$, and the mth order Fréchet differential (or the mth order variation) $d^mF(x)(z_1, \ldots, z_m)$.

The mth order Fréchet derivative $d^mF(x)$ is said to be bounded, if $d^mF(x)(z_1, \ldots, z_m) : \mathbb{X}^m \to \mathbb{Y}$ is a bounded m linear map.

Gâteaux Derivatives — An Extension of Directional Derivatives

Definition

F is said to be Gâteaux differentiable at $x \in \Omega$ in the direction $z \in \mathbb{X}$, if the following limit exists:

$$DF(x; z) = \lim_{t \to 0} \frac{F(x + tz) - F(x)}{t}.$$

$DF(x; z)$ is called the Gâteaux differential of F at x in the direction $z \in \mathbb{X}$. if the map $DF(x; z)$ is linear with respect to z, *i.e.* there exists a linear map $A : \mathbb{X} \to \mathbb{Y}$ such that $DF(x; z) = Az$, then the map A is called the Gâteaux derivative of F at x, and is denoted as $DF(x) = A$.

Gâteaux Derivatives — An Extension of Directional Derivatives

- The Gâteaux derivative is an extension of the directional directives in the multidimensional calculus;

- Fréchet differentiable implies Gâteaux differentiable, the inverse is not true in general.

Higher Order Gâteaux Derivatives

Definition

If for a given $z \in \mathbb{X}$, $DF(x; z)$ is Gâteaux differentiable at $x \in \Omega$ in the direction $y \in \mathbb{X}$, then the corresponding differential is called the second order mixed Gâteaux differential of F at x in the directions z and y, and is denoted as $D^2F(x; z, y)$.

If $D^2F(x; z, y)$ is bilinear with respect to (z, y), then the bilinear form $D^2F(x)$, with $D^2F(x)(z, y) \triangleq D^2F(x; z, y)$, is called the second order Gâteaux derivative of F at x.

We can recursively define the mth order mixed Gâteaux differential $D^mF(x; z_1, \dots, z_m) \triangleq D(D^{m-1}F)(x; z_1, \dots, z_{m-1}; z_m)$, and the mth order Gâteaux derivative $D^mF(x) \triangleq D(D^{m-1}F)(x)$.

Higher Order Gâteaux Derivatives — Commutability

1. If the Gâteaux differential $DF(\cdot)$ of F exists in a neighborhood of x and is continuous at x, then, the Fréchet differential of F at x exists and $dF(x)z = DF(x)z = \frac{d}{dt}F(x+tz)\Big|_{t=0}$.

   (notice that $F(x+z) - F(x) = \int_0^1 \frac{d}{dt}F(x+tz)\,dt$).

2. In general, $D^2F(x;z,y) \neq D^2F(x;y,z)$, *i.e.* the map is not necessarily symmetric with respect to (y,z).

   (counter examples can be found in multi-dimensional calculus).

Higher Order Gâteaux Derivatives — Commutability

3. If the mth order Gâteaux differential $D^m F(\cdot)$ is a uniformly bounded m linear map in a neighborhood of x_0 and is uniformly continuous with respect to x, then $D^m F(\cdot)$ is indeed symmetric with respect to $(z_1, \ldots, z_m)$,

   in addition the mth order Fréchet differential exists and

   $F^{(m)}(x_0) = d^m F(x_0) = D^m F(x_0)$ with

$$F^{(m)}(x)(z_1, \ldots, z_m) = \frac{d}{dt_m}\Big[\cdots\Big[\frac{d}{dt_1} F(x + t_1 z_1 + \cdots + t_m z_m)\Big|_{t_1=0}\Big]\cdots\Big]\Big|_{t_m=0}.$$

A Necessary Condition for a Functional to Attain an Extremum at x

Let $F : \mathbb{X} \to \mathbb{R}$ be Fréchet differentiable, and F attains a local extremum at x. Then

1. For fixed $z \in \mathbb{X}$, $f(t) \triangleq F(x + tz)$, as a differentiable function of $t \in \mathbb{R}$, attains a same type of local extremum at $t = 0$.
2. Hence, $F'(x)z = f'(0) = 0$, $\forall z \in \mathbb{X}$.
3. Therefore, a necessary condition for a Fréchet differentiable functional F to attain a local extremum at x is
$$F'(x)z = 0, \quad \forall z \in \mathbb{X},$$
which is called the weak form (or variational form) of the Euler-Lagrange equation $F'(x) = 0$ of the extremum problem.

A Typical Example on Energy Minimization Problem

1. $J(v) = \frac{1}{2}\, a(v,\ v) - f(v)$. $a(\cdot,\cdot)$ symmetric, a, f continuous.
2. $t^{-1}(J(u+tv) - J(u)) = a(u,v) - f(v) + \frac{t}{2} a(v,v)$.

   (Since $a(u+tv, u+tv) = a(u,u) + t(a(u,v) + a(v,u)) + t^2 a(v,v)$ and $f(u+tv) = f(u) + tf(v)$.)
3. Gâteaux differential $DJ(u)v = a(u,v) - f(v)$.
4. Continuity $\Rightarrow$ Fréchet differential $J'(u)v = a(u,v) - f(v)$.
5. $t^{-1}(J'(u+tw,v) - J'(u,v)) = a(w,v)$.
6. $J''(u)(v,w) = a(w,v)$. $J^{(k)}(u) = 0$, for all $k \geq 3$.

A Typical Example on Energy Minimization Problem

7. Suppose that $u \in \mathbb{U}$ satisfies $J'(u)v = 0$, $\forall v \in \mathbb{U}$. Then
8. $J(u+tv) = J(u) + tJ'(u)v + \frac{t^2}{2}J''(u)(v,v) = J(u) + \frac{t^2}{2}a(v,v)$.
9. If, in addition, $\exists$ const. $\alpha > 0$, s.t. $a(v,v) \geq \alpha\|v\|^2$, $\forall v \in \mathbb{U}$, then $J(u+tv) \geq J(u) + \frac{1}{2}\alpha t^2\|v\|^2$.

Under the conditions that $a(\cdot,\cdot)$ is a symmetric, continuous and uniformly elliptic bilinear form, and f is a continuous linear form,

$$u \textbf{ is the unique minimum of } J \Leftrightarrow J'(u) = 0.$$

Abstract Variational Problem Corresponding to the Virtual Work Principle

Various forms of variational principles, such as the virtual work principle in elasticity, etc., lead to the following abstract variational problem:

$$\begin{cases} \text{Find} \quad u \in \mathbb{V} \text{ such that} \\ A(u)v = 0, \quad \forall v \in \mathbb{V}, \end{cases}$$

where $A \in \mathfrak{L}(\mathbb{V}; \mathbb{V}^*)$, *i.e.* $A(\cdot)$ is a linear map from $\mathbb{V}$ to its dual space $\mathbb{V}^*$.

- In an energy minimization problem, a necessary condition for $u \in \mathbb{U}$ to be a minimizer is that $J'(u)v = 0$, $\forall v \in \mathbb{U}$.
- In the case when $a(\cdot, \cdot)$ is uniformly elliptic, the two problems are equivalent.

Lax-Milgram Lemma — Existence and Uniqueness of a Solution

Theorem

Let $\mathbb{V}$ be a Hilbert space. let $a(\cdot,\ \cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ be a continuous bilinear form satisfying the $\mathbb{V}$-elliptic condition (also known as the coerciveness condition):

$$\exists \alpha > 0, \quad \textit{such that} \ \ a(u,u) \geq \alpha \|u\|^2, \quad \forall u \in \mathbb{V},$$

$f : \mathbb{V} \to \mathbb{R}$ be a continuous linear form. Then, the abstract variational problem

$$\begin{cases} \textit{Find} \ \ u \in \mathbb{V} \ \textit{such that} \\ a(u,v) = f(v), \quad \forall v \in \mathbb{V}, \end{cases}$$

has a unique solution.

Proof of the Lax-Milgram Lemma

1. Continuity of $a(\cdot,\cdot) \Rightarrow \exists$ const. $M > 0$ such that
$$a(u,v) \le M\|u\|\|v\|, \quad \forall u,\ v \in \mathbb{V}.$$
2. $v \in \mathbb{V} \to a(u,v)$ continuous linear $\Rightarrow \exists|\ A(u) \in \mathbb{V}^*$ such that
$$A(u)v = a(u,v), \quad \forall v \in \mathbb{V}.$$
3. $\|A\|_{\mathfrak{L}(\mathbb{V},\mathbb{V}^*)} = \sup_{u\in\mathbb{V},\|u\|=1} \sup_{v\in\mathbb{V},\|v\|=1} |A(u)v| \le M.$
4. $\tau : \mathbb{V}^* \to \mathbb{V}$, the Riesz map: $f(v) = \langle \tau f,\ v\rangle, \forall v \in \mathbb{V}.$

Proof of the Lax-Milgram Lemma (continue)

5. The abstract variational problem is equivalent to

$$\begin{cases} \text{Find } \ u \in \mathbb{V} \text{ such that} \\ \tau A(u) = \tau f. \end{cases}$$

6. Define $F : \mathbb{V} \to \mathbb{V}$ as $F(v) = v - \rho(\tau A(v) - \tau f)$.
7. Then, u is a solution $\Leftrightarrow$ $F(u) = u$. (*i.e.* u is a fix point of F.)
8. Since $\langle \tau A(v),\ v\rangle = A(v)v = a(v, v) \geq \alpha \|v\|^2$,
9. $\|\tau A(v)\| = \|A(v)\|^* \leq \|A\|_{\mathfrak{L}(\mathbb{V},\mathbb{V}^*)}\|v\| \leq M\|v\|$, and
10. $\|F(w + v) - F(w)\|^2 = \|v\|^2 - 2\rho\langle \tau A(v),\ v\rangle + \rho^2\|\tau A(v)\|^2$,

    ($\because$ $F(w + v) = w + v - \rho(\tau A(w + v) - \tau f) = F(w) + v - \rho\tau A(v)$,)

Proof of the Lax-Milgram Lemma (continue)

10. therefore, for any given $\rho \in (0,\, 2\alpha/M^2)$, we have

$$\|F(w+v) - F(w)\|^2 \leq (1 - 2\rho\alpha + \rho^2 M^2)\|v\|^2 < \|v\|^2,$$

11. $F : \mathbb{V} \to \mathbb{V}$ is a contractive map, for $\rho \in (0,\, 2\alpha/M^2)$.
12. In addition, if $\|v\| > (2\alpha - M^2\rho)^{-1}\|f\|$, then $\|F(v)\| < \|v\|$.
13. By the contractive-mapping principle, F has a unique fixed point in $\mathbb{V}$. ■

Remark: In applications, the Hilbert space $\mathbb{V}$ in the variational problem usually consists of functions with derivatives in some weaker sense. Sobolev spaces are important in studying variational forms of PDE and the finite element method.

Definition of Generalized Derivatives for Functions in $\mathbb{L}^1_{loc}(\Omega)$

Let $u \in \mathbb{C}^m(\Omega)$, then, for any $\phi \in \mathbb{C}_0^\infty(\Omega)$, it follows from the Green's formula that

$$\int_\Omega (\partial^\alpha u)\,\phi\,dx = (-1)^{|\alpha|}\int_\Omega u\,(\partial^\alpha \phi)\,dx.$$

Definition

Let $u \in \mathbb{L}^1_{loc}(\Omega)$, if there exists $v_\alpha \in \mathbb{L}^1_{loc}(\Omega)$ such that

$$\int_\Omega v_\alpha\,\phi\,dx = (-1)^{|\alpha|}\int_\Omega u\,(\partial^\alpha \phi)\,dx, \qquad \forall \phi \in \mathbb{C}_0^\infty(\Omega),$$

then v_α is called a $|\alpha|$th order generalized partial derivative (or weak partial derivative) of u with respect to the multi-index α, and is denoted as $\partial^\alpha u = v_\alpha$.

An Important Property of Generalized Derivatives

The concept of the generalized derivatives are obviously an extension of that of the classical derivatives.

In addition, the generalized derivatives also inherit some important properties of the classical derivatives. In particular, we have

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a connected open set. Let all of the generalized partial derivatives of order $|\alpha| = m + 1$ of u are zero, then, u is a polynomial of degree no greater than m on Ω.

An Important Property of Generalized Derivatives

Remark: Two functions in $\mathbb{L}^1_{loc}(\Omega)$ are considered to be the same (or in the same equivalent class of functions), if they are different only on a set of zero measure.

The theorem above is understood in the sense that there exists a representative in the equivalent class of u such that the conclusion holds.

Definition of the Sobolev Spaces

Definition

Let m be a nonnegative integer, let $1 \le p \le \infty$, define

$$\mathbb{W}^{m,p}(\Omega) = \{u \in \mathbb{L}^p(\Omega) : \partial^\alpha u \in \mathbb{L}^p(\Omega),\ \forall \alpha \text{ s.t. } 0 \le |\alpha| \le m\},$$

where $\mathbb{L}^p(\Omega)$ is the Banach space consists of all Lebesgue p integrable functions on Ω with norm $\|\cdot\|_{0,p,\Omega}$. Then, the set $\mathbb{W}^{m,p}(\Omega)$ endowed with the following norm

$$\|u\|_{m,p,\Omega} = \Big(\sum_{0\le|\alpha|\le m} \|\partial^\alpha u\|_{0,p,\Omega}^p\Big)^{1/p}, \quad 1 \le p < \infty;$$

$$\|u\|_{m,\infty,\Omega} = \max_{0\le|\alpha|\le m} \|\partial^\alpha u\|_{0,\infty,\Omega}$$

is a normed linear space, and is called a Sobolev space, denoted again as $\mathbb{W}^{m,p}(\Omega)$.

Some Basic Inequalities of $\mathbb{L}^p(\Omega)$ Functions

The following inequalities are very important for analysis in the Sobolev spaces.

Minkowski inequality: For any $1 \leq p \leq \infty$ and f, $g \in \mathbb{L}^p(\Omega)$,

$$\|f+g\|_{0,p,\Omega} \leq \|f\|_{0,p,\Omega} + \|g\|_{0,p,\Omega}.$$

Hölder inequality: Let $1 \leq p,\ q \leq \infty$ satisfy $1/p + 1/q = 1$, then, for any $f \in \mathbb{L}^p(\Omega)$ and $g \in \mathbb{L}^q(\Omega)$, we have $f \cdot g \in \mathbb{L}^1(\Omega)$, and

$$\|f \cdot g\|_{0,1,\Omega} \leq \|f\|_{0,p,\Omega} \|g\|_{0,q,\Omega}.$$

Cauchy-Schwarz inequality: In particular, for $p = q = 2$, it follows from the Hölder inequality that

$$\|f \cdot g\|_{0,1,\Omega} \leq \|f\|_{0,2,\Omega} \|g\|_{0,2,\Omega}.$$

Some important Facts of Sobolev Spaces

- $\mathbb{W}^{m,p}(\Omega)$ is a Banach space.
- If $p = 2$, $\mathbb{W}^{m,p}(\Omega)$ is a Hilbert space, denoted as $\mathbb{H}^m(\Omega)$, and its norm is often denoted as $\|\cdot\|_{m,\Omega}$.

Theorem

If the boundary $\partial\Omega$ of the domain Ω is Lipschitz continuous, then, for $1 \leq p < \infty$, $\mathbb{C}^\infty(\overline{\Omega})$ is dense in $\mathbb{W}^{m,p}(\Omega)$.

- $\mathbb{W}^{m,p}(\Omega)$ is a closure of $\mathbb{C}^\infty(\overline{\Omega})$ w.r.t the norm $\|\cdot\|_{m,p}$.

Some important Facts of Sobolev Spaces

Definition

The closure of $\mathbb{C}_0^\infty(\Omega)$ w.r.t. the norm $\|\cdot\|_{m,p}$ is a subspace of the Sobolev space $\mathbb{W}^{m,p}(\Omega)$, and is denoted as $\mathbb{W}_0^{m,p}(\Omega)$.

- $\mathbb{H}_0^m(\Omega) \triangleq \mathbb{W}_0^{m,2}(\Omega)$ is a Hilbert space.

Poincaré-Friedrichs Inequality

Theorem

Let the domain Ω *be of finite width,* i.e. *it is located between two parallel hyperplanes. Then, there exist a constant* $K(n, m, d, p)$*, which depends only on the space dimension* n*, the order* m *of the partial derivatives, the distance* d *between the two hyperplanes and the Sobolev index* $1 \le p < \infty$*, such that*

$$|u|_{m,p} \le \|u\|_{m,p} \le K(n, m, d, p)|u|_{m,p}, \quad \forall u \in W_0^{m,p}(\Omega),$$

where
$$|u|_{m,p} = \Big(\sum_{|\alpha|=m} \|\partial^\alpha u\|_{0,p,\Omega}^p \Big)^{1/p}, \quad 1 \le p < \infty$$

is a semi-norm of the Sobolev space $\mathbb{W}^{m,p}(\Omega)$*. The inequality is usually called the Poincaré-Friedrichs inequality.*

Proof of the Poincaré-Friedrichs Inequality

1. Assume the domain Ω is between $x_n = 0$ and $x_n = d$.

2. Denote $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1})$. For any given $u \in \mathbb{C}_0^\infty(\Omega)$, we have $u(x) = \int_0^{x_n} \frac{d}{dt} u(x', t)\, dt$.

3. For $p' = \frac{p}{p-1}$, by the Hölder inequality,

$$|u(x)| = \left| \int_0^{x_n} \partial_n u(x', t)\, dt \right| \leq \left(\int_0^{x_n} 1^{p'} \right)^{1/p'} \left(\int_0^{x_n} |\partial_n u(x', t)|^p \right)^{1/p}$$

Proof of the Poincaré-Friedrichs Inequality

4.

$$\|u\|_{0,p,\Omega}^p = \int_{R^{n-1}} \int_0^d |u(x)|^p dx_n dx'$$
$$\leq \int_0^d x_n^{p-1} dx_n \int_{R^{n-1}} \int_0^d |\partial_n u(x',t)|^p dt\, dx' \leq (d^p/p)|u|_{1,p,\Omega}^p.$$

5. $\|u\|_{1,p,\Omega} \leq \|u\|_{0,p,\Omega} + |u|_{1,p,\Omega} \leq K(d,p)|u|_{1,p,\Omega}$, $\forall u \in \mathbb{C}_0^\infty(\Omega)$.

6. $|u|_{m,p} \leq \|u\|_{m,p} \leq K(n,m,d,p)|u|_{m,p}$, $\forall u \in \mathbb{C}_0^\infty(\Omega)$.

7. For $u \in \mathbb{W}_0^{m,p}(\Omega)$, recall that $\mathbb{C}_0^\infty(\Omega)$ is dense in $\mathbb{W}_0^{m,p}(\Omega)$. ■

(5) used the inequality $a^p + b^p \leq (a+b)^p$ for $a, b \geq 0$ and $p \geq 1$; while (6) used induction.

Embedding Operator and Embedding Relation of Banach Spaces

1. $\mathbb{X}$, $\mathbb{Y}$: Banach spaces with norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$.
2. If $x \in \mathbb{X} \Rightarrow x \in \mathbb{Y}$, & $\exists$ const. $C > 0$ independent of x s.t. $\|x\|_{\mathbb{Y}} \leq C\|x\|_{\mathbb{X}}$, $\forall x \in \mathbb{X}$, then the identity map $I : \mathbb{X} \to \mathbb{Y}$, $I\,x = x$ is called an embedding operator, and the corresponding embedding relation is denoted by $\mathbb{X} \hookrightarrow \mathbb{Y}$.
3. The embedding operator $I : \mathbb{X} \to \mathbb{Y}$ is a bounded linear map.
4. If, in addition, I is happened to be a compact map, then, the corresponding embedding is called a compact embedding, and is denoted by $\mathbb{X} \overset{c}{\hookrightarrow} \mathbb{Y}$.

Some embedding relations exist in Sobolev spaces, which play an very important role in the theory of partial differential equations and finite element analysis.

The Sobolev Embedding Theorem

Theorem

Let Ω be a bounded connected domain with a Lipschitz continuous boundary $\partial\Omega$, then

$$\begin{array}{lll} \mathbb{W}^{m+k,p}(\Omega) \hookrightarrow \mathbb{W}^{k,q}(\Omega), & \forall\, 1 \le q \le \dfrac{np}{n-mp},\ k \ge 0, & \text{if } m < n/p; \\ \mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega), & \forall\, 1 \le q < \dfrac{np}{n-mp},\ k \ge 0, & \text{if } m < n/p; \\ \mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{W}^{k,q}(\Omega), & \forall\, 1 \le q < \infty,\ k \ge 0, & \text{if } m = n/p; \\ \mathbb{W}^{m+k,p}(\Omega) \overset{c}{\hookrightarrow} \mathbb{C}^{k}(\overline{\Omega}), & \forall\, k \ge 0, & \text{if } m > n/p. \end{array}$$

Remark: The last embedding relation implies that for every u in $\mathbb{W}^{m+k,p}(\Omega)$, there is a $\tilde{u} \in \mathbb{C}^k(\overline{\Omega})$ such that $u - \tilde{u} = 0$ almost everywhere.

习题 5：2，3，6.

Thank You!