Itô Formula

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Theorem (Itô 公式 I)

若F为二次连续可微，则

$$F(B_t) - F(B_0) = \int_0^t F'(B_s)dB_s + \frac{1}{2}\int_0^t F''(B_s)ds.$$

其形式地记为

$$dF(B_t) = F'(B_t)dB_t + \frac{1}{2}F''(B_t)dt.$$

Theorem (Itô 公式 II)

设$F(t,x)$ 关于t 一次连续可微，关于x 二次连续可微，则

$$\begin{aligned} & F(t,B_t)-F(0,B_0) \\ = & \int_0^t \frac{\partial F(s,B_s)}{\partial s}ds+\int_0^t \frac{\partial F(s,B_s)}{\partial x}dB_s+\frac{1}{2}\int_0^t \frac{\partial^2 F(s,B_s)}{\partial x^2}ds \\ = & \int_0^t F_s'(s,B_s)ds+\int_0^t F_x'(s,B_s)dB_s+\frac{1}{2}\int_0^t F_{xx}''(s,B_s)ds \end{aligned}$$

其形式地记为

$$dF(t,B_t)=F_t'(t,B_t)dt+\frac{1}{2}F_{xx}''(t,B_t)dt+F_x'(t,B_t)dB_t$$

Theorem (Itô 公式 III)

设$X_t = x_0 + \int_0^t \phi_s dB_s + \int_0^t \psi_s ds$, 即$dX_t = \phi_t dB_t + \psi_s ds$;
$F(t,x)$关于t 一次连续可微，关于x 二次连续可微，则

$$F(t, X_t) - F(0, x_0) = \int_0^t \frac{\partial F}{\partial s}(s, X_s)ds + \int_0^t \frac{\partial F}{\partial x}(t, X_s)dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s)(dX_s)^2.$$

其形式地记为

$$dF(t, X_t) = \frac{\partial F}{\partial t}(t, X_t)dt + \frac{\partial F}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t, X_t)(dX_t)^2,$$

其中dt, dB_t 的平方和乘积由如下的乘法表给出

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Theorem (Itô 公式 III)

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乘积	dt	dB_t
dt	0	0
dB_t	0	dt

,

即

$$
\begin{aligned}
dF(t,X_t) &= \left[\frac{\partial F}{\partial t}(t,X_t) + \frac{\partial F}{\partial x}(t,X_t)\psi_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,X_t)\phi_t^2\right]dt \\
&\quad + \frac{\partial F}{\partial x}(t,X_t)\phi_t dB_t.
\end{aligned}
$$

写成真正的积分形式

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Theorem (Itô 公式 III)

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$$
\begin{aligned}
& F(t, X_t) - F(0, x_0) \\
= \ & \int_0^t \left[\frac{\partial F}{\partial s}(s, X_s) + \frac{\partial F}{\partial x}(s, X_s)\psi_s + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(s, X_s)\phi_s^2\right] ds \\
& + \int_0^t \frac{\partial F}{\partial x}(s, X_s)\phi_s dB_s.
\end{aligned}
$$

设$\mathbb{B}_t = \begin{pmatrix} B_t^{(1)} \\ \vdots \\ B_t^{(m)} \end{pmatrix}$为一个$m$-维Brown运动,

多维Brown运动的Itô过程

$$\xi_t = x + \int_0^t \Phi_s^T d\mathbb{B}_s + \int_0^t \psi_s ds,$$

其中$\Phi_t = \begin{pmatrix} \phi_t^{(1)} \\ \vdots \\ \phi_t^{(m)} \end{pmatrix}$,$\xi_t = x + \sum_{k=1}^{m} \int_0^t \phi_s^{(k)} dB_s^{(k)} + \int_0^t \psi_s ds,$

而$(\phi_t^{(i)})$, $(\psi_t)_t$ 分别是$\mathcal{L}_T^{2,loc}$, $\mathcal{L}_T^{1,loc}$ 中的元，那么称ξ_t 为初值为x的多维B.M. 的Itô 过程.

Theorem (Itô 公式 IV)

$$d\xi_t = \phi_t^T d\mathbb{B}_t + \psi_t dt = \sum_{i=1}^{m} \phi_t^{(i)} dB_t^{(i)} + \psi_t dt,$$

$F(t,x)$关于t 一次连续可微，关于x 二次连续可微，则

$$\begin{aligned} dF(t,\xi_t) &= \frac{\partial F(t,\xi_t)}{\partial t}dt + \frac{\partial F(t,\xi_t)}{\partial x}d\xi_t \\ &\quad + \frac{1}{2}\frac{\partial^2 F(t,\xi_t)}{\partial x^2}(d\xi_t)^2, \end{aligned}$$

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Theorem (Itô 公式 IV)

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$$F(t,\xi_t)-F(0,\xi_0) = \int_0^t \frac{\partial F(s,\xi_s)}{\partial s}ds+\int_0^t \frac{\partial F(s,\xi_s)}{\partial x}d\xi_s + \frac{1}{2}\int_0^t \frac{\partial^2 F(s,\xi_s)}{\partial x^2}(d\xi_s)^2$$

其中dt, $dB_t^{(i)}$的平方和乘积由如下的乘法表给出

乘积	dt	$dB_t^{(i)}$
dt	0	0
$dB_t^{(j)}$	0	$\delta_{ij}dt$

,

$$\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}.$$

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Theorem (Itô 公式 IV)

(接上页) 即

$$
\begin{aligned}
&F(t,\xi_t) - F(0,\xi_0) \\
= \; & \int_0^t \Big[\frac{\partial F(s,\xi_s)}{\partial s} + \frac{\partial F(s,\xi_s)}{\partial x}\psi_s + \frac{1}{2}\frac{\partial^2 F(s,\xi_s)}{\partial x^2}\sum_{i=1}^m (\phi_s^{(i)})^2\Big] ds \\
& + \sum_{i=1}^m \int_0^t \frac{\partial F(s,\xi_s)}{\partial x}\phi_s^{(i)} dB_s^{(i)}.
\end{aligned}
$$

Corollary (乘积公式)

ξ_t, η_t 都是多维*B.M.*, $\{\mathbb{B}_t,\ t \geq 0\}$的*Itô* 过程，则

$$d(\xi_t \eta_t) = \xi_t d\eta_t + \eta_t d\xi_t + (d\xi_t)(d\eta_t).$$

进一步，若$\xi_t = (\xi_t^{(1)}, \xi_t^{(2)}, \cdots, \xi_t^{(d)})^T$, $(\xi_t^{(i)})$, $i \le d$ 都是同一个m-维Brown运动$\mathbb{B}_t = \begin{pmatrix} B_t^{(1)} \\ \vdots \\ B_t^{(m)} \end{pmatrix}$ 的Itô过程，即

$$d\xi_t^{(i)} = (\Phi_t^{(i)})^T d\mathbb{B}_t + \psi_t^{(i)} dt, \quad 1 \le i \le d,$$

$$\begin{pmatrix} d\xi_t^{(1)} \\ \vdots \\ d\xi_t^{(d)} \end{pmatrix} = \begin{pmatrix} \phi_t^{11} & \phi_t^{12} & \cdots & \phi_t^{1m} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_t^{d1} & \phi_t^{d2} & \cdots & \phi_t^{dm} \end{pmatrix} \begin{pmatrix} dB_t^{(1)} \\ \vdots \\ dB_t^{(m)} \end{pmatrix} + \begin{pmatrix} \psi_t^{(1)} \\ \vdots \\ \psi_t^{(d)} \end{pmatrix} dt,$$

$$d\xi = \Phi_t^T \, d\mathbb{B}_t + \Psi \, dt,$$

$$\Phi = (\Phi^{(1)}, \cdots, \Phi^{(d)}), \quad \Phi^{(i)} = \begin{pmatrix} \phi^{i1} \\ \vdots \\ \phi^{im} \end{pmatrix}, \quad 1 \le i \le d.$$

Theorem (Itô 公式 V)

$F(x_1,\cdots,x_d)$ 为一个二次连续可微的函数

$$\begin{aligned} & dF(\xi_t^{(1)},\cdots,\xi_t^{(d)}) \\ = & \sum_{k=1}^{d}\frac{\partial}{\partial x_k}F(\xi_t^{(1)},\cdots,\xi_t^{(d)})d\xi_t^{(k)} \\ & +\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial^2}{\partial x_i\partial x_j}F(\xi_t^{(1)},\cdots,\xi_t^{(d)})d\xi_t^{(i)}d\xi_t^{(j)}, \end{aligned}$$

其中

$$\begin{aligned} d\xi_t^{(i)}d\xi_t^{(j)} &= a_{ij}dt, \\ (a_{ij})_{i,j\le d} &= (\Phi^T\Phi)_{i,j\le d} = (\phi^{ij})^T_{\substack{i\le m\\ j\le d}}(\phi^{ij})_{\substack{i\le m\\ j\le d}}. \end{aligned}$$