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LIMIT THEOREMS FOR SEQUENCES OF JUMP MARKOV PROCESSES APPROXIMATING ORDINARY DIFFERENTIAL PROCESSES

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1. Introduction

In [3] this author gave conditions under which a sequence of jump Markov processes $X_n(t)$ will converge to the solution $X(t)$ of a system of first order ordinary differential equations, in the sense that

$$\lim_{n\to\infty} P\left\{\sup_{s\leq t}\left|X_n(s) - X(s)\right| > \delta\right\} = 0 \tag{1.1}$$

for every $\delta > 0$.

As was indicated in the earlier paper, this result gives the relationship between the stochastic and deterministic models that have been proposed for many physical, chemical and biological processes.

The implication is that the deterministic model, which is frequently fairly easy to analyze, is for all practical purposes as good as the stochastic model provided that the population or number of particles involved is sufficiently large. The question then becomes what is "sufficiently large". In this paper we consider this question in two ways. In Section 2, we give bounds on the probability in (1.1) using martingale theory and in Section 3 we prove two "central limit theorems", which can be used to estimate the above probability. In Section 4 we specialize the results to what we call density dependent Markov chains, giving examples from epidemiology and chemistry.

The following notation and assumptions will be used throughout: for each n, $X_n(t)$ is a right continuous, temporally homogeneous, jump Markov process with state space (E_n, B_n) where E_n is a Borel measurable subset of r-dimensional Euclidean space ($\boldsymbol{R}^r$), and B_n is the σ-algebra of Borel measurable subsets of E_n; $\lambda_n(x)$ and $\mu_n(x,\Gamma)$ are the waiting time parameter function and the jump distribution function for $X_n(t)$. That is, if

$$\tau = \inf\{t: X_n(t) \neq X_n(0)\},$$

then

$$P\{\tau > t \mid X_n(0) = x\} \equiv P_x\{\tau > t\} = \exp\{-\lambda_n(x)t\}$$

and

$$P_x\{X_n(\tau)\in\Gamma\} = \mu_n(x,\Gamma), \text{ for every } \Gamma\in B_n.$$

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We will assume that $\lambda_n(x)$ is bounded on bounded subsets of E_n, and that

$$\lambda_n(x)\int_{E_n}|z-x|\mu_n(x,dz)<\infty$$

for each $x\in E_n$.

Define

$$F_n(x)=\lambda_n(x)\int_{E_n}(z-x)\mu_n(x,dz),$$

and note $F_n(\cdot): E_n\to \boldsymbol{R}^r$.

The following lemma, a standard result in the theory of ordinary differential equations, is basic to our considerations.

Lemma (1.2). Let E be an open subset of $\boldsymbol{R}^r$ with $E_n\subset E$ and suppose

$$F(\cdot): E\to \boldsymbol{R}^r$$

satisfies

$$|F(x)-F(y)|\leqq L|x-y|\quad x,y\in E$$

for some constant L. Let $X(s,x)$ denote the solution of

$$\frac{\partial}{\partial s}X(s,x)=F(X(s,x))\quad s\leqq t$$
$$X(0,x)=x.$$

Then

$$\sup_{s\leqq t}\left|X_n(s)-X_n(0)-\int_0^s F_n(X_n(u))du\right|\leqq\delta$$

implies

$$\sup_{s\leqq t}|X_n(s)-X(s,x)|\leqq\Big(\delta+|X_n(0)-x|+t\sup_{x\in E_n}|F(x)-F_n(x)|\Big)e^{Lt}.$$

Consequently, if

$$\lim_{n\to\infty}X_n(0)=x$$

and

$$\lim_{n\to\infty}\sup_{x\in E_n}|F(x)-F_n(x)|=0,$$

in order to show that

$$\text{(1.3)}\qquad \lim_{n\to\infty}P\left\{\sup_{s\leqq t}|X_n(s)-X(s,x)|>\delta\right\}=0$$

for every $\delta>0$, it suffices to show that

$$\text{(1.4)}\qquad \lim_{n\to\infty}P\left\{\sup_{s\leqq t}\left|X_n(s)-X_n(0)-\int_0^s F_n(X_n(u))du\right|>\delta\right\}=0$$

for every $\delta>0$.

Since the probabilities in (1.3) depend only upon $\lambda_n(x)$ and $\mu_n(x, \Gamma)$ for x in $K_\delta = \{y: \inf_{s \leqq t} |y - X(s,x)| \leqq \delta\}$ (a compact subset of $\boldsymbol{R}^r$), we may make strong global assumptions about $\lambda_n(x)$ and $\mu_n(x,\Gamma)$ without significantly restricting the applicability of our results.

In particular we will now assume

$$\sup_x \lambda_n(x) < \infty \tag{1.5}$$

and

$$\sup_x \lambda_n(x) \int |z - x| \mu_n(x, dz) < \infty. \tag{1.6}$$

2. Martingale inequalities

In this section we will obtain bounds for the probabilities in (1.4) in terms of $\lambda_n(x)$ and $\mu_n(x,\Gamma)$. Since the fact that we are interested in a sequence of processes is not of particular importance here we will drop the subscript n.

Proposition (2.1). Let $X(t)$ be a jump Markov process with state space (E, B), $E \subset \boldsymbol{R}^r$, life time parameter function $\lambda(x)$, and jump distribution $\mu(x,\Gamma)$ satisfying

$$\sup_x \lambda(x) < \infty \tag{2.2}$$

and

$$\sup_x \lambda(x) \int |z - x| \mu(x, dz) < \infty. \tag{2.3}$$

Then the process

$$Z(t) = X(t) - X(0) - \int_0^t F(X(s))\, ds$$

is a (vector valued) martingale.

Proof. Since $\lambda(x)$ is bounded, every bounded measurable function is in the domain of the infinitesimal operator A for $X(t)$, and

$$Af(x) = \lambda(x) \int_E (f(z) - f(x)) \mu(x, dz).$$

Let

$$f_{m,x}(y) = m \wedge |x - y|.$$

A basic theorem of the theory of semigroups (see Dynkin [2], p. 33) gives

$$E_x(f_{m,x}(X(t))) - f_{m,x}(x)$$
$$= \int_0^t E_x(\lambda(X(s)) \int_E (m \wedge |x - z| - m \wedge |x - X(s)|) \mu(X(s), dz)).$$

By (2.3)

$$M \equiv \sup_x \lambda(x) \int_E |z-x|\mu(x,dz) < \infty$$

and noting that

$$|(m \wedge |x-z| - m \wedge |x - X_n(s)|)| \leqq |z - X(s)|,$$

we have

$$\lim_{m\to\infty} E_x(f_{m,x}(X(t))) = E_x(|X(t)-x|) \leqq tM.$$

Using the fact that $E_x(|X(t)-X(0)|)$ is finite, a similar argument gives

$$E_x(X(t)-X(0)) = E_x\left(\int_0^t F(X(s))\,ds\right)$$

and the fact that $Z(t)$ is a martingale follows from the Markov property.

We will now assume $r=1$. The proofs of the inequalities for a general r-dimensional process are essentially the same as the proofs in the one-dimensional case.

Standard martingale results give

$$\text{(2.5)} \qquad \begin{aligned} &P_x\left\{\sup_{s\leqq t}\left|X(s)-X(0)-\int_0^s F(X(u))\,du\right| > \delta\right\} \\ &\leqq [\phi(\delta)]^{-1}E_x\left(\phi\left(\left|X(t)-X(0)-\int_0^t F(X(u))\,du\right|\right)\right) \end{aligned}$$

for every convex function ϕ. The problem is to estimate the expectation on the right.

In order to do this we consider the Markov process

$$\xi(t) = (X(t), Z(t)).$$

The following is easy to verify:

Lemma (2.6). Let $f(x,z)\equiv f(z)$ be a bounded continuously differentiable function of z. Then f is in the domain of the weak infinitesimal operator $\tilde{A}$ of ξ, and

$$\text{(2.7)} \qquad \begin{aligned} \tilde{A}f(x,z) &= \lambda(x)\int_E (f(w-x+z)-f(z))\mu(x,dw) - F(x)f'(z) \\ &= \lambda(x)\int_E (f(w-x+z)-f(z)-(w-x)f'(z))\mu(x,dw). \end{aligned}$$

Lemma (2.6) implies

(2.8) $E_x(f(Z(t))) - f(0) =$

$$\int_0^t E_x(\lambda(X(s)) \int_E (f(w-X(s)+Z(s)) - f(Z(s)) - (w-X(s))f'(Z(s)))\mu(X(s),dw))ds.$$

This result can be extended to more general functions f.

Lemma (2.9). Let $f(z)$ be any non-negative, continuously differentiable function. If

$$\mathscr{A}f(x,z) = \lambda(x) \int_E (f(w-x+z) - f(z) - (w-x)f'(z))\mu(x,dw) \tag{2.10}$$

is bounded, then

$$E_x(f(Z(t))) \leqq f(0) + \int_0^t E_x(\mathscr{A}f(X(s),Z(s)))\,ds. \tag{2.11}$$

If in addition $f'(z)$ is absolutely continuous, and $f''(z) \geqq 0$ then equality holds in (2.11).

Proof. Let $\{f_m(z)\}$ be an increasing sequence of non-negative bounded functions having bounded continuous derivatives satisfying

$$f_m(z) = f(z) \text{ for } |z| \leqq m.$$

Let

$$\tau_k = \inf\{t: |Z(t)| \geqq k\}.$$

The fact that $\xi(t)$ is a strong Markov process and τ_k is a stopping time implies

$$E_x(f_m(Z(t \wedge \tau_k))) - f(0) = E_x\left(\int_0^{t\wedge\tau_k} \tilde{A}f_m(X(s),Z(s))\,ds\right). \tag{2.12}$$

If $s < \tau_k$ and $m \geqq k$ then

$$\begin{aligned}\tilde{A}f_m(X(s),Z(s)) &= \lambda(X(s)) \int_E [f_m(w - X(s) + Z(s)) - f(Z(s)) \\ &\qquad - (w - X(s))f'(Z(s))]\mu(X(s),dw) \\ &\leqq \lambda(X(s)) \int_E [f(w - X(s) + Z(s)) - f(Z(s)) \\ &\qquad - (w - X(s))f'(Z(s))]\mu(X(s),dw) \\ &= \mathscr{A}f(X(s),Z(s)),\end{aligned}$$

and as m goes to infinity $\tilde{A}f_m$ increases to $\mathscr{A}f$. This gives

$$E_x(f(Z(t \wedge \tau_k))) - f(0) = E_x\left(\int_0^{t\wedge\tau_k} \mathscr{A}f(X(s),Z(s))\,ds\right). \tag{2.13}$$

Finally, since $E(|Z(t)|) < \infty$, (2.5) with $\phi(z) = |z|$ implies $\lim_{k\to\infty} t \wedge \tau_k = t$. Applying Fatou's Lemma to the left hand side of (2.13) and the Dominated Convergence Theorem to the right, (2.11) follows.

If $f'(z)$ is absolutely continuous then

$$\mathscr{A}f(x,y) = \lambda(x) \int_E \int_0^{w-x} \int_0^u f''(z+v)\,dv\,du\,\mu(x,dw)$$

and for appropriately chosen f_m

$$\tilde{A}f_m(x,y) = \lambda(x) \int_E \int_0^{w-x} \int_0^u f''_m(z+v)\,dv\,du\,\mu(x,dw).$$

If $f''(z) > 0$ we can select f_m so that f'_m is absolutely continuous and f''_m increases to f'', and we obtain equality in (2.11).

To obtain bounds on the right hand side of (2.5) we consider two cases.

Case 1. ($\phi(0) = \phi'(0) = 0$, $\phi'' \geqq 0$ and decreasing, for example $\phi(u) = u^\alpha$, $1 < \alpha \leqq 2$.)

Setting
$$f(z) = \phi(|z|)$$

$$\begin{aligned} f(z+u) - f(z) - uf'(z) &= \int_0^u \int_0^v f''(z+w)\,dw\,dv \\ &\leqq 2\int_0^{|u|} \int_0^{v/2} f''(w)\,dw\,dv \\ &= 4\phi(\tfrac{1}{2}|u|). \end{aligned}$$

Consequently

$$P_x\left\{\sup_{s\leqq t} |Z(s)| > \delta\right\} \leqq [\phi(\delta)]^{-1} 4t \sup_x \lambda(x) \int_E \phi(\tfrac{1}{2}|w-x|)\mu(x,dw).$$

In the multidimensional case one can obtain the corresponding inequality

$$P_x\left\{\sup_{s\leqq t} |Z(s)| > \delta\right\} \leqq [\phi(\delta)]^{-1} 4t \sup_x \lambda(x) \int_E \sum_{i=1}^r \phi(\tfrac{1}{2}|w_i - x_i|)\mu(x,dw).$$

Case 2. ($\phi(0) = \phi'(0) = 0$, $\phi'' \geqq 0$ and bounded, for example

$$\phi(u) = \begin{cases} u^\alpha, & u \leqq 1 \\ \alpha u - (\alpha - 1), & u > 1 \end{cases}$$

for $\alpha \geqq 2$. Note, in this example $\phi(u) \leqq u^\alpha$.) Let

$$\phi_\varepsilon(u) = \begin{cases} \phi(u-\varepsilon), & u \geqq \varepsilon \\ 0, & u < \varepsilon \end{cases}$$

and

$$f_\varepsilon(z) = \phi_\varepsilon(|z|).$$

Trivially

$$f_\varepsilon(z+u) - f_\varepsilon(z) - uf_\varepsilon'(z) \leqq \begin{cases} \phi(|u|), & |z| \leqq \varepsilon \\ \frac{1}{2}\|\phi''\|u^2 & |z| > \varepsilon. \end{cases}$$

Consequently for $\varepsilon < \delta$

$$P\left\{\sup_{s\leqq t} |Z(s)| > \delta\right\} \leqq [\phi(\delta-\varepsilon)]^{-1} \left[t \sup_x \lambda(x) \int_E \phi(|w-x|)\mu(x,dw) P\left\{\sup_{s\leqq t} |Z(s)| \leqq \varepsilon\right\}\right.$$

$$\left. + \; t\|\phi''\| \sup_x \lambda(x) \int_E \tfrac{1}{2}|w-x|^2 \mu(x,dw) P\left\{\sup_{s\leqq t} |Z(s)| > \varepsilon\right\}\right].$$

Let

$$M_\phi = \sup_x \lambda(x) \int_E \phi(|w-x|)\mu(x,dw)$$

and

$$M_2 = \sup_x \lambda(x) \int_E \tfrac{1}{2}|w-x|^2 \mu(x,dw).$$

Replacing δ by ε and ε by $\varepsilon/2$, one obtains

$$P\left\{\sup_{s\leqq t} |Z(s)| > \delta\right\} \leqq \frac{t}{\phi(\delta-\varepsilon)} M_\phi + \frac{t}{\phi(\delta-\varepsilon)}\|\phi''\| M_2 \frac{1}{\phi(\varepsilon/2)}[tM_\phi + t\|\phi''\|M_2].$$

More generally

$$P\left\{\sup_{s\leqq t} |Z(s)| > \delta\right\} \leqq \frac{t}{\phi(\delta-\varepsilon)} M_\phi \left(1 + \sum_{l=1}^{k} \frac{(\|\phi''\|M_2 t)^l}{\Pi_{m=1}^{l}\phi(\varepsilon/2^m)}\right)$$

$$+ \frac{1}{\phi(\delta-\varepsilon)} \frac{(\|\phi''\|M_2 t)^{k+1}}{\Pi_{m=1}^{k}\phi(\varepsilon/2^m)} P\left\{\sup_{s\leqq t} |Z(s)| > \varepsilon/2^k\right\}.$$

The same inequality holds in the multidimensional case.

Inequalities of the type derived above are explored in more detail in [4].

3. A central limit theorem

We now return to the consideration of a sequence of processes taking values in $\boldsymbol{R}^r$. We define

$$F_n(x) = \lambda_n(x) \int_{E_n} (z-x)\mu_n(x,dz),$$

$$g_{ij}^n(x) = \alpha_n^2 \lambda_n(x) \int_{E_n} (z_i - x_i)(z_j - x_j)\mu_n(x,dz),$$

and

$$G_n(x) = ((g_{ij}^n(x))) \text{ (the } r \times r \text{ matrix with elements } g_{ij}^n(x)),$$

where α_n is a sequence of positive numbers going to infinity.

Theorem (3.1) Suppose $F_n(x)$ and $G_n(x)$ converge uniformly to $F(x)$ and $G(x)$ where $F(x)$ is bounded and Lipschitz continuous with constant L, and $G(x)$ is bounded and uniformly continuous. Suppose there is a sequence ε_n decreasing to zero such that

$$\text{(3.2)} \qquad \lim_{n\to\infty} \sup_{x\in E_n} \alpha_n^2 \lambda_n(x) \int_{\alpha_n|z-x|>\varepsilon_n} |z-x|^2 \mu_n(x, dz) = 0.$$

Then

$$W_n(t) = \alpha_n(X_n(t) - X_n(0) - \int_0^t F_n(X_n(u))\,du) = \alpha_n Z_n(t)$$

is tight in D, the space of right continuous functions with left limits (see Billingsley [1]) and if $X_n(0) \to x$, $W_n(t)$ converges to the diffusion $W(t)$ with characteristic function

$$E(\exp\{i\theta W(t)\}) \equiv \phi(t,\theta) = \exp\left\{ -\tfrac{1}{2} \sum_{i,j} \theta_i \theta_j \int_0^t g_{ij}(X(s,x))\,ds \right\}.$$

Proof. The tightness of the sequence $W_n(t)$ follows from the Markov property and the inequality

$$\text{(3.3)} \qquad P\left\{ \sup_{s\leq t} |W_n(t)| > \delta \right\} \leq \frac{t\alpha_n^2}{\delta^2} \sup_x \lambda_n(x) \int_{E_n} |z-x|^2 \mu_n(x, dz)$$

which holds by the results of the previous section.

Using the results in Section 1 and Section 2 we obtain

$$P\left\{ \sup_{s\leq t} |X_n(s) - X(s,x)| > \left(\delta + |X_n(0) - x| + \sup_z |F(z) - F_n(z)|\right) e^{Lt} \right\}$$

$$\leq \frac{t}{\delta^2} \sup_x \lambda_n(x) \int_{E_n} |z-x|^2 \mu_n(x, dz)$$

and hence if $\lim_{n\to\infty} X_n(0) = x$

$$\text{(3.4)} \qquad \lim_{n\to\infty} P\left\{ \sup_{s\leq t} |X_n(s) - X(s,x)| > \varepsilon \right\} = 0$$

for every $\varepsilon > 0$. Letting

$$\theta x = \sum_{j=1}^{r} \theta_j x_j \text{ if } \theta, x \in \mathbf{R}^r,$$

define

$$\phi_n(t,\theta) = E(\exp\{i\theta W_n(t)\}).$$

By the natural extension of (2.7) to the r-dimensional case we have

$$\phi_n(t,\theta) - 1 = \int_0^t E(\lambda_n(X_n(s))\int_{E_n} [\exp\{i\alpha_n\theta(w - X_n(s) + Z_n(s))\}$$

$$- \exp\{i\alpha_n\theta Z_n(s)\} - i\alpha_n\theta(w - X_n(s))\exp\{i\alpha_n\theta Z_n(s)\}]\mu_n(X_n(s), dw))$$

$$= - \int_0^t E\left(\tfrac{1}{2} \sum_{j,k} \theta_j\theta_k g^n_{jk}(X_n(s))\exp\{i\theta W_n(s)\}\right) ds$$

$$+ \int_0^t E(\exp\{i\theta W_n(s)\}\lambda_n(X_n(s)) \int_{E_n} \psi(\alpha_n\theta(w - X_n(s)))(\alpha_n\theta(w - X_n(s)))^2\mu_n(X_n(s), dw))ds,$$

where

$$\psi(u) = (e^{iu} - 1 - iu + \tfrac{1}{2}u^2)/u^2.$$

Noting that $\psi(u)$ is bounded and $\lim_{u\to 0}\psi(u) = 0$, (3.2) implies the second term on the right, call it $K_n(\theta)$, goes to zero as n goes to infinity. This leaves

$$\phi_n(t,\theta) - 1 = - \int_0^t \tfrac{1}{2} \sum_{j,k} \theta_j\theta_k g_{jk}(X(s,x))\phi_n(s,\theta)\, ds$$

$$+ E\left(\int_0^t \tfrac{1}{2} \sum_{j,k} \theta_j\theta_k(g_{jk}(X(s,x)) - g^n_{jk}(X_n(s)))\exp\{i\theta W_n(s)\}ds\right) + K_n(\theta).$$

The second term on the right, call it $J_n(\theta)$, goes to zero by the uniform convergence of g^n_{jk} to g_{jk}, the uniform continuity of g_{jk}, and (3.4).

Letting

$$\Gamma(\theta) = \sup_x \tfrac{1}{2} \sum_{j,k} \theta_j\theta_k g_{jk}(x),$$

we have

$$|\phi_n(t,\theta) - \phi(t,\theta)| \leqq |J_n(\theta) + K_n(\theta)|\exp\{\Gamma(\theta)t\},$$

and the theorem follows.

Theorem (3.5). In addition to the hypotheses of Theorem (3.1), suppose

$$\lim_{n\to\infty} \sup_x \alpha_n|F_n(x) - F(x)| = 0$$

and that $F(x)$ has uniformly continuous first partial derivatives. Then

$$\lim_{n\to\infty} \alpha_n(X_n(0) - x) = v$$

implies

$$V_n(t) = \alpha_n(X_n(t) - X(t,x))$$

converges weakly to the diffusion $V(t)$ with $V(0) = v$ and characteristic function $\Psi(t,\theta)$ satisfying

$$(3.6)\quad \frac{\partial \Psi}{\partial t}(t,\theta) = -\tfrac{1}{2}\sum_{j,k}\theta_j\theta_k g_{jk}(X(t,x))\Psi(t,\theta) + \sum_{j,k}\theta_j\partial_k F^j(X(t,x))\frac{\partial \Psi}{\partial \theta_k}(t,\theta).$$

Noting that under an appropriate change of variable (3.6) becomes

$$\frac{\partial \Psi}{\partial u} = \Gamma\Psi,$$

the proof of Theorem (3.5) is essentially the same as the proof of Theorem (3.1).

4. Density dependent Markov chains

We call a one-parameter family of Markov chains $X_A(t)$, $A > 0$, taking values in $\boldsymbol{Z}^r$ (r-vectors with integer components), density dependent if the infinitesimal parameters for $X_A(t)$ can be represented in the form

$$q^A_{k,k+l} = Af(A^{-1}k,l),\ \ k,l \in \boldsymbol{Z}^r.$$

We will refer to the functions $f(x,l)$ as the parameter functions.

In addition to the prey-preditor model discussed in [3], examples of density dependent Markov chains include models for chemical reactions and the epidemic model (appropriately formulated).

Chemical Reaction. Consider the simple reaction $B + C \to D$ taking place in a solution of volume A. The reaction occurs if a molecule of B and a molecule of C come "sufficiently close" together. It is reasonable to assume that the chance of this occurring in a small interval of time is proportional to the number of molecules of B, the number of molecules of C and inversely proportional to the volume of the container. This leads one to formulate the following Markov chain model.

Let k_1 be the number of molecules of B and k_2 the number of molecules of C. The only possible transitions are

$$(k_1,k_2) \to (k_1 - 1, k_2 - 1)$$

and the corresponding infinitesimal parameters are given by

$$\frac{\lambda k_1 k_2}{A} = A\lambda \frac{k_1}{A}\frac{k_2}{A}.$$

In other words, the models form a density dependent family of Markov chains with

$$f(x,(-1,-1)) = \lambda x_1 x_2$$

and

$$f(x,l) = 0 \quad l \neq (-1,-1).$$

Epidemic Model. Let i be the number of infectious individuals and s the number of susceptible individuals in a total population A. (The remaining

$A-(i+s)$ individuals are immune.) In order for a susceptible to become infected during a short interval of time, there must be an infectious individual among the people he encounters during that interval of time. Therefore it is reasonable to assume that the chances of one of the susceptibles becoming infected in a short interval of time is proportional to the number of susceptibles and to the fraction of the population that is infectious. Assuming that there are only two possible transitions, the infection of a susceptible and the recovery of an infectious, we formulate the model with parameters given as follows.

$$(i,s) \to (i+1, s-1) \qquad \lambda s \frac{i}{A} = A\lambda \frac{s}{A}\frac{i}{A},$$

$$(i,s) \to (i-1, s) \qquad \mu i = A\mu \frac{i}{A}.$$

The parameter functions are

$$f(x,(1,-1)) = \lambda x_1 x_2$$

$$f(x,(-1,0)) = \mu x_1.$$

We also note that a multitype Markov branching process is another example of a density dependent family. In this case all of the parameter functions are of the form $f(x,l) = \Sigma \alpha_i(l) x_i$.

To apply the results of the previous sections to density dependent families we consider $A^{-1}X_A(t)$. The waiting time parameter function and jump distribution are given by

$$\lambda_A(x) = A \sum_l f(x,l)$$

and

$$\mu(x, \{x + A^{-1}l\}) = \frac{f(x,l)}{\Sigma_l f(x,l)}.$$

The parameter α_A used in Section 3 is just $\sqrt{A}$. This gives

$$F(x) = \sum_l l f(x,l)$$

and

$$g_{ij}(x) = \sum_l l_i l_j f(x,l).$$

Defining

$$Z_A(t) = A^{-1}X_A(t) - A^{-1}X_A(0) - \int_0^t F(A^{-1}X_A(s))\,ds,$$

the inequalities of Section 2 become

Case 1. ($\phi(0) = \phi'(0) = 0$, $\phi'' \geqq 0$ and decreasing.)

$$P\left\{\sup_{s \leqq t} |Z_A(t)| > \delta\right\} \leqq [\phi(\delta)]^{-1} 4tA \sup_x \sum_l \sum_{i=1}^r \phi((2A)^{-1}|l_i|)f(x,l). \tag{4.1}$$

Case 2. ($\phi(0) = \phi'(0) = 0$, $\phi'' \geqq 0$ and bounded.)

$$\begin{aligned} P\left\{\sup_{s \leqq t} |Z_A(t)| > \delta\right\} &\leqq \frac{t}{\phi(\delta-\varepsilon)} M_\phi\left(1 + \sum_{l=1}^k \frac{(\|\phi''\|M_2 t)^l}{\Pi_{m=1}^l \phi(\varepsilon/2^m)}\right) \\ &\quad + \frac{1}{\phi(\delta-\varepsilon)} \frac{(\|\phi''\|M_2 t)^{k+1}}{\Pi_{m=1}^k \phi(\varepsilon/2^m)} P\left\{\sup_{s \leqq t} |Z_A(s)| > \varepsilon/2^k\right\}, \end{aligned} \tag{4.2}$$

where

$$M_\phi = A \sup_x \sum_l \phi(A^{-1}|l|)f(x,l)$$

and

$$M_2 = (2A)^{-1} \sup_x \sum_l |l|^2 f(x,l).$$

Using the inequality (4.1) for

$$\phi(u) = \begin{cases} u^2, & u < 1 \\ 2u - 1, & u \geqq 1 \end{cases}$$

it is easy to see

$$\lim_{A \to \infty} P\left\{\sup_{s \leqq t} |Z_A(s)| > \delta\right\} = 0 \tag{4.3}$$

provided

$$\lim_{d \to \infty} \sup_x \sum_{|l| > d} |l| f(x,l) = 0. \tag{4.4}$$

We note that if

$$\lim_{A \to \infty} A^{-1} X_A(0) = x_0$$

and $F(x)$ is Lipschitz continuous in a neighborhood K of the trajectory

$$X(s, x_0) \qquad 0 \leqq s \leqq t,$$

$$X(0, x_0) = x_0,$$

$$\frac{\partial X}{\partial s}(s, x_0) = F(X(s, x_0)),$$

then in (4.4) the supremum may be taken over K and one concludes from (4.3) that

$$\lim_{A \to \infty} P\left\{\sup_{s \leqq t} |A^{-1} X_A(s) - X(s, x_0)| > \delta\right\} = 0.$$

Theorem (3.1) is valid for $W_A(t) = \sqrt{A}Z_A(t)$ if in addition

$$\lim_{d \to \infty} \sup_{x \in K} \sum_{|l| > d} |l|^2 f(x, l) = 0. \tag{4.5}$$

Then $W_A(t)$ converges weakly to the diffusion process with independent increments having the characteristic function

$$E(\exp\{i\theta W(t)\}) = \exp\left\{-\tfrac{1}{2}\sum_{i,j}\theta_i\theta_j\int_0^t g_{ij}(X(s,x))\,ds\right\}.$$

Remark. In some sense this appears to mean that $X_A(t)$ is "close" to the solution of

$$X(t) = X(0) + A^{-\frac{1}{2}}\int_0^t \sigma(X(s))dB + \int_0^t F(X(s))\,ds,$$

where $\sigma(x)$ is the square root of the $r \times r$ matrix $G(x)$ and dB denotes integration with respect to r-dimensional Brownian motion.

Theorem (3.5) is valid if $F(x)$ has uniformly continuous first partial derivatives.

References

[1] BILLINGSLEY, P. (1968) *Convergence of Probability Measures.* John Wiley and Sons, New York.

[2] DYNKIN, E. B. (1965) *Markov Processes* I. Academic Press, New York.

[3] KURTZ, T. G. (1970) Solutions of ordinary differential equations as limits of pure jump Markov processes. *J. Appl. Prob.* **7**, 49–58.

[4] KURTZ, T. G. (1970) Inequalities for the law of large numbers. (To appear)