Homework 09

1. Prove that

$$\int_0^\infty \frac{e^{-t}dt}{1+xt^2} \sim \sum_{n=0}^{\infty} (-1)^n (2n)! x^n, \quad x \to 0+.$$

2. We define a general asymptotic expansion as follows. Let $\phi_n(x)$ be a sequence of functions satisfying $\phi_{n+1} \ll \phi_n$ $(x \to x_0)$ for all n. Then we write $y(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$ $(x \to x_0)$ and say that the series is asymptotic to $y(x)$ as $x \to x_0$ if $y(x) - \sum_{n=1}^{N} a_n \phi_n(x) \ll \phi_N(x)$ $(x \to x_0)$ for all N.

   (a). Show that if $y \sim \sum a_n \phi_n(x)$ and $z \sim \sum b_n \phi_n(x)$ as $x \to x_0$, then $c_1 y + c_2 z \sim \sum (c_1 a_n + c_2 b_n) \phi_n$ $(x \to x_0)$.

   (b). Show that if $\psi_n(x) = \int_{x_0}^{x} \phi_n(t)dt$ exists for each n and if all the functions $\phi_n(x)$ are positive for $x_0 < x < x_1$ for some x_1, then $y \sim \sum a_n \phi_n$ as $x \to x_0$ implies $\int_{x_0}^{x} y(t)dt \sim \sum_{n=1}^{\infty} a_n \psi_n(x)(x \to x_0+)$.

3. Show that if $y(x), p(x), p'(x), q(x)$ are expandable in asymptotic power series as $x \to x_0$ and if y satisfies $y'' + p(x)y' + q(x)y = 0$, then y' and y'' are also expandable in asymptotic series which may be obtained by differentiating the series for y termwise.

4. Show that if $f(x) \sim x^p$ $(x \to +\infty)$ with $p \geq 1$ and $f''(x) > 0$ for sufficiently large x, then $f'(x) \sim px^{p-1}(x \to +\infty)$.