Lect 10 Multigrid method

一. 简述:

1. 历史: 1960's Fedorenko (前苏联)

1970's A. Brandt (以色列):

Math. Comp. 31(1977), 333-90: Multilevel adaptive solutions to boundary value problems.

其它: Nicolaides, Hackbush, Xu.

2. 基本手段:

A. Smoothing: 减缩高频误差 (JOR, Gauss-Seidel, ...)

B. Restriction: 残量误差限制于粗网格.

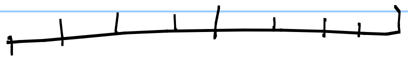
C. Prolongation (或 interpolation): 从粗网格插值到细网格.

3. 结果: 以 $O(N)$ 计算量得到近似解.

二. 问题设置及简单迭代法:

1. 背景 (1D)

$$\begin{cases} -\dfrac{d^2u}{dx^2} = f(x) & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$



$x_j = jh \quad j=0,1,\cdots,n \quad h=1/n$

$$-\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = f(x_j), \quad j=1,2,\cdots,n-1$$

得 $A_h u_h = f_h$

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2\end{pmatrix}, \quad f_h = (f(x_1),\cdots,f(x_{n-1}))^T$$

A. 基本性质:

特征值 λ_k^h 及特征向量 W_k^h

$A_h u = \lambda u \Rightarrow -u_{j+1}+2u_j-u_{j-1} = \lambda h^2 u_j \quad j=1,\cdots,n-1$

令 $u_j=\mu^j \Rightarrow \mu^2+(\lambda h^2-2)\mu+1=0$

由 $u_j = C_+\mu_+^j + C_-\mu_-^j$ 及 $u_0=0 \Rightarrow C_+=-C_-$

由 $u_n=0 \Rightarrow \mu_+^n-\mu_-^n=0 \Rightarrow \mu_+=\omega_n^k\mu_- \quad k=0,1,\cdots,n-1$

$\omega_n = \exp(i\cdot 2\pi/n)$ 为n次单位根.

如$k=0 \Rightarrow \mu_+=\mu_- \Rightarrow u_j\equiv 0$ 舍去! 故 $k=1,\cdots,n-1$

又 $\mu_++\mu_-=2-\lambda h^2, \quad \mu_+\cdot\mu_-=1=\omega_n^k\mu_-^2$

$$\Rightarrow \lambda_k = \frac{1}{h^2}\left[2-(\omega_n^{k/2}+\omega_n^{-k/2})\right] = 4/h^2 \sin^2(k\pi/2n)$$

且 $W_k^h(j) = \mu_+^j - \mu_-^j \sim \sin(jk\pi/n) \quad j,k=1,\cdots,n-1$

B. 简单迭代法:

$$u^{n+1} = B u^n + g$$

· Jacobi 迭代:

$$B_J = D^{-1}(L+L^T)$$

特征值 $\mu_k^J = 1-2\sin^2\frac{k\pi h}{2} \quad k=1,\cdots,n-1$

特征向量与 A_h 相同.

· JOR:

$$B_\omega = (1-\omega)I + \omega B_J \quad \omega\in(0,1)$$

特征值 $\mu_k^\omega = 1-2\omega\sin^2\frac{k\pi h}{2} \quad k=1,\cdots,n-1$

特征向量与 A_h 相同.

· Gauss_Seidel:

$$B_{GS} = (D-L)^{-1}\cdot L^T$$

特征值 $\mu_k^{GS} = \cos^2\frac{k\pi}{n}$

特征向量 $W_k^{GS}(j) = \left(\cos\frac{k\pi}{n}\right)^j \sin\frac{jk\pi}{n}$.

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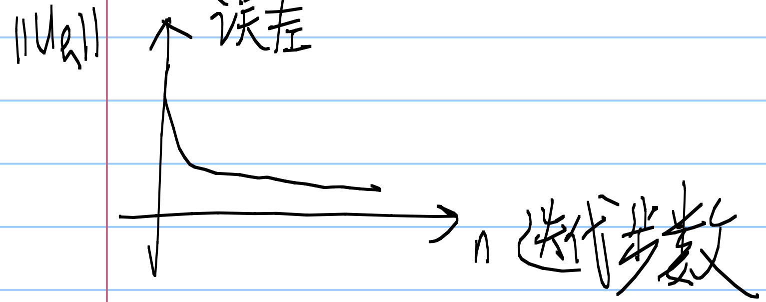
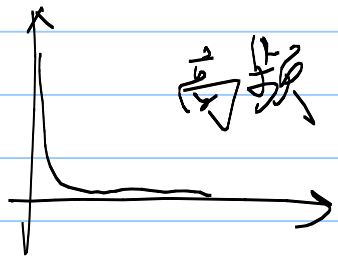
C. 计算结果的典型特征:

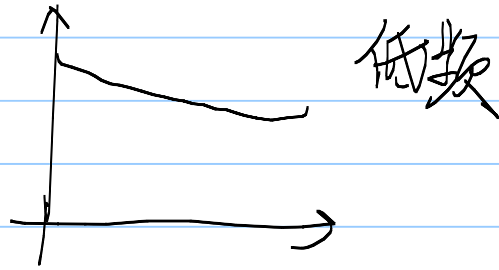
设 $f_h=0$, 即 $u_h=0$, 任取 u_h^0, 画出迭代过程:



特征: 开始迅速下降, 然后缓慢下降.

特别:





D. 初步分析:

误差 $e^n \triangleq u^n - u^*$, $e^{n+1}=B\cdot e^n$

设B特征向量 $\{W_k\}_k$, 特征值 μ_k, 则

设 $e^0=\sum_k C_k W_k \Rightarrow e^n=\sum_k C_k(\mu_k)^n W_k$

则 $\|e^n\| \sim (\rho(B))^n$, $\rho(B)$ 为B的谱半径.

· 设误差界 $\varepsilon \ll 1$, 则迭代步 $N \sim \log\varepsilon / \log\rho(B)$

$$\sim O(-\log\rho(B))^{-1}$$

Jacobi 迭代 $\rho(B_J)=\mu_1^J=1-2\sin^2(\pi h/2)\sim 1-\pi^2h^2/2$

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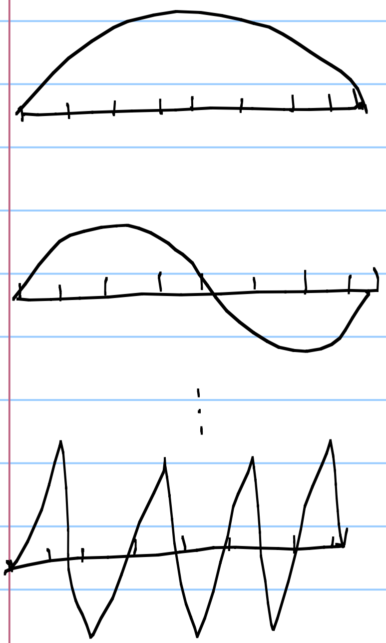
$$(-\log \rho(B_J))^{-1} \sim O(h^{-2})$$

· G-S: $\rho(B_{GS}) = \cos^2(\pi h) \sim 1-\pi^2 h^2$，类似！

· JOR: $\rho(B_\omega) = 1-2\omega\sin^2(\pi h/2) \sim 1-\omega\pi^2 h^2/2$，类似！

E. 以上为最坏情形，如考察不同频率的初值：

设 $e_k^0(j) = \sin(jk\pi h)$，则



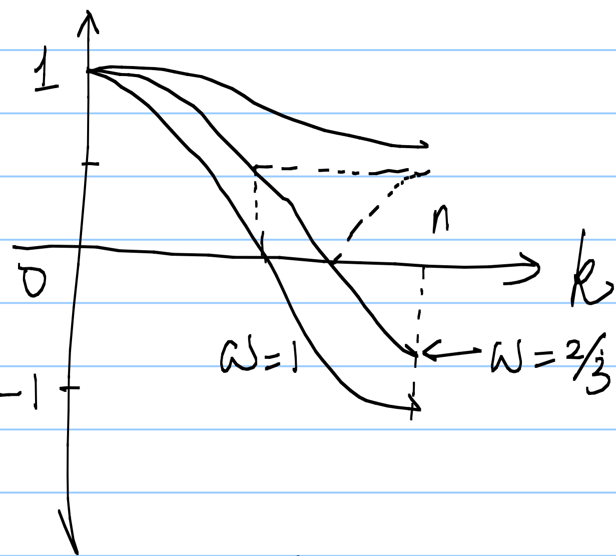
$k=1$

$k=2$

$\vdots$

$k=n-1$

对JOR



$$\lambda_k(\omega) = 1-2\omega\sin^2\frac{k\pi h}{2}$$

为使高频有效衰减（$n/2 \le k \le n-1$），取ω使得

$$\lambda_{n/2}(\omega) = -\lambda_n(\omega) \Rightarrow \omega = 2/3$$

且 $|\lambda_k| < 1/3 \qquad n/2 \le k \le n-1$. 不依赖于$h$.

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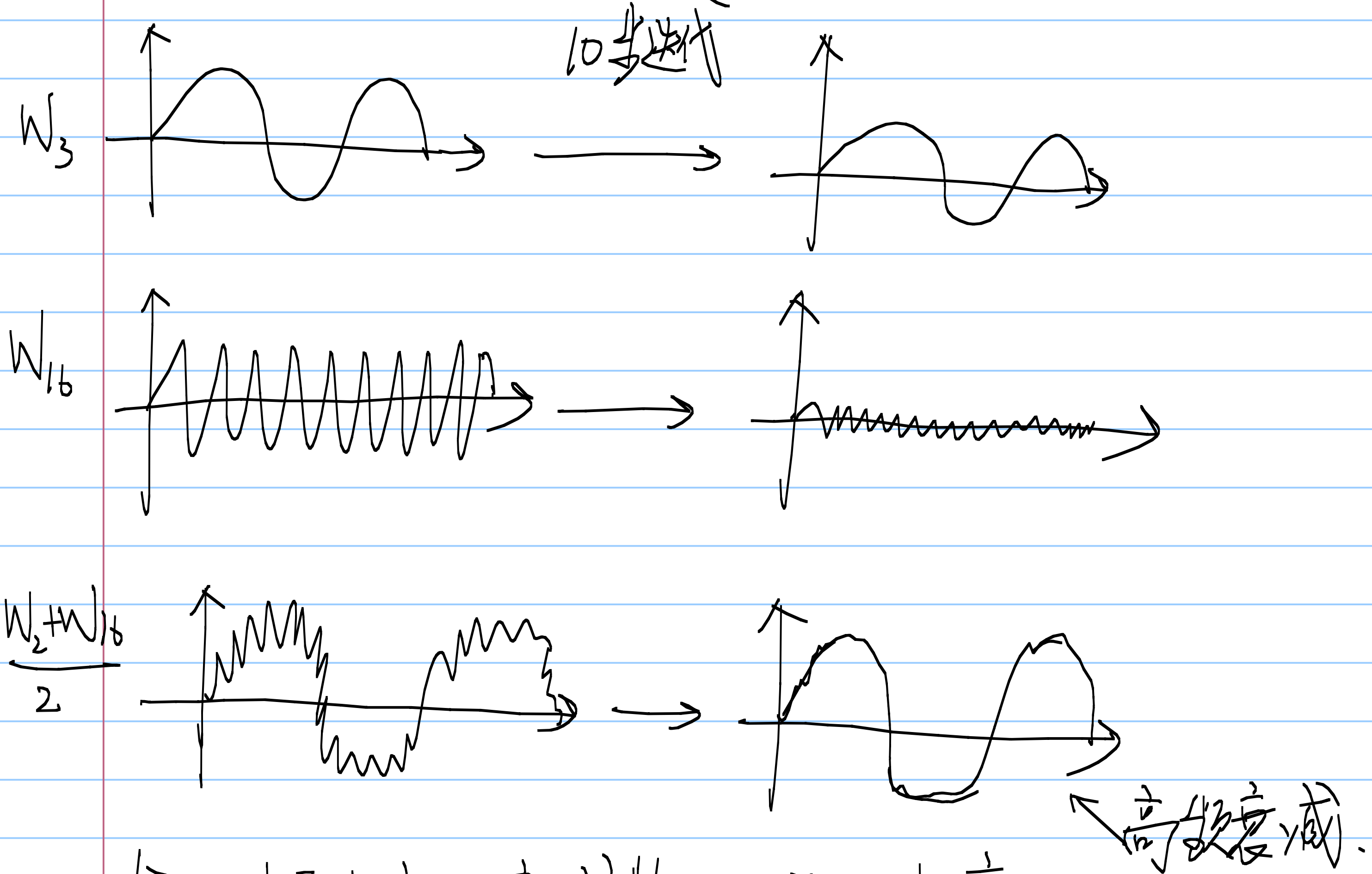
·对G-S有类似现象.

F.光滑化的直观:

取 $\omega=2/3$, $f_h=0$ 初值为 W_3, W_{16} 及 $(W_2+W_{16})/2$
$(n=64)$



直观: 振荡高频衰减快, 误差迅速磨光;

光滑低频衰减慢, 误差难以衰减.

三. 多重网格法:

1. 基本观察: 设 n 偶数, 细网格 Ω_h, 偶点对应的粗网格 Ω_{2h}.

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细网格上的低频成分 $(1\le k<n/2)$ 限制于 Ω_{2h}:

$$W^{h}_{k,2j}=\sin(2jk\pi h)=\sin[jk\pi(2h)]=W^{2h}_{k,j} \qquad j=1,\cdots,n-1$$

即 Ω_h 上的低频在 Ω_{2h} 上看为高频!

2.基本思想与技巧:

A.校正格式 (Correction scheme)

Step1: Ω_h 上迭代求解 $A_h u_h=f_h$ 得近似 v_h (JOR或GS)

step2: 求残量 $r_h=f_h-A_h v_h$

step3: Ω_{2h} 上解 $A_{2h}\cdot e_{2h}=$ "r_{2h}" 得 e_{2h}

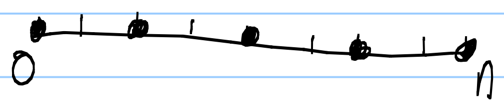
step4: Ω_h 上校正 $v_h\longleftarrow v_h+$ "e_{2h}"

B.限制与插值:

$\Omega_h\to\Omega_{2h}$, $I_h^{2h}: R^{n-1}\longrightarrow R^{n/2-1}$, Restriction

$\Omega_{2h}\to\Omega_h$, $I_{2h}^{h}: R^{n/2-1}\longrightarrow R^{n-1}$, Interpolation

典型选择:



$n=$ 偶数

$\Omega_h\longrightarrow\Omega_{2h}$: Restriction:

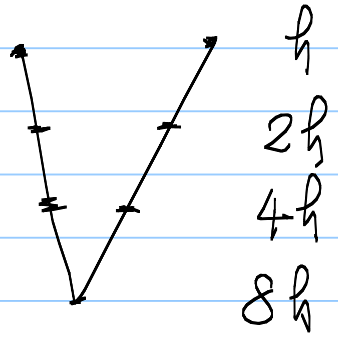
$$I_h^{2h} = \begin{pmatrix} \frac14 & \frac12 & \frac14 \\ & & \frac14 & \frac12 & \frac14 \\ & & & & \ddots \\ & & & & & \frac14 & \frac12 & \frac14 \end{pmatrix}$$

$\Omega_{2h} \to \Omega_h$: Interpolation

$$I_{2h}^h = \begin{pmatrix} \frac12 \\ 1 \\ \frac12 & \frac12 \\ & 1 & \ddots \\ & \frac12 & & \ddots \\ & & & 1 \\ & & & \frac12 \end{pmatrix}$$

此时 $I_{2h}^h = c\cdot I_h^{2h}$

将上述过程递归得V-cycle



3. 简单分析：（二尺度迭代）

易证 $B_{MG} = \left(I - I_{2h}^h A_{2h}^{-1} I_h^{2h} A_h\right)\cdot B_\omega^m$

其中 A_{2h}^{-1} 为 Ω_{2h}上离散的方程，B_ω为JOR的迭代矩阵，m为在细网格上磨光的次数。

设 I_h^{2h}，I_{2h}^h 为如上所示，则有：

性质1: $I_h^{2h} W_k^h = C_k W_k^{2h}$ $C_k = \cos^2 k\pi/2n$

$I_h^{2h} W_{k'}^h = -S_k W_k^{2h}$ $S_k = \sin^2 k\pi/2n$

$W_k^h(j) = \sin(jk\pi h)$, $k' = n-k$, $1 \le k < n/2$

$I_h^{2h} W_{n/2}^h = 0$

性质2: $I_{2h}^h W_k^{2h} = C_k W_k^h - S_k W_{k'}^h$, $1 \le k < n/2$, $k' = n-k$

k_1

$$B_{MG} \cdot W_k^h = \left(1 - 2\omega \sin^2 \frac{k\pi h}{2}\right)^m \cdot S_k \cdot (W_k^h + W_{k'}^h)$$

$$B_{MG} \cdot W_{k'}^h = \left(1 - 2\omega \sin^2 \frac{k\pi h}{2}\right)^m \cdot C_k \cdot (W_k^h + W_{k'}^h)$$

$1 \le k \le n/2$, $k' = n-k$.

由于 $\left|1 - 2\omega \sin^2 \frac{k\pi h}{2}\right| \le 1/3$, $C_k \le 1$, $S_k \le 1/2$

$1 \le k \le n/2$

故高,低频都有效衰减且与h无关!

Ref: W.L. Briggs 书, A multigrid tutorial, SIAM.