Homework 06

1. (a) Show that if $f(x) \sim a(x - x_0)^{-b}$ as $x \to x_0+$, then

   (i) $\int^x f dx \sim [a/(1-b)](x - x_0)^{1-b}$ $(x \to x_0+)$ if $b > 1$ and the path of integration does not pass through x_0;

   (ii) $\int^x f dx \sim c(x \to x_0+)$, where c is a constant if $b < 1$; if $c = 0$, then $\int^x f dx \sim [a/(1-b)] \cdot (x - x_0)^{1-b}$ $(x \to x_0+)$;

   (iii) $\int_{x_0}^x f dx \sim [a/(1-b)] \cdot (x - x_0)^{1-b}$ $(x \to x_0+)$ if $b < 1$;

   (iv) $\int^x f dx \sim a ln(x - x_0)$ $(x \to x_0+)$ if $b = 1$.

   (b) Show that if $f(x) \sim g(x)$ as $x \to x_0$ and $g(x)$ is one-signed in a neighborhood of x_0, then $\int^x f(x)dx \sim \int^x g(x)dx + c$ $(x \to x_0)$, where c is some integration constant.

2. Find the leading behaviors as $x \to 0+$ of the following equations:

   (a) $x^4 y''' = y$;

   (b) $x^4 y''' - 3x^2 y' + 2y = 0$;

   (c) $x^5 y''' - 2xy' + y = 0$.

3. (a) Give an example of an asymptotic relation $f(x) \sim g(x)$ $(x \to \infty)$ that cannot be exponentiated; that is, $e^{f(x)} \sim e^{g(x)}$ $(x \to \infty)$ is false.
   (b) Show that if $f(x) - g(x) \ll 1 \quad (x \to \infty)$, then $e^{f(x)} \sim e^{g(x)}$ $(x \to \infty)$.

4. Find the leading asymptotic behaviors as $x \to \infty$ of the following equation: $xy''' = y'$.

5. What is the leading behavior of solutions to $y'' + x^{-3/2} y' - x^{-2} y = 0$ as $x \to +\infty$? Show that it is inconsistent to assume that $S'' \ll (S')^2$ $(x \to +\infty)$. However, show that the approximate equation $S'' + (S')^2 \sim x^{-2}$ $(x \to +\infty)$ can be solved exactly by assuming a solution of the form $S' = c/x$.

6. Find the leading behavior of solutions to $y' - y/x = \cos x$ as $x \to 0+$. Show that the leading behavior is determined by a three-term dominant balance. Compare this leading behavior with the exact solution.

7. Show that although $[x/(1+x)]\cos x \sim \cos x$ $(x \to \infty)$, it does not follow that $\int_0^x [t/(1+t)]\cos t dt \sim \int_0^x \cos t dt$ $(x \to \infty)$.