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ANALYSIS OF EXPLICIT TAU-LEAPING SCHEMES FOR SIMULATING CHEMICALLY REACTING SYSTEMS*

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Abstract. This paper builds a convergence analysis of explicit tau-leaping schemes for simulating chemical reactions from the viewpoint of stochastic differential equations. Mathematically, the chemical reaction process is a pure jump process on a lattice with state-dependent intensity. The stochastic differential equation form of the chemical master equation can be given via Poisson random measures. Based on this form, different types of tau-leaping schemes can be proposed. In order to make the problem well-posed, a modified explicit tau-leaping scheme is considered. It is shown that the mean square strong convergence is of order 1/2 and the weak convergence is of order 1 for this modified scheme. The novelty of the analysis is to handle the non-Lipschitz property of the coefficients and jumps on the integer lattice.

Key words. tau-leaping scheme, jump process, state-dependent intensity, convergence analysis, non-Lipschitz coefficient

AMS subject classifications. 60H35, 60J75, 62P10, 65C30

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1. Introduction.

Traditional modeling of chemical reactions is based on the deterministic formulation, that is, the ordinary differential equations (ODEs) of the species concentration with the reaction rates as parameters. But this approach encounters difficulties when the number of some species is very small such that the stochastic effect plays a crucial role in some biological processes [2, 18, 19, 32]. In order to capture the correct stochasticity, Gillespie first proposed a stochastic simulation algorithm (SSA) for simulating the spatially homogeneous or well-stirred chemical systems [21, 22]. It was improved numerically in the next reaction method by Gibson and Bruck [20] in 2000, while a recent study by Cao, Li, and Petzold [13] shows that an optimized direct SSA is more efficient, and it is announced as the best way to implement SSA for large systems. The SSA is considered to be exact because it is rigorously based on the same microphysical principles that underlie the chemical master equation (CME). In condensed matter physics, there is a similar algorithm called the kinetic Monte Carlo method or BKL algorithm. It was proposed by Bortz, Kalos, and Lebowitz [4] (see also [31]) for simulating spin dynamics and crystal growth in 1975.

Though SSA simulates the discrete stochastic microscopic model exactly, it is inefficient when some of the reactions fire frequently and the number of molecules is in the intermediate scale. Since in SSA each chemical reaction event is stimulated one at a time, the simulation of frequent firing will be quite time-consuming. Gillespie realized that it was possible to construct an approximate scheme which could advance the process with a deterministic time stepsize, instead of sampling each reaction. He introduced the following leap condition:

"*Leap Condition*: Require the leap time τ to be small enough that the change

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in the state during $[t, t+\tau)$ will be so slight that no propensity function will suffer an appreciable (i.e., macroscopically noninfinitesimal) change in its value."

All of the nomenclatures here will be made precise in the next section. Based on the above idea, Gillespie proposed his ingenious "tau-leaping" scheme [24] in 2001. This scheme consists of finding an appropriate leaping time stepsize τ and then sampling the number of reactions for each chemical reaction channel with independent Poisson random variables. The method is much more efficient than the direct SSA, according to the numerical simulations in [24]. Furthermore, the tau-leaping method provides a natural bridge from the SSA in the discrete stochastic regime, to the chemical Langevin equation in the kinetic regime, and to the reaction rate equation in the continuous deterministic regime [24]. Thus it could be used to build seamless multiscale methods for simulating chemically reacting systems. Many papers followed after it was proposed. For example, Cao, Gillespie, and Petzold have developed more robust and efficient leaping time stepsize selection strategies [12, 25]. The implicit tau-leaping method [36] and nested SSA method [16, 17] are introduced for stiff reacting systems. The hybrid simulations are introduced for diffusion approximation of some fast reaction channels [33]. The theoretical analysis of the tau-leaping method is also investigated by Rathinam et al. [37] in 2005. Some rigorous analysis related to the multiscales in chemical reaction networks is presented in [3].

Since the tau-leaping scheme has so many applications, its rigorous mathematical analysis is important. So far, there are at least two explicit fundamental mathematical problems:

- *Is there a stochastic differential equation (SDE) form for a chemically reacting system? What is the relation between the tau-leaping scheme and this SDE form?* SDE is a basis for developing more advanced numerical schemes. What we have now is the CME, the forward Kolmogorov equation. It is not a natural one for designing schemes. In this aspect, we should remark that this problem has been partially answered in [33].
- *What about the strong convergence and weak convergence of the tau-leaping scheme?* There is no result on strong convergence, and the weak convergence was proved only for linear propensity functions [37].

The aim of this paper is to address these two questions. We first formulate an SDE form via Poisson random measures for jump processes. This is the same as that obtained in [33]. Then we found that the tau-leaping scheme is just the explicit Euler scheme for this SDE. Based on this form, different types of tau-leaping schemes can be proposed quite naturally, such as the implicit schemes, trapezoidal schemes, Runge–Kutta schemes, and so on. In order to make the problem well-posed, we modify the explicit tau-leaping scheme by setting the values of propensity functions with negative populations to zero. We prove that the strong convergence of the modified scheme is of order $1/2$ and the weak convergence is of order 1 for general nonlinear propensity functions through the backward Kolmogorov equation. We should comment that our result does not include the open system case like $\emptyset \to S$ since we make assumptions that the total states are finite.

We hope that this is a bridge step towards finally building the rigorous mathematical foundation of tau-leaping schemes. Compared to the previous convergence analysis specific to tau-leaping by Rathinam et al. [37], they proved that both explicit and implicit tau-leaping methods converge weakly in the sense of moments under the same assumption of a closed system and the extra assumption of linear propensity functions. But through a totally different strategy from stochastic analysis, we prove the weak convergence for general nonlinear propensity functions and, more impor-

tantly, the mean square strong convergence. However, our method is limited only to the explicit tau method. To further advance the result, we would consider the analysis for the implicit method and the analysis of tau-leaping schemes based on the binomial distribution [15, 41] or some acceptance-rejection strategy. There is also potential to construct new methods with the current SDE formulation. In this regard, we would like to comment that an accelerated leap method based on the ideas from SDE driven by Poisson noise was recently proposed in [8], which is quite related to this paper.

This paper is organized as follows. In section 2, the background of the tau-leaping scheme and the SDE formulation of the reacting system are introduced. In sections 3 and 4, we give the strong and weak convergence proof for the explicit tau-leaping scheme. Numerical experiments are shown in section 5, and, finally, we add some conclusions.

2. Mathematical formulation of tau-leaping scheme. Consider a well-stirred system of N molecular species $\{S_1, S_2, \ldots, S_N\}$ interacting through M chemical reaction channels $\{R_1, R_2, \ldots, R_M\}$. The state of the system is described by the vector

$$\boldsymbol{X}_t = (X_t^1, X_t^2, \ldots, X_t^N). \tag{2.1}$$

Each reaction channel R_j is characterized by its propensity function $a_j(\boldsymbol{x})$ and its state change vector

$$\boldsymbol{\nu}_j = (\nu_j^1, \nu_j^2, \ldots, \nu_j^N), \tag{2.2}$$

where $a_j(\boldsymbol{x}) \geq 0$ for physical states. Here $a_j(\boldsymbol{x})dt$ gives the probability that the system will experience an R_j reaction in the next infinitesimal time dt when the current state $\boldsymbol{X}_t = \boldsymbol{x}$. ν_j^i is the change in the number of S_i molecules caused by one R_j reaction.

Let us define $\mathbb{Z}_0^+ = \mathbb{N} \cup \{0\}$ to be the set of nonnegative integers. Mathematically, well-stirred chemical reaction system can be accurately described by a discrete state continuous time jump process on the lattice $(\mathbb{Z}_0^+)^N$. The conditions on the propensity functions $a_j(\boldsymbol{x})$ must be supplemented to prevent the physically meaningless negative population of molecules. In this regard we will adopt Remark 1 in [37]. These conditions are automatically satisfied in real applications.

Define the transition probability $P(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0)$ that the state variable $\boldsymbol{X}_t$ will equal $\boldsymbol{x}$, given that $\boldsymbol{X}_{t_0} = \boldsymbol{x}_0$. The CME for the system is

$$\partial_t P(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0) = \sum_{j=1}^{M} a_j(\boldsymbol{x} - \boldsymbol{\nu}_j) P(\boldsymbol{x} - \boldsymbol{\nu}_j, t|\boldsymbol{x}_0, t_0) - \sum_{j=1}^{M} a_j(\boldsymbol{x}) P(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0). \tag{2.3}$$

This is also the forward Kolmogorov equation or Fokker–Planck equation for transition probability function. For simulation, define $p(\tau, j|\boldsymbol{x}, t)$ as the probability density, given $\boldsymbol{X}_t = \boldsymbol{x}$, that the next reaction in the system will occur after τ and will be an R_j reaction. Then we have

$$\begin{aligned} p(\tau, j|\boldsymbol{x}, t) &= a_j(\boldsymbol{x}) \exp(-a_0(\boldsymbol{x})\tau) \\ &= \frac{a_j(\boldsymbol{x})}{a_0(\boldsymbol{x})} \cdot a_0(\boldsymbol{x}) \exp(-a_0(\boldsymbol{x})\tau) \end{aligned} \tag{2.4}$$

through analysis in [23], where $a_0(\boldsymbol{x}) = \sum_{j=1}^{M} a_j(\boldsymbol{x})$. This induces the so-called SSA.

- Step 1: sampling the waiting time τ as an exponentially distributed random variable (R.V.) with rate $a_0(\boldsymbol{X}_t)$;
- Step 2: sampling an M point R.V. k with probability $\frac{a_j(\boldsymbol{X}_t)}{a_0(\boldsymbol{X}_t)}$ for the jth reaction;
- Step 3: update $\boldsymbol{X}_{t+\tau} = \boldsymbol{X}_t + \boldsymbol{\nu}_k$; then return to Step 1.

Let us first observe that $\boldsymbol{X}_t$ is actually a *compound Poisson process with state-dependent intensity*. While Lévy processes have been intensively investigated in probability theory, the formulation of CME drives us to find its SDE counterpart. This can be realized via Poisson random measure, which has been obtained in [33]. Further theory about Lévy processes may be referenced in [1].

Given any initial state $\boldsymbol{X}_0 \in (\mathbb{Z}_0^+)^N$, we define the set $\Omega_{\boldsymbol{X}_0}$ as all of the possible physical states generated from $\boldsymbol{X}_0$

$$\Omega_{\boldsymbol{X}_0} = \Big\{ \boldsymbol{Y} \;\Big|\; \boldsymbol{Y} \in (\mathbb{Z}_0^+)^N,\ \boldsymbol{Y} = \boldsymbol{X}_0 + \sum_{j=1}^{M} k_j \boldsymbol{\nu}_j,\quad k_j \in \mathbb{Z}_0^+ \Big\}, \tag{2.5}$$

and the set $\Omega^t_{\boldsymbol{X}_0}$ as all of the possible states generated from $\boldsymbol{X}_0$

$$\Omega^t_{\boldsymbol{X}_0} = \Big\{ \boldsymbol{Y} \;\Big|\; \boldsymbol{Y} \in \mathbb{Z}^N,\ \boldsymbol{Y} = \boldsymbol{X}_0 + \sum_{j=1}^{M} k_j \boldsymbol{\nu}_j,\quad k_j \in \mathbb{Z}_0^+ \Big\}. \tag{2.6}$$

We have $\boldsymbol{X}_t \in \Omega_{\boldsymbol{X}_0}$ for the exact solution but $\boldsymbol{X}_n \in \Omega^t_{\boldsymbol{X}_0}$ for the tau-leaping scheme. For most of the chemically reacting systems, we can make the following assumptions.

Assumption 2.1 (condition on propensity functions). The propensity function $a_j(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \Omega_{\boldsymbol{X}_0}$, and $a_j(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in \Omega_{\boldsymbol{X}_0}$ but $\boldsymbol{x} + \boldsymbol{\nu}_j \notin \Omega_{\boldsymbol{X}_0}$.

This assumption is natural. Otherwise the negative states will appear in the physical process!

Assumption 2.2 (bound on $\boldsymbol{X}_t$). The number of elements in $\Omega_{\boldsymbol{X}_0}$ is finite; i.e., $\boldsymbol{X}_t$ is in a bounded lattice.

This assumption is reasonable because the number of the molecules could not be arbitrary large in realistic chemical reactions.

In order to perform the analysis, we make the following assumption on $a_j(\boldsymbol{x})$.

Assumption 2.3 (local Lipschitz condition on $a_j(\boldsymbol{x})$). The function $a_j(\boldsymbol{x})$ is Lipschitz continuous in a bounded domain. That is, $|a_j(\boldsymbol{x}) - a_j(\boldsymbol{y})| \leq L_j|\boldsymbol{x} - \boldsymbol{y}|$ for any bounded $\boldsymbol{x}$ and $\boldsymbol{y}$, where L is a fixed positive real number.

Define the upper bound of total propensity

$$A = \max \Big\{ a_0(\boldsymbol{x}),\quad \boldsymbol{x} \in \Omega_{\boldsymbol{X}_0} \Big\}. \tag{2.7}$$

Suppose that $\mathcal{S} = \mathbb{R}^+ \times (0, A]$ equipped with Borel σ-algebra $\mathfrak{B}(\mathbb{R}^+ \times (0, A])$, and define the reference Poisson random measure as $\lambda(dt \times da)$ associated with a Poisson point process $(q_t,\, t \geq 0)$ taking values in $(0, A]$. That is,

$$\int_0^t \int_{\mathcal{B}} \lambda(dt \times da) = \#\{0 \leq s < t;\ q_s \in \mathcal{B}\}, \tag{2.8}$$

where $\mathcal{B}$ is a Borel set in $(0, A]$. And we assume $\lambda(dt \times da)$ has Lebesgue intensity measure $m(dt \times da) = dt \times da$. Then we define the following thinning of measure λ with acceptance-rejection strategy:

$$\mu(dt) = \int_0^A 1_{\{0 < a \leq a_0(\boldsymbol{X}_{t-})\}} \lambda(dt \times da), \tag{2.9}$$

where $\boldsymbol{X}_{t-} = \lim_{s\to t-0} \boldsymbol{X}_s$.

PROPOSITION 2.1 (Poisson random measure with state-dependent intensity). *The random measure $\mu(dt)$ defined above is a Poisson random measure with state-dependent intensity measure*

$$a_0(\boldsymbol{X}_t)dt. \tag{2.10}$$

Proof. The proof is a simple consequence of Theorem 3.1 in [34]. An elementary argument is as follows. Suppose that $a_0(\boldsymbol{X}_t)$ is a constant, and consider a sequence of jumps

$$J_1, J_2, \ldots, J_N$$

with constant jump intensity A at time $\tau_1, \tau_2, \ldots, \tau_N$. Then $\tau_1, \tau_2 - \tau_1, \ldots, \tau_N - \tau_{N-1}$ are independently, identically, and exponentially distributed random variables with rate A. We have τ_N, a Gamma-distributed random variable, and the probability density function is

$$\frac{A^N t^{N-1}}{\Gamma(N)} e^{-At}.$$

Here $\Gamma(N) = (N-1)!$. A successful jump after acceptance-rejection strategy will make the distribution density of the waiting time be

$$\sum_{N=1}^{\infty} (1-\eta)^{N-1}\eta \cdot \frac{A^N t^{N-1}}{\Gamma(N)} e^{-At} = \eta A e^{-\eta A t}.$$

In our case, $\eta = \frac{a_0(\boldsymbol{X}_t)}{A}$ according to the uniform distribution. Thus the new process is again a Poisson process with rate $a_0(\boldsymbol{X}_t)$. ∎

Furthermore, in order to describe M chemical reactions, we define the function

$$c_j(a; \boldsymbol{X}_t) = \begin{cases} 1 & \text{if } a \in (h_{j-1}(\boldsymbol{X}_t), h_j(\boldsymbol{X}_t)], \\ 0 & \text{otherwise,} \end{cases} \quad j = 1, 2, \ldots, M, \tag{2.11}$$

where

$$h_j(\boldsymbol{X}_t) = \sum_{i=1}^{j} a_i(\boldsymbol{X}_t), \quad j = 0, 1, \ldots, M. \tag{2.12}$$

Then we have the SDE form for the CME

$$d\boldsymbol{X}_t = \sum_{j=1}^{M} \int_0^A \boldsymbol{\nu}_j c_j(a; \boldsymbol{X}_{t-}) \lambda(dt \times da). \tag{2.13}$$

It is worth noting that the jump adapted version of (2.13) is just the SSA [6].

REMARK 2.1. *Note that the coefficient $c_j(a; \boldsymbol{X})$ is not Lipschitz in $\boldsymbol{X}$, which is different from the assumptions made in* [5, 6].

LEMMA 2.2. *Under Assumption* 2.1*, the SDE* (2.13) *is well-posed in the sense that there exists a unique physical solution $\boldsymbol{X}_t \in \Omega_{\boldsymbol{X}_0}$ in $[0, +\infty)$.*

Proof. The idea of the proof is obtained through the SSA simulation directly. That is the essence of the proof in [34]. Let us first define processes

$$N_t^j = \int_0^t \int_0^A c_j(a; \boldsymbol{X}_{t-}) \lambda(dt \times da), \quad j = 1, 2, \ldots, M. \tag{2.14}$$

Define a sequence of processes $\boldsymbol{X}_t^n$, stopping times T^n, and indices I^n $(n = 0, 1, \ldots)$ as $\boldsymbol{X}_t^0 = \boldsymbol{X}_0$, $T^0 = 0$, $I^0 = 1$, and

$$\boldsymbol{X}_t^{n+1} = \boldsymbol{X}_0 + \sum_{i=1}^{n} \boldsymbol{\nu}_{I^i} 1_{\{t \geq T_i\}}, \tag{2.15}$$

$$T^{n+1} = \min\Big\{\inf\{t : N_t^j(\boldsymbol{X}^{n+1}) > N_{T^n}^j(\boldsymbol{X}^{n+1})\}, \quad j = 1, 2, \ldots, M\Big\}, \tag{2.16}$$

$$I^{n+1} = \text{Index } j \in \{1, 2, \ldots, M\} \text{ such that } \Delta N_{T^{n+1}}^j(\boldsymbol{X}^{n+1}) = 1, \tag{2.17}$$

where the symbol "Δ" is defined in (3.1). It is easy to find that under Assumption 2.1 $\boldsymbol{X}_t^n$ stays in $\Omega_{\boldsymbol{X}_0}$ all the time. $\boldsymbol{X}_t^n = \boldsymbol{X}_t^{n-1}$ in $[0, T^{n-1})$, and the stopping time T^n can be extended to ∞. The rest is similar as that in [34], and this ends the proof. ∎

From the computational point of view, we can take another form of (2.13):

$$\begin{aligned} d\boldsymbol{X}_t &= \sum_{j=1}^{M} \int_0^A \boldsymbol{\nu}_j c_j(a; \boldsymbol{X}_{t-}) m(dt \times da) \\ &\quad + \sum_{j=1}^{M} \int_0^A \boldsymbol{\nu}_j c_j(a; \boldsymbol{X}_{t-})(\lambda - m)(dt \times da) \\ &= \boldsymbol{P}_1 + \boldsymbol{P}_2. \end{aligned} \tag{2.18}$$

Actually we have

$$\boldsymbol{P}_1 = \sum_{j=1}^{M} \boldsymbol{\nu}_j a_j(\boldsymbol{X}_{t-}) dt. \tag{2.19}$$

And we can observe that $\boldsymbol{P}_2$ is a martingale with mean zero. We will call $\boldsymbol{P}_1$ the drift term, and $\boldsymbol{P}_2$ is the jump term later.

In the explicit tau-leaping scheme, it is assumed that in each time step, the variation of propensity function $a_j(\boldsymbol{x})$ can be ignored. This is equivalent to an explicit Euler discretization of (2.18). We have

$$\begin{aligned} \boldsymbol{X}_{n+1} &= \boldsymbol{X}_n + \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \int_0^A \boldsymbol{\nu}_j c_j(a; \boldsymbol{X}_n) m(dt \times da) \\ &\quad + \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \int_0^A \boldsymbol{\nu}_j c_j(a; \boldsymbol{X}_n)(\lambda - m)(dt \times da) \qquad (2.20) \\ &= \boldsymbol{X}_n + \sum_{j=1}^{M} \boldsymbol{\nu}_j a_j(\boldsymbol{X}_n)\delta t_n + \Big(\sum_{j=1}^{M} \boldsymbol{\nu}_j \mathcal{P}(a_j(\boldsymbol{X}_n)\delta t_n) - \sum_{j=1}^{M} \boldsymbol{\nu}_j a_j(\boldsymbol{X}_n)\delta t_n\Big) \\ &= \boldsymbol{X}_n + \sum_{j=1}^{M} \boldsymbol{\nu}_j \mathcal{P}(a_j(\boldsymbol{X}_n)\delta t_n), \qquad (2.21) \end{aligned}$$

where $\delta t_n = t_{n+1} - t_n$, and $\mathcal{P}(a_j(\boldsymbol{X}_n)\delta t_n)$ is a Poisson random variable with mean and variance $a_j(\boldsymbol{X}_n)\delta t_n$.

Furthermore, we can define the semi-implicit tau-leaping scheme which is implicit for the drift term and explicit for the jump term

$$\begin{aligned}\boldsymbol{X}_{n+1} &= \boldsymbol{X}_n + \sum_{j=1}^{M}\int_{t_n}^{t_{n+1}}\int_0^A \boldsymbol{\nu}_j c_j(a;\boldsymbol{X}_{n+1})m(dt\times da)\\ &\quad + \sum_{j=1}^{M}\int_{t_n}^{t_{n+1}}\int_0^A \boldsymbol{\nu}_j c_j(a;\boldsymbol{X}_n)(\lambda - m)(dt\times da)\\ &= \boldsymbol{X}_n + \sum_{j=1}^{M}\boldsymbol{\nu}_j a_j(\boldsymbol{X}_{n+1})\delta t_n + \Big(\sum_{j=1}^{M}\boldsymbol{\nu}_j\mathcal{P}(a_j(\boldsymbol{X}_n)\delta t_n) - \sum_{j=1}^{M}\boldsymbol{\nu}_j a_j(\boldsymbol{X}_n)\delta t_n\Big). \qquad (2.22)\end{aligned}$$

This is nothing but the implicit method proposed in [36]. Similarly we have the stochastic theta-methods for the tau-leaping scheme as the numerical solution of SDE

$$\begin{aligned}\boldsymbol{X}_{n+1} &= \boldsymbol{X}_n + \sum_{j=1}^{M}\boldsymbol{\nu}_j\Big(\theta a_j(\boldsymbol{X}_{n+1}) + (1-\theta)a_j(\boldsymbol{X}_n)\Big)\delta t_n\\ &\quad + \Big(\sum_{j=1}^{M}\boldsymbol{\nu}_j\mathcal{P}(a_j(\boldsymbol{X}_n)\delta t_n) - \sum_{j=1}^{M}\boldsymbol{\nu}_j a_j(\boldsymbol{X}_n)\delta t_n\Big)\\ &= \boldsymbol{X}_n + \sum_{j=1}^{M}\theta\boldsymbol{\nu}_j\Big(a_j(\boldsymbol{X}_{n+1}) - a_j(\boldsymbol{X}_n)\Big)\delta t_n + \sum_{j=1}^{M}\boldsymbol{\nu}_j\mathcal{P}(a_j(\boldsymbol{X}_n)\delta t_n), \qquad (2.23)\end{aligned}$$

where $\theta \in [0,1]$. This parameter may be used to control the balance between the instability caused by the explicit scheme and the damping effect caused by the implicit scheme. Different kinds of tau-leaping schemes could be constructed similarly with the numerical solution of SDEs, for example, the trapezoidal methods [14], Runge–Kutta methods [10, 9], and so on. A recipe for the numerical methods of SDE may be referenced in Schurz [39].

REMARK 2.2. *An important numerical issue must be addressed is that all of the numerical schemes proposed so far are of first order weak convergence* [9, 11, 37]. *This can be explained as follows. Now all of the methods take Runge–Kutta-type form. From the results for SDEs driven by Wiener process in* [7, 38], *the mean square strong order barrier is* 0.5 *for the schemes taking only the single integral of white noise in general. If the multiple integral is taken into account, the convergence order could be raised. If we extend their results to SDEs driven by Poisson noise here, we may expect that the Milstein-type scheme will give higher order methods. This will be done in a future work.*

REMARK 2.3. *In writing* (2.21), (2.22), *and* (2.23), *we have implicitly assumed the following condition:*

$$a_j(\boldsymbol{x}) \geq 0 \quad \forall\ \boldsymbol{x}\in\Omega^t_{\boldsymbol{X}_0} \quad \textit{and} \quad \max\Big\{a_0(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega^t_{\boldsymbol{X}_0}\Big\} \leq A, \qquad (2.24)$$

which is not satisfied for almost all realistic applications. For example, for the decaying-dimerizing reaction cited in section 5, $a_2(\boldsymbol{x}) = \frac{c_2}{2}x_1(x_1-1)$ *will be infinitely large as* x_1 *tends to* $-\infty$. *Violation of this condition will not give the discretization form* (2.21), (2.22), *and* (2.23) *because some parts of* $h_j(\boldsymbol{X}_n)$ *will exceed the prescribed upper bound* A. *Since the tau-leaping algorithm may generate arbitrarily large Poisson random variables, the violation of condition* (2.24) *seems inevitable. This dilemma can be resolved in the following key redefinition of propensity functions.*

PROPOSITION 2.3 (redefinition of $a_j(\boldsymbol{x})$). *We define the modification of* $a_j(\boldsymbol{x})$ *as*

$$\tilde{a}_j(\boldsymbol{x}) = \begin{cases} a_j(\boldsymbol{x}), & \boldsymbol{x} \in (\mathbb{Z}_0^+)^N, \\ 0, & \boldsymbol{x} \in \mathbb{Z}^N \backslash (\mathbb{Z}_0^+)^N. \end{cases} \tag{2.25}$$

We have

$$|\tilde{a}_j(\boldsymbol{x}) - \tilde{a}_j(\boldsymbol{y})| \le L_j |\boldsymbol{x} - \boldsymbol{y}| \quad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \Omega^t_{\boldsymbol{X}_0} \cup \left(\mathbb{Z}^N \backslash (\mathbb{Z}_0^+)^N\right), \quad j = 1, 2, \ldots, M. \tag{2.26}$$

Proof. The condition (2.26) holds because we consider only the finite set $\Omega_{\boldsymbol{X}_0}$ and the set of negative populations, in which the propensity function has been redefined to zero. The local Lipschitz condition on $a_j(\boldsymbol{x})$ (Assumption 2.3) gives the results immediately. In addition we have $L_j \le 2A$. This ends the proof. ∎

Applying this redefinition to the statement before, we actually consider a *modified tau-leaping scheme*. Suppose the state is $\boldsymbol{X}_n$ at time t_n, and the next state is a nonphysical state $\boldsymbol{X}_{n+1}$ with negative populations. Since $\tilde{a}_j(\boldsymbol{X}_{n+1}) = 0$, the tau-leaping scheme will give

$$\boldsymbol{X}_{n+k} = \boldsymbol{X}_{n+1} \quad \forall\, k \in \mathbb{N}.$$

This means the modified tau-leaping scheme permits the appearance of negative states but freezes them afterward. We will show that this scheme also has strong and weak convergence in this paper.

We will still denote $\tilde{a}_j(\boldsymbol{x})$ as $a_j(\boldsymbol{x})$ in the continuing text for simplicity.

3. Strong convergence. We will prove the strong convergence of the modified scheme (2.20) in this section. First, we establish some technical lemmas.

Define the jump operator

$$\Delta \boldsymbol{X}_t = \boldsymbol{X}_t - \boldsymbol{X}_{t-}; \tag{3.1}$$

the following lemma describes the jumps of $\boldsymbol{X}_t$.

LEMMA 3.1. *For any fixed $s > 0$, $\Delta \boldsymbol{X}_s = 0$ (a.s.).*

Proof. The Poisson random measure $\lambda(dt \times dx)$ is generated through a Poisson point process q_t. From the definition of $\boldsymbol{X}_t$ in (2.13), $\Delta \boldsymbol{X}_s \neq 0$ implies $\Delta q_s \neq 0$, while from the properties of the Lévy process (see Applebaum [1, page 86]), $\Delta q_s = 0$ (*a.s.*). This ends the proof. ∎

For the deterministic integral, we may obtain the following lemma.

LEMMA 3.2. *For any continuous function $a(\boldsymbol{x})$ and two positive reals $d > c$, we have*

$$\int_c^d \Delta a(\boldsymbol{X}_t) dt = 0. \tag{3.2}$$

Proof. We know $\boldsymbol{X}_t$ is a càdlàg process (continuous on the right and always having the limit on the left). Thus the set of jump sites

$$S = \left\{ t \mid t \in [c, d],\ \Delta a(\boldsymbol{X}_t) > 0 \right\}$$

is at most countable (see Klebaner [29, page 3]). This means the integrand in (3.2) is zero, except a set with Lebesgue measure 0. Hence the integral must be zero. ∎

For notational convenience, we define Lipschitz constant $L = \sum_{j=1}^M L_j$ and

$$K = \max\{\ |\boldsymbol{\nu}_j|, \quad j = 1, 2, \ldots, M\ \}. \tag{3.3}$$

Now suppose the tau-leaping scheme is posed in a time interval $[0,T]$ with N_T steps:

$$0 = t_0 < t_1 < \cdots < t_{N_T} = T.$$

We have the following strong convergence theorem.

THEOREM 3.3 (mean square convergence). *With Assumptions* 2.1–2.3 *and Proposition* 2.3 *we have*

$$\sup_{n \le N_T} \mathbb{E}|\boldsymbol{X}_n - \boldsymbol{X}_{t_n}|^2 \le C\tau, \tag{3.4}$$

where $\tau = \max_n \delta t_n$*, and* C *is a constant that depends on* L*,* K*,* T*,* M*, and* A*.*

Proof. In order to prove the strong convergence of the explicit tau-leaping scheme, we integrate both sides of (2.18) from t_n to t_{n+1}:

$$\begin{aligned}
\boldsymbol{X}_{t_{n+1}} &= \boldsymbol{X}_{t_n} + \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \boldsymbol{\nu}_j a_j(\boldsymbol{X}_{t-}) dt \\
&\quad + \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \int_0^A \boldsymbol{\nu}_j c_j(a; \boldsymbol{X}_{t-})(\lambda - m)(dt \times da) \\
&= \boldsymbol{X}_{t_n} + \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \boldsymbol{\nu}_j a_j(\boldsymbol{X}_{t_n-}) dt \\
&\quad + \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \int_0^A \boldsymbol{\nu}_j c_j(a; \boldsymbol{X}_{t_n-})(\lambda - m)(dt \times da) \\
&\quad + \boldsymbol{R}_1 + \boldsymbol{R}_2,
\end{aligned} \tag{3.5}$$

where

$$\boldsymbol{R}_1 = \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \boldsymbol{\nu}_j \Big(a_j(\boldsymbol{X}_{t-}) - a_j(\boldsymbol{X}_{t_n-}) \Big) dt,$$

$$\boldsymbol{R}_2 = \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \int_0^A \boldsymbol{\nu}_j \Big(c_j(a; \boldsymbol{X}_{t-}) - c_j(a; \boldsymbol{X}_{t_n-}) \Big)(\lambda - m)(dt \times da)$$

are remainder terms. Furthermore, we have

$$\begin{aligned}
\boldsymbol{R}_1 &= \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \boldsymbol{\nu}_j \Big(a_j(\boldsymbol{X}_t) - a_j(\boldsymbol{X}_{t_n}) \Big) dt + \tilde{\boldsymbol{R}}_1 \\
&= \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \boldsymbol{\nu}_j \Big(a_j(\boldsymbol{X}_t) - a_j(\boldsymbol{X}_{t_n}) \Big) dt,
\end{aligned} \tag{3.6}$$

where

$$\tilde{\boldsymbol{R}}_1 = \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \boldsymbol{\nu}_j \Delta a_j(\boldsymbol{X}_{t_n}) dt - \sum_{j=1}^{M} \int_{t_n}^{t_{n+1}} \boldsymbol{\nu}_j \Delta a_j(\boldsymbol{X}_t) dt. \tag{3.7}$$

From Lemma 3.1, we know the first part of $\tilde{\boldsymbol{R}}_1$ is zero a.s. This result holds for finite time steps. From Lemma 3.2, we know the second part of $\tilde{\boldsymbol{R}}_1$ is zero, too. Combining these gives (3.6).

Now we subtract (3.5) and (2.20). Define the error

$$\boldsymbol{E}_n = \boldsymbol{X}_{t_n} - \boldsymbol{X}_n.$$

Then we get

$$\begin{aligned}\boldsymbol{E}_{n+1} &= \boldsymbol{E}_n + \sum_{j=1}^{M}\int_{t_n}^{t_{n+1}} \boldsymbol{\nu}_j\Big(a_j(\boldsymbol{X}_{t_n-}) - a_j(\boldsymbol{X}_n)\Big)dt \\ &\quad + \sum_{j=1}^{M}\int_{t_n}^{t_{n+1}}\int_0^A \boldsymbol{\nu}_j\Big(c_j(a;\boldsymbol{X}_{t_n-}) - c_j(a;\boldsymbol{X}_n)\Big)(\lambda - m)(dt\times da) + \boldsymbol{R}_1 + \boldsymbol{R}_2 \\ &= \boldsymbol{E}_n + \boldsymbol{P}_3 + \boldsymbol{P}_4 + \boldsymbol{R}_1 + \boldsymbol{R}_2. \qquad (3.8)\end{aligned}$$

Squaring both sides of (3.8) we obtain

$$\begin{aligned}\boldsymbol{E}_{n+1}^2 &= \boldsymbol{E}_n^2 + \boldsymbol{P}_3^2 + \boldsymbol{P}_4^2 + \boldsymbol{R}_1^2 + \boldsymbol{R}_2^2 + 2\Big(\boldsymbol{E}_n\cdot\boldsymbol{P}_3 + \boldsymbol{E}_n\cdot\boldsymbol{P}_4 + \boldsymbol{E}_n\cdot\boldsymbol{R}_1 + \boldsymbol{E}_n\cdot\boldsymbol{R}_2 \\ &\quad + \boldsymbol{P}_3\cdot\boldsymbol{P}_4 + \boldsymbol{P}_3\cdot\boldsymbol{R}_1 + \boldsymbol{P}_3\cdot\boldsymbol{R}_2 + \boldsymbol{P}_4\cdot\boldsymbol{R}_1 + \boldsymbol{P}_4\cdot\boldsymbol{R}_2 + \boldsymbol{R}_1\cdot\boldsymbol{R}_2\Big). \qquad (3.9)\end{aligned}$$

We will consider each term in what follows.

By Proposition 2.3 and Lemma 3.1 we have

$$\mathbb{E}\boldsymbol{P}_3^2 \le K^2L^2\delta t_n^2\mathbb{E}\boldsymbol{E}_n^2. \qquad (3.10)$$

By Itô's isometry of stochastic integral for Lévy processes [1] and Lemma 3.1, we get

$$\begin{aligned}&\mathbb{E}\left(\int_{t_n}^{t_{n+1}}\int_0^A \Big(c_j(a;\boldsymbol{X}_{t_n-}) - c_j(a;\boldsymbol{X}_n)\Big)(\lambda-m)(dt\times da)\right)^2 \\ &= \mathbb{E}\int_{t_n}^{t_{n+1}}\int_0^A \Big(c_j(a;\boldsymbol{X}_{t_n-}) - c_j(a;\boldsymbol{X}_n)\Big)^2 m(dt\times da) \\ &\le \delta t_n\mathbb{E}\Big(|h_{j-1}(\boldsymbol{X}_{t_n-}) - h_{j-1}(\boldsymbol{X}_n)| + |h_j(\boldsymbol{X}_{t_n-}) - h_j(\boldsymbol{X}_n)|\Big) \\ &= \delta t_n\mathbb{E}\Big(|h_{j-1}(\boldsymbol{X}_{t_n}) - h_{j-1}(\boldsymbol{X}_n)| + |h_j(\boldsymbol{X}_{t_n}) - h_j(\boldsymbol{X}_n)|\Big) \\ &\le 2L\delta t_n\mathbb{E}|\boldsymbol{E}_n|. \qquad (3.11)\end{aligned}$$

Here the term $\mathbb{E}|\boldsymbol{E}_n|$ reflects the non-Lipschitz nature of this state-dependent intensity problem. A coarse estimate

$$\mathbb{E}|\boldsymbol{E}_n| \le (\mathbb{E}|\boldsymbol{E}_n|^2)^{\frac{1}{2}}$$

suggests that the scheme may *not* be convergent. To circumvent this issue, it is enough to observe that the processes $\boldsymbol{X}_t$ and $\boldsymbol{X}_n$ are on lattice $\mathbb{Z}^N$. We have

$$\text{either} \quad |\boldsymbol{E}_n| = 0 \quad \text{or} \quad |\boldsymbol{E}_n| \ge 1, \qquad (3.12)$$

which means that

$$|\boldsymbol{E}_n| \le |\boldsymbol{E}_n|^2. \qquad (3.13)$$

From the estimate (3.13) and Cauchy's inequality we obtain

$$\mathbb{E}\boldsymbol{P}_4^2 \le M^2K^2 \cdot 2L\delta t_n \mathbb{E}|\boldsymbol{E}_n| \le 2M^2K^2L\delta t_n \mathbb{E}|\boldsymbol{E}_n|^2. \tag{3.14}$$

From Itô's formula for jump SDE (see Lemma 4.4.5 on page 223 of [1]), we have

$$\begin{aligned}
\boldsymbol{R}_1 &= \sum_{j=1}^M \int_{t_n}^{t_{n+1}} \boldsymbol{\nu}_j \Big(a_j(\boldsymbol{X}_t) - a_j(\boldsymbol{X}_{t_n})\Big)dt \\
&= \sum_{j=1}^M \boldsymbol{\nu}_j \int_{t_n}^{t_{n+1}} \int_{t_n}^t \int_0^A \Big(a_j\Big(\boldsymbol{X}_{s-} + \sum_{k=1}^M \boldsymbol{\nu}_k c_k(a;\boldsymbol{X}_{s-})\Big) - a_j(\boldsymbol{X}_{s-})\Big)\lambda(ds\times da)dt \\
&= \sum_{j=1}^M \boldsymbol{\nu}_j \int_{t_n}^{t_{n+1}} \int_{t_n}^t \int_0^A \Big(a_j\Big(\boldsymbol{X}_{s-} + \sum_{k=1}^M \boldsymbol{\nu}_k c_k(a;\boldsymbol{X}_{s-})\Big) - a_j(\boldsymbol{X}_{s-})\Big)m(ds\times da)dt \\
&+ \sum_{j=1}^M \boldsymbol{\nu}_j \int_{t_n}^{t_{n+1}} \int_{t_n}^t \int_0^A \Big(a_j\Big(\boldsymbol{X}_{s-} + \sum_{k=1}^M \boldsymbol{\nu}_k c_k(a;\boldsymbol{X}_{s-})\Big) - a_j(\boldsymbol{X}_{s-})\Big) \\
&\cdot\ (\lambda - m)(ds\times da)dt \\
&= \boldsymbol{R}_{11} + \boldsymbol{R}_{12}.
\end{aligned}$$

From Assumption 2.3 we have

$$\mathbb{E}\boldsymbol{R}_{11}^2 \le A^2K^4L^2\delta t_n^4 \tag{3.15}$$

and

$$\begin{aligned}
\mathbb{E}|\boldsymbol{R}_{12}|^2 &\le MK^2\delta t_n^2 \sum_{j=1}^M \mathbb{E}\left[\int_{t_n}^{t_{n+1}} \int_0^A L_j \left|\sum_{k=1}^M \boldsymbol{\nu}_k c_k(a;\boldsymbol{X}_{s-})\right| (\lambda - m)(ds\times da)\right]^2 \\
&\le AMK^4\delta t_n^3 \sum_{j=1}^M L_j^2. \qquad (3.16)
\end{aligned}$$

Combining (3.15) and (3.16) we obtain

$$\mathbb{E}\boldsymbol{R}_1^2 \le C_1\delta t_n^3, \tag{3.17}$$

where C_1 is a bounded constant that depends on K, L, A, and M.

For $\boldsymbol{R}_2$, we have by Itô's isometry

$$\begin{aligned}
&\mathbb{E}\left(\int_{t_n}^{t_{n+1}} \int_0^A \Big(c_j(a;\boldsymbol{X}_{t-}) - c_j(a;\boldsymbol{X}_{t_n-})\Big)(\lambda - m)(dt\times da)\right)^2 \\
&= \mathbb{E}\int_{t_n}^{t_{n+1}} \int_0^A \Big(c_j(a;\boldsymbol{X}_{t-}) - c_j(a;\boldsymbol{X}_{t_n-})\Big)^2 m(dt\times da) \\
&\le \int_{t_n}^{t_{n+1}} \mathbb{E}\Big(|h_{j-1}(\boldsymbol{X}_{t-}) - h_{j-1}(\boldsymbol{X}_{t_n-})| + |h_j(\boldsymbol{X}_{t-}) - h_j(\boldsymbol{X}_{t_n-})|\Big)dt \\
&\le L\int_{t_n}^{t_{n+1}} \mathbb{E}|\boldsymbol{X}_{t-} - \boldsymbol{X}_{t_n-}|dt = L\int_{t_n}^{t_{n+1}} \mathbb{E}|\boldsymbol{X}_t - \boldsymbol{X}_{t_n}|dt \qquad (3.18) \\
&\le L\int_{t_n}^{t_{n+1}} \mathbb{E}|\boldsymbol{X}_t - \boldsymbol{X}_{t_n}|^2 dt \qquad (3.19)
\end{aligned}$$

$$
\begin{aligned}
&\le L\int_{t_n}^{t_{n+1}}\mathbb{E}\left|\int_{t_n}^{t}\int_0^A\sum_{k=1}^{M}\boldsymbol{\nu}_j c_k(a;\boldsymbol{X}_{s-})\lambda(ds\times da)\right|^2 dt\\
&\le 2L\int_{t_n}^{t_{n+1}}\mathbb{E}\left|\int_{t_n}^{t}\int_0^A\sum_{k=1}^{M}\boldsymbol{\nu}_j c_k(a;\boldsymbol{X}_{s-})m(ds\times da)\right|^2 dt\\
&+ 2L\int_{t_n}^{t_{n+1}}\mathbb{E}\left|\int_{t_n}^{t}\int_0^A\sum_{k=1}^{M}\boldsymbol{\nu}_j c_k(a;\boldsymbol{X}_{s-})(\lambda-m)(ds\times da)\right|^2 dt\\
&\le 2LA^2K^2\delta t_n^3 + 2LAK^2\delta t_n^2. \qquad (3.20)
\end{aligned}
$$

In estimate (3.18), we have used Lemmas 3.1 and 3.2. And we apply a similar trick in (3.13) to (3.19). Thus we obtain

$$\mathbb{E}|\boldsymbol{R}_2|^2 \le C_2\delta t_n^2. \qquad (3.21)$$

For $\boldsymbol{E}_n\cdot\boldsymbol{P}_4$, we have

$$
\begin{aligned}
\mathbb{E}\boldsymbol{E}_n\cdot\boldsymbol{P}_4 &= \mathbb{E}\Delta\boldsymbol{X}_{t_n}\cdot\boldsymbol{P}_4 + \mathbb{E}(\boldsymbol{X}_{t_n-}-\boldsymbol{X}_n)\cdot\boldsymbol{P}_4\\
&= \sum_{j=1}^{M}\mathbb{E}\int_{t_n}^{t_{n+1}}\int_0^A(\boldsymbol{X}_{t_n-}-\boldsymbol{X}_n)\\
&\cdot\ \boldsymbol{\nu}_j\Big(c_j(a;\boldsymbol{X}_{t_n-})-c_j(a;\boldsymbol{X}_n)\Big)(\lambda-m)(dt\times da)\\
&= 0
\end{aligned}
$$

because of Lemma 3.1 and the independence between $\boldsymbol{X}_{t_n-}$, $\boldsymbol{X}_n$, and the Poisson random measure λ in $[t_n,t_{n+1}]\times[0,A]$. A similar idea applies to $\boldsymbol{E}_n\cdot\boldsymbol{R}_2$. That is, $\mathbb{E}\boldsymbol{E}_n\cdot\boldsymbol{R}_2=0$.

For the other terms, we have

$$\mathbb{E}\boldsymbol{E}_n\cdot\boldsymbol{P}_3 \le \Big(\mathbb{E}\boldsymbol{E}_n^2\cdot\mathbb{E}\boldsymbol{P}_3^2\Big)^{\frac{1}{2}} \le C_3\delta t_n\mathbb{E}\boldsymbol{E}_n^2,$$

$$\mathbb{E}\boldsymbol{E}_n\cdot\boldsymbol{R}_1 \le \delta t_n\mathbb{E}\boldsymbol{E}_n^2 + \frac{1}{4\delta t_n}\mathbb{E}\boldsymbol{R}_1^2 \le \delta t_n\mathbb{E}\boldsymbol{E}_n^2 + C_4\delta t_n^2,$$

$$\mathbb{E}\boldsymbol{P}_3\cdot\boldsymbol{P}_4 \le \frac{1}{2}(\mathbb{E}\boldsymbol{P}_3^2+\mathbb{E}\boldsymbol{P}_4^2),\quad \mathbb{E}\boldsymbol{P}_3\cdot\boldsymbol{R}_1 \le \frac{1}{2}(\mathbb{E}\boldsymbol{P}_3^2+\mathbb{E}\boldsymbol{R}_1^2),$$

$$\mathbb{E}\boldsymbol{P}_3\cdot\boldsymbol{R}_2 \le \frac{1}{2}(\mathbb{E}\boldsymbol{P}_3^2+\mathbb{E}\boldsymbol{R}_2^2),\quad \mathbb{E}\boldsymbol{P}_4\cdot\boldsymbol{R}_1 \le \frac{1}{2}(\mathbb{E}\boldsymbol{P}_4^2+\mathbb{E}\boldsymbol{R}_1^2),$$

$$\mathbb{E}\boldsymbol{P}_4\cdot\boldsymbol{R}_2 \le \frac{1}{2}(\mathbb{E}\boldsymbol{P}_4^2+\mathbb{E}\boldsymbol{R}_2^2),\quad \mathbb{E}\boldsymbol{R}_1\cdot\boldsymbol{R}_2 \le \frac{1}{2}(\mathbb{E}\boldsymbol{R}_1^2+\mathbb{E}\boldsymbol{R}_2^2).$$

Finally we have the following energy estimate:

$$\mathbb{E}\boldsymbol{E}_{n+1}^2 \le \mathbb{E}\boldsymbol{E}_n^2 + C\delta t_n\mathbb{E}\boldsymbol{E}_n^2 + C\delta t_n^2. \qquad (3.22)$$

The discrete Gronwall inequality shows that

$$\sup_{n\le N_T}\mathbb{E}|\boldsymbol{X}_n-\boldsymbol{X}_{t_n}|^2 \le C\tau \qquad (3.23)$$

immediately. That ends the proof of Theorem 3.3. ∎

REMARK 3.1. *The estimates* (3.13) *and* (3.19) *are essential for the convergence analysis. It sufficiently utilizes the property of integer jump size. If there is no* (3.13)*, it might not be convergent. If there is no* (3.19)*, the convergence order would be* $\frac{1}{4}$*. This case can be easily generalized into the jump process on a scaled version of the integer lattices with finite* M*. But it is still a problem on how to prove the mean square convergence for the unrounded implicit tau-leaping scheme* [37] *and the jump diffusion process with evolving intensity.*

4. Weak convergence. The weak convergence of Euler discretization for some types of Lévy processes has been considered in [35, 27]. Our result here for the explicit tau-leaping scheme has much fewer restrictions on the coefficients and function $g(\boldsymbol{x})$ (see Theorem 4.3) because of its particularity. The basic idea is the same as that in [35] and that for the weak convergence of numerical SDEs [30, 40] through backward equations. We should comment that our method cannot deal with the convergence for semi-implicit discretization, while the weak convergence was proven for both explicit and implicit tau-leaping methods under linear propensity assumptions in [37]. The analysis of the implicit tau method for general nonlinear propensity functions is still an issue.

We have the backward equation for $P(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0)$ from the Chapman–Kolmogorov equation of the Markov process

$$\begin{aligned} P(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0) &= \sum_{j=1}^{M} P(\boldsymbol{x}, t|\boldsymbol{x}_0 + \boldsymbol{\nu}_j, t_0 + dt) \cdot a_j(\boldsymbol{x}_0)dt \\ &\quad + \Big(1 - \sum_{j=1}^{M} a_j(\boldsymbol{x}_0)dt\Big) P(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0 + dt). \end{aligned} \tag{4.1}$$

Subtracting $P(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0 + dt)$ from both sides, dividing dt, and taking dt to 0, we obtain

$$-\partial_{t_0} P(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0) = \sum_{j=1}^{M} a_j(\boldsymbol{x}_0)\Big(P(\boldsymbol{x}, t|\boldsymbol{x}_0 + \boldsymbol{\nu}_j, t_0) - P(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0)\Big). \tag{4.2}$$

Actually, the operator on $P(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0)$ on the right-hand side is the adjoint operator that appeared in the CME.

From the backward equation (4.2), we have the infinitesimal generator of this process:

$$\mathcal{A}f = \lim_{t\to 0} \frac{\mathbb{E}^{\boldsymbol{x}} f(\boldsymbol{X}_t) - f(\boldsymbol{x})}{t} = \sum_{j=1}^{M} a_j(\boldsymbol{x})\Big(f(\boldsymbol{x} + \boldsymbol{\nu}_j) - f(\boldsymbol{x})\Big), \tag{4.3}$$

where $\mathbb{E}^x$ is the conditional expectation started from $\boldsymbol{X}_0 = \boldsymbol{x}$.

Define the operator $\mathcal{L}$ as

$$\mathcal{L}v(t, \boldsymbol{x}) = \partial_t v(t, \boldsymbol{x}) + \mathcal{A}v(t, \boldsymbol{x}); \tag{4.4}$$

then it is straightforward to have the following lemma by Dynkin's formula.

LEMMA 4.1. *For any continuous function* $v(t, \boldsymbol{x})$*,*

$$\mathbb{E}v(t, \boldsymbol{X}_t) = \mathbb{E}v(0, \boldsymbol{X}_0) + \mathbb{E}\int_0^t \mathcal{L}v(s, \boldsymbol{X}_s)ds. \tag{4.5}$$

Given $g : \mathbb{R}^N \to \mathbb{R}$ define the function

$$u(t, \boldsymbol{x}) = \mathbb{E}^{\boldsymbol{x}} g(\boldsymbol{X}_{T-t}); \tag{4.6}$$

then we have

$$\mathcal{L}u = 0, \quad u(T, \boldsymbol{x}) = g(\boldsymbol{x}). \tag{4.7}$$

Furthermore, we have the following lemma on the well-posedness of $u(t, \boldsymbol{x})$ on $\mathbb{Z}^N$.

LEMMA 4.2. *If $g(\boldsymbol{x})$ is locally bounded in $\mathbb{R}^N$, then for any $\boldsymbol{x} \in \mathbb{Z}^N \backslash (\mathbb{Z}_0^+)^N$, $u(t, \boldsymbol{x}) = g(\boldsymbol{x})$. And for any $\boldsymbol{x}$ in lattice $(\mathbb{Z}_0^+)^N$, $T > 0$, there exists a unique*

$$u(t, \boldsymbol{x}) \in C^1[0, T] \tag{4.8}$$

such that $u(t, \boldsymbol{x})$ is the solution of (4.7). For $\boldsymbol{x} \in (\mathbb{Z}_0^+)^N$, define

$$\Omega_{\boldsymbol{x}} + \boldsymbol{\nu}_j := \Big\{ \boldsymbol{y} + \boldsymbol{\nu}_j, \ \boldsymbol{y} \in \Omega_{\boldsymbol{x}} \Big\} \quad \textit{and} \quad \Omega_{\boldsymbol{x}}^s = \bigcup_{j=1}^{M} \Big\{ \Omega_{\boldsymbol{x}} + \boldsymbol{\nu}_j \Big\};$$

then we have the following estimate:

$$\max_{\boldsymbol{y} \in \Omega_{\boldsymbol{x}}} |u(t, \boldsymbol{y})| \le (AMT + 1) \max_{\boldsymbol{y} \in \Omega_{\boldsymbol{x}} \cup \Omega_{\boldsymbol{x}}^s} |g(\boldsymbol{y})| \exp(2AMT), \quad t \in [0, T]. \tag{4.9}$$

Proof. At first it is straightforward to observe that

$$u(t, \boldsymbol{x}) = g(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \mathbb{Z}^N \backslash (\mathbb{Z}_0^+)^N, \quad t \in [0, T], \tag{4.10}$$

because $a_j(\boldsymbol{x})$ is zero.

For any fixed $\boldsymbol{x}$ in lattice $(\mathbb{Z}_0^+)^N$, (4.7) may be viewed as an infinite linear ODE system on $\Omega_{\boldsymbol{x}}^t$ with each $u(t, \boldsymbol{x})$ being a function only of t and $\boldsymbol{x}$ being just a label. But it is lucky to have condition (2.25) after the redefinition of $a_j(\boldsymbol{x})$, which means the real dynamics is on $\Omega_{\boldsymbol{x}}$ with only finite equations. It is a trivial result for the existence of the solution for a finite linear ODE system.

In order to prove (4.9), we define $\tilde{u}(t, \boldsymbol{x}) = u(T - t, \boldsymbol{x})$. It is enough to observe that we have

$$\max_{\boldsymbol{y} \in \Omega_{\boldsymbol{x}}} |\tilde{u}(t, \boldsymbol{y})| \le \max_{\boldsymbol{y} \in \Omega_{\boldsymbol{x}}} |g(\boldsymbol{y})| + AMt \max_{\boldsymbol{y} \in \Omega_{\boldsymbol{x}}^s} |g(\boldsymbol{y})| + 2AM \int_0^t \max_{\boldsymbol{y} \in \Omega_{\boldsymbol{x}}} |\tilde{u}(s, \boldsymbol{y})| ds$$

from (4.7). The Gronwall inequality gives the desired estimate. ∎

In order to prove weak convergence for the tau-leaping scheme, we define a *pathwise continuous time extension* $\bar{\boldsymbol{X}}_t$ of $\boldsymbol{X}_n$ as

$$\bar{\boldsymbol{X}}_t = \boldsymbol{X}_n + \sum_{j=1}^{M} \int_{t_n}^{t} \int_0^A \boldsymbol{\nu}_j c_j(a; \boldsymbol{X}_n) \lambda(ds \times da), \quad t \in [t_n, t_{n+1}), \tag{4.11}$$

and so we have $\bar{\boldsymbol{X}}_{t_{n+1}} = \boldsymbol{X}_{n+1}$.

THEOREM 4.3 (weak convergence). *Under Assumptions 2.1–2.3 and Proposition 2.3, for any continuous function $g(\boldsymbol{x})$ satisfying exponential growth condition*

$$|g(\boldsymbol{x})| \le C_g B^{|\boldsymbol{x}|}, \quad \boldsymbol{x} \in \mathbb{R}^N, \quad C_g, B > 0,$$

we have

$$\Big|\mathbb{E}g(\boldsymbol{X}_{N_T}) - \mathbb{E}g(\boldsymbol{X}_T)\Big| \le C\tau, \tag{4.12}$$

where $T = t_{N_T}$, $\tau = \max_n \delta t_n$, *and* C *is a constant that depends on* A, M, K, T, B, *and* C_g.

Proof. Consider $u(t, \boldsymbol{x})$ as that in (4.7); then from Lemma 4.1 we have

$$\begin{aligned}
\mathbb{E}g(\boldsymbol{X}_{N_T}) - \mathbb{E}g(\boldsymbol{X}_T) &= \mathbb{E}u(T, \boldsymbol{X}_{N_T}) - \mathbb{E}u(T, \boldsymbol{X}_T) = \mathbb{E}u(T, \boldsymbol{X}_{N_T}) - \mathbb{E}u(0, \boldsymbol{X}_0) \\
&= \sum_{n=0}^{N_T-1} \Big(\mathbb{E}u(t_{n+1}, \boldsymbol{X}_{n+1}) - \mathbb{E}u(t_n, \boldsymbol{X}_n)\Big) \\
&= \sum_{n=0}^{N_T-1} \Big(\mathbb{E}u(t_{n+1}, \bar{\boldsymbol{X}}_{t_{n+1}}) - \mathbb{E}u(t_n, \bar{\boldsymbol{X}}_{t_n})\Big) \\
&= \sum_{n=0}^{N_T-1} \mathbb{E}\int_{t_n}^{t_{n+1}} \mathcal{L}^n u(t, \bar{\boldsymbol{X}}_t)dt. \qquad (4.13)
\end{aligned}$$

Here the tau-leaping backward operator

$$\mathcal{L}^n u(t, \boldsymbol{x}) = \partial_t u(t, \boldsymbol{x}) + \sum_{j=1}^{M} a_j(\boldsymbol{X}_n)\Big(u(t, \boldsymbol{x} + \boldsymbol{\nu}_j) - u(t, \boldsymbol{x})\Big). \tag{4.14}$$

We have

$$\begin{aligned}
\mathbb{E}\int_{t_n}^{t_{n+1}} \mathcal{L}^n u(t, \bar{\boldsymbol{X}}_t)dt &= \mathbb{E}\int_{t_n}^{t_{n+1}} \Big(\mathcal{L}^n u(t, \bar{\boldsymbol{X}}_t) - \mathcal{L}u(t, \bar{\boldsymbol{X}}_t)\Big)dt \\
&= \sum_{j=1}^{M} \mathbb{E}\Bigg[\int_{t_n}^{t_{n+1}} a_j(\boldsymbol{X}_n) \cdot \Big(u(t, \bar{\boldsymbol{X}}_t + \boldsymbol{\nu}_j) - u(t, \bar{\boldsymbol{X}}_t)\Big)dt \\
&\quad - \int_{t_n}^{t_{n+1}} a_j(\boldsymbol{X}_n) \cdot \Big(u(t, \boldsymbol{X}_n + \boldsymbol{\nu}_j) - u(t, \boldsymbol{X}_n)\Big)dt \\
&\quad + \int_{t_n}^{t_{n+1}} a_j(\boldsymbol{X}_n) \cdot \Big(u(t, \boldsymbol{X}_n + \boldsymbol{\nu}_j) - u(t, \boldsymbol{X}_n)\Big)dt \\
&\quad - \int_{t_n}^{t_{n+1}} a_j(\bar{\boldsymbol{X}}_t) \cdot \Big(u(t, \bar{\boldsymbol{X}}_t + \boldsymbol{\nu}_j) - u(t, \bar{\boldsymbol{X}}_t)\Big)dt\Bigg]. \qquad (4.15)
\end{aligned}$$

Define $h_1(\boldsymbol{x}) = u(t, \boldsymbol{x} + \boldsymbol{\nu}_j) - u(t, \boldsymbol{x})$ and $h_2(\boldsymbol{x}) = a_j(\boldsymbol{x}) \cdot (u(t, \boldsymbol{x} + \boldsymbol{\nu}_j) - u(t, \boldsymbol{x}))$; then for h_2 we have

$$\mathbb{E}\Big(h_2(\bar{\boldsymbol{X}}_t) - h_2(\boldsymbol{X}_n)\Big) = \sum_{j=1}^{M} \mathbb{E}\left(\int_{t_n}^{t} a_j(\boldsymbol{X}_n)\Big(h_2(\bar{\boldsymbol{X}}_s + \boldsymbol{\nu}_j) - h_2(\bar{\boldsymbol{X}}_s)\Big)ds\right)$$

by Lemma 4.1. From Proposition 2.3, $h_2(\boldsymbol{x}) \neq 0$ if and only if $\boldsymbol{x} \in \Omega_{\boldsymbol{X}_0}$. Thus we have

$$\Big|\mathbb{E}\Big(h_2(\bar{\boldsymbol{X}}_t) - h_2(\boldsymbol{X}_n)\Big)\Big| \le 4MA^2C_u\delta t_n, \tag{4.16}$$

where $C_u = \max|u(t,\boldsymbol{x})|$, $\boldsymbol{x} \in \Omega_{\boldsymbol{X}_0} \cup \Omega^s_{\boldsymbol{X}_0}$, and $t \in [0,T]$. It is straightforward that C_u depends on A, M, T, and g from the estimate (4.9) in Lemma 4.2.

For h_1, we have

$$\begin{aligned}
&\mathbb{E}\int_{t_n}^{t_{n+1}} a_j(\boldsymbol{X}_n)\cdot\Big(h_1(\bar{\boldsymbol{X}}_t)-h_1(\boldsymbol{X}_n)\Big)dt\\
&=\mathbb{E}\bigg(\mathbb{E}\bigg(\int_{t_n}^{t_{n+1}} a_j(\boldsymbol{X}_n)\cdot\Big(h_1(\bar{\boldsymbol{X}}_t)-h_1(\boldsymbol{X}_n)\Big)dt\Big|\boldsymbol{X}_n\bigg)\bigg)\\
&=\mathbb{E}\bigg(\int_{t_n}^{t_{n+1}} a_j(\boldsymbol{X}_n)\cdot\mathbb{E}\Big(h_1(\bar{\boldsymbol{X}}_t)-h_1(\boldsymbol{X}_n)\Big|\boldsymbol{X}_n\Big)dt\bigg) \qquad (4.17)
\end{aligned}$$

and

$$\mathbb{E}\Big(h_1(\bar{\boldsymbol{X}}_t)-h_1(\boldsymbol{X}_n)\Big|\boldsymbol{X}_n\Big)=\sum_{j=1}^{M}\mathbb{E}\bigg(\int_{t_n}^{t} a_j(\boldsymbol{X}_n)\Big(h_1(\bar{\boldsymbol{X}}_s+\boldsymbol{\nu}_j)-h_1(\bar{\boldsymbol{X}}_s)\Big)ds\Big|\boldsymbol{X}_n\bigg). \qquad (4.18)$$

Because $h_1(\boldsymbol{x})$ may be unbounded, we have to estimate its expectation. We have

$$\begin{aligned}
&\mathbb{E}\Big(|h_1(\bar{\boldsymbol{X}}_s)|\Big|\boldsymbol{X}_n\Big)\\
&\le 2C_u+\sum_{k_1,k_2,\ldots,k_M=0}^{\infty} 2C_gB^{|\boldsymbol{X}_n+\sum_{j=1}^{M}k_j\boldsymbol{\nu}_j|}\frac{(A\delta t_n)^{k_1}}{k_1!}\frac{(A\delta t_n)^{k_2}}{k_2!}\cdots\frac{(A\delta t_n)^{k_M}}{k_M!}\\
&\le 2C_u+2C_gB^{|\boldsymbol{X}_n|}\prod_{j=1}^{M}\Big(\sum_{k_j=0}^{\infty}\frac{(B^KA\delta t_n)^{k_j}}{k_j!}\Big)\\
&= 2C_u+2C_gB^{|\boldsymbol{X}_n|}\exp\Big(MAB^K\delta t_n\Big). \qquad (4.19)
\end{aligned}$$

Here $\boldsymbol{X}_n \in \Omega_{\boldsymbol{X}_0}$; otherwise $a_j(\boldsymbol{X}_n)=0$.

Combining (4.13), (4.15), (4.16), (4.17), (4.18), and (4.19), we obtain the final convergence result:

$$|\mathbb{E}g(\boldsymbol{X}_{N_T})-\mathbb{E}g(\boldsymbol{X}_T)|\le C\tau, \qquad (4.20)$$

where C depends on A, M, K, T, B, and C_g. This ends the proof of weak convergence. ∎

REMARK 4.1. *The assumption made on $g(\boldsymbol{x})$ in Theorem 4.3 is not restricted in real applications. In most cases, one is interested in computing the finite order moments of $\boldsymbol{X}_t$. It is satisfied with the exponential growth condition.*

5. Numerical experiments. We will consider two examples to compare the numerical results and the theoretical analysis. Because the weak convergence of mean and variance of tau-leaping schemes has been tested in [37], we will demonstrate only the strong convergence of the scheme here. The results show the sharpness of half order convergence.

EXAMPLE 5.1 (isomerization reaction). *The chemical reaction is*

$$X \longrightarrow \emptyset,$$

with the propensity function $a(x)=cx$, $X(0)=X_0$ in the time interval $[0,T]$. We choose $c=0.1$, $X_0=100$, and $T=1$ in the following.

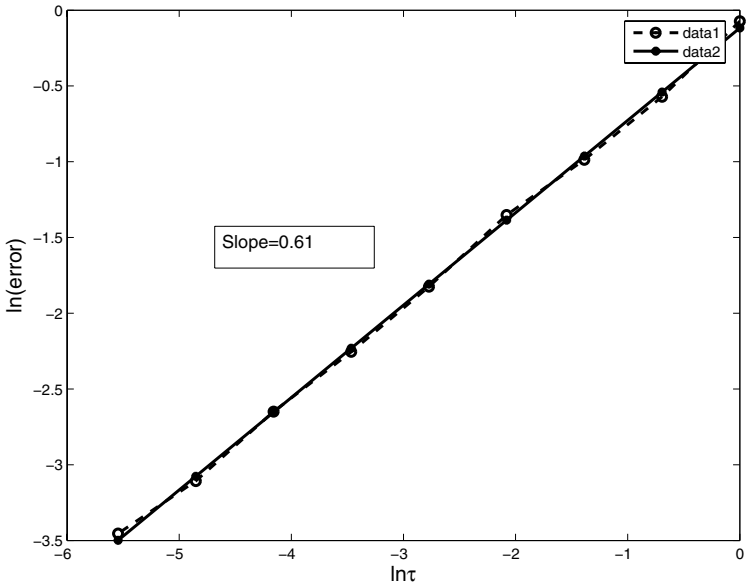

FIG. 1. *The log-log plot of the mean square error versus time stepsize τ and linear fitting in the explicit tau-leaping scheme for the isomerization process. The dashed line is numerical results. The solid line is the linear fitted result. This fitted line has slope* 0.61*, which is a little larger than* 0.5 *in the theoretical analysis.*

For this process, we take $A = 10$ in (2.7). Because the Poisson random measure $\lambda(dt \times dx)$ has intensity measure $dt \times dx = Adt \cdot \frac{1}{A}dx$, we generate the independent random increments $(\Delta t_n, \Delta x_n)$ such that Δt_n is exponentially distributed with rate A, and Δx_n is uniformly distributed in $[0, A]$. We take the uniform time stepsize $\tau = 1, \frac{1}{2}, \ldots, \frac{1}{2^8}$ in the tau-leaping scheme and approximate the mean square error $\left(\mathbb{E}|\boldsymbol{X}_{N_T} - \boldsymbol{X}_T|^2\right)^{\frac{1}{2}}$ with the Monte Carlo method. The sampling size is 1000. The log-log plot of the error versus τ is shown in Figure 1. We may observe that the fitted line has slope 0.61, which is a little larger than 0.5 in the theoretical analysis. We speculate that the numerical higher order arises from fluctuations. This verifies that the strong order is at least 0.5 in the mean square sense.

EXAMPLE 5.2 (decaying-dimerizing reaction). *The chemical reaction is*

$$\begin{aligned} S_1 &\longrightarrow \emptyset, \\ S_1 + S_1 &\longrightarrow S_2, \\ S_2 &\longrightarrow S_1 + S_1, \\ S_2 &\longrightarrow S_3, \end{aligned}$$

with the propensity functions

$$a_1(\boldsymbol{x}) = c_1 x_1, \quad a_2(\boldsymbol{x}) = \frac{c_2}{2} x_1(x_1 - 1), \quad a_3(\boldsymbol{x}) = c_3 x_2, \quad a_4(\boldsymbol{x}) = c_4 x_2$$

and $\boldsymbol{X}(0) = \boldsymbol{X}_0$ in the time interval $[0, T]$. We choose

$$c_1 = 1, \quad c_2 = 0.002, \quad c_3 = 0.5, \quad c_4 = 0.04, \quad \boldsymbol{X}_0 = (1000, 0, 0)$$

and $T = 1$ in the following.

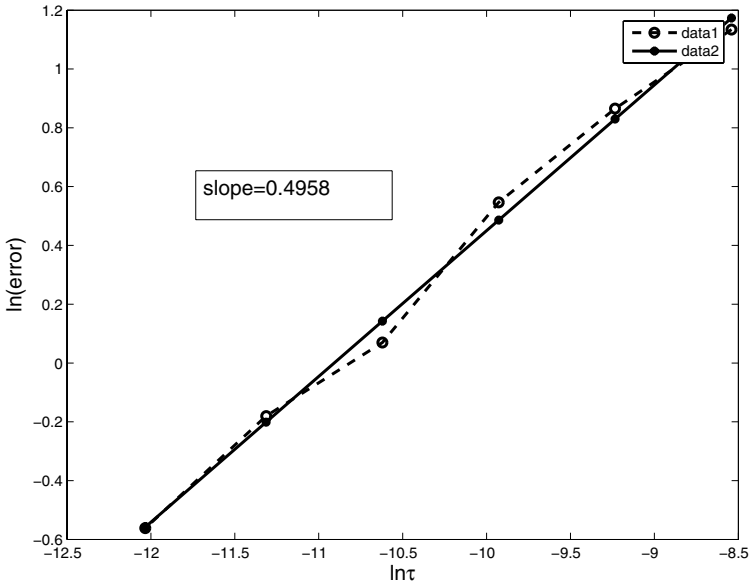

FIG. 2. *The log-log plot of the mean square error versus time stepsize τ and linear fitting in the explicit tau-leaping scheme for the decaying-dimerizing process. The dashed line is numerical results. The solid line is the linear fitted result. This fitted line has slope* 0.4958.

For this process, we take $A = 10000$ in (2.7). The independent random increments $(\Delta t_n, \Delta x_n)$ are similarly generated as in Example 5.1. We take the uniform time stepsize $\tau = 0.1\times\frac{1}{2^9}, 0.1\times\frac{1}{2^{10}}, \ldots, 0.1\times\frac{1}{2^{14}}$ in the tau-leaping scheme and approximate the mean square error $\left(\mathbb{E}|\boldsymbol{X}_{N_T} - \boldsymbol{X}_T|^2\right)^{\frac{1}{2}}$ with the Monte Carlo method. The sampling size is 1000. The log-log plot of the error versus τ is shown in Figure 2. We observe that the fitted line has slope 0.4958, which is quite close to 0.5 in the theoretical analysis. This further verifies that the strong order is at least 0.5 in the mean square sense.

6. Conclusion. This paper builds a convergence analysis of explicit tau-leaping schemes for simulating chemical reactions from the viewpoint of SDEs. Mathematically, the chemical reaction process is a pure jump process on a lattice with state-dependent intensity. The SDE form of the CME can be given via Poisson random measures. Based on this form, different types of tau-leaping schemes can be proposed similar to the construction of numerical methods for SDE driven by Brownian motion. For the theoretical analysis, a modified explicit tau-leaping scheme is considered in order to make the problem well-posed. It is shown that the mean square strong convergence is of order 1/2 and the weak convergence is of order 1 for the modified scheme. This result generalizes the weak convergence analysis in [37]. The analysis depends heavily on that it is a pure jump process on lattice $(\mathbb{Z}_0^+)^N$. It would be very interesting to investigate the convergence result in the jump-diffusion case. In this regard, the paper [28] might be helpful. Furthermore, the current scheme permits the appearance of the nonphysical states, which is not satisfactory in some aspects. To analyze a tau-leaping scheme avoiding negative populations will be our next step.

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