Homework 12

1. Why is $x^3 - x^2 + \varepsilon = 0$ a singular perturbation problem? That is, in what sense does the exact solution undergo an abrupt change in character in the limit $\varepsilon \to 0$? Use the perturbation theory to approximate the roots for small ε.

2. Analyze in the limit $\varepsilon \to 0$ the roots of the polynomial
$$\varepsilon x^3 + x^2 - 2x + 1 = 0.$$

3. Compute all of the coefficients in the perturbation series solution to the initial-value problem $y' = y + \varepsilon xy$ (y(0)=1). Show that the series converges fro all values of ε. Also compute the perturbation series indirectly by expanding the explicit exact solution in powers of ε.

4. (a) Explain the following paradox. We can use perturbation theory to solve the initial-value problem $dy^n/dx^n = \varepsilon y$ $[y(0) = y_0,\ y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}]$ as a power series in ε [That is, substitute $y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$ into the ODE and get the solution $y_n(x)$ from the initial values as $y_0(0) = y_0,\ y_0'(0) = y_1, \ldots, y_0^{(n-1)}(0) = y_{n-1}; y_k(0) = 0,\ y_k'(0) = 0, \ldots, y_k^{(n-1)}(0) = 0$ for $k \geq 1$]. On the other hand, solutions to $dy^n/dx^n = \varepsilon y$ have the form $e^{\omega \varepsilon^{1/n} x}$, where ω is an nth root of unity. Such solutions may be expanded in powers of $\varepsilon^{1/n}$. Which expansion is correct?
(b) Carefully contrast this perturbation problem, which is regular, with the polynomial perturbation problem $x^n = \varepsilon f(x)$, where $f(x)$ is a polynomial of degree at most $n-1$ and $f(0) = 1$. The latter problem is singular.

5. (a) Apply regular perturbation theory to first order in ε to estimate the effect of the εx^{19} perturbation upon the roots of the Wilkinson's polynomial. Show that the root at $x = k$ changes by the amount
$$(-1)^{k+1} \frac{k^{19}}{(k-1)!(20-k)!} + O(\varepsilon^2), \quad \varepsilon \to 0.$$
(b) Show that the unperturbed root at $x = 16$ is most sensitive to ε. Estimate the magnitude of ε necessary to perturb each of the roots by 1 percent.