Homework 11

1. Examine the behavior of the trajectories in the phase plane for the following systems of equations:

   (a). $\dot{y}_1 = y_2, \dot{y}_2 = \pm a^2(1 - y_1^2)y_2 - y_1$ (Van der Pol's equation);

   (b). $\dot{y}_1 = y_2, \dot{y}_2 = (\sin y_2) - y_1$ (Show that this system has an infinite number of limit cycles which are alternately stable and unstable).

2. Show that the Jacobian

$$J(t) = \frac{\partial[\mathbf{p}(t), \mathbf{q}(t)]}{\partial[\mathbf{p}(0), \mathbf{q}(0)]}$$

   satisfies $J(t) = 1$ for all t if $\mathbf{p}$ and $\mathbf{q}$ satisfy the Hamiltonian system

$$\begin{aligned} \frac{dq_j}{dt} &= \frac{\partial H}{\partial p_j} \\ \frac{dp_j}{dt} &= -\frac{\partial H}{\partial q_j}. \end{aligned}$$

3. (a). Prove that when $r < 1$ the only critical point of the Lorenz model

$$\begin{aligned} \frac{dx}{dt} &= -3(x - y) \\ \frac{dy}{dt} &= -xz + rx - y \\ \frac{dz}{dt} &= xy - z \end{aligned}$$

   is at $x = y = z = 0$ and it is stable. Show also that when $r > 1$ the origin becomes an unstable critical point. What happens when $r = 1$?

   (b). Show that when $r > 1$ there are critical points at $x = y = \pm\sqrt{r - 1}, z = r - 1$. Prove that these critical points are stable if $1 < r < 21$ and unstable if $r > 21$.