Lecture 16 Numerical SDEs: Basics *

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1 Schemes

As most of the SDEs can not be solved in analytical form, we should appeal to numerical computations for practical purpose. Below we illustrate the basic idea of constructing the numerical schemes for solving the SDEs

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t. \tag{1.1}$$

Most of the ideas can be extended to the multidimensional SDEs with coefficients involving t explicitly.

With Ito's formula, we define

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2 = (L_1 f)(X_t)dt + (L_2 f)(X_t)dW_t, \tag{1.2}$$

where

$$(L_1 f)(x) = b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x), \quad (L_2 f)(x) = \sigma(x)f'(x).$$

Taking integral from t_n to t_{n+1} to both sides of (1.1), and taking $f(x) = b(x)$ and $\sigma(x)$, we have

$$\begin{aligned}
X_{t_{n+1}} =& X_{t_n} + \int_{t_n}^{t_{n+1}} b(X_s)ds + \int_{t_n}^{t_{n+1}} \sigma(X_s)dW_s \\
=& X_{t_n} + b(X_{t_n})\delta t_n + \sigma(X_{t_n})(W_{t_{n+1}} - W_{t_n}) && (1.3) \\
&+ \int_{t_n}^{t_{n+1}} dW_s \int_{t_n}^{s} (L_2\sigma)(X_\tau)dW_\tau && (1.4) \\
&+ \int_{t_n}^{t_{n+1}} dW_s \int_{t_n}^{s} (L_1\sigma)(X_\tau)d\tau + \int_{t_n}^{t_{n+1}} ds \int_{t_n}^{s} (L_2 b)(X_\tau)dW_\tau && (1.5) \\
&+ \int_{t_n}^{t_{n+1}} ds \int_{t_n}^{s} (L_1 b)(X_\tau)d\tau, && (1.6)
\end{aligned}$$

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where $\delta t_n = t_{n+1} - t_n$. The above procedure can be further carried on by replacing $L_i b(X_\tau)$, $L_i\sigma(X_\tau)$ as $L_i b(X_t)$, $L_i\sigma(X_t)$ and getting higher order iterative integrals correspondingly. The obtained series is usually called Itô-Taylor expansion for SDEs. It is not difficult to find that each term in the Ito-Taylor expansion has the form

$$I_{\boldsymbol{i}}(g) = \int_{t_n}^{t_{n+1}} dW_{s_1}^{i_1} \int_{t_n}^{s_1} dW_{s_2}^{i_2} \cdots \int_{t_n}^{s_{k-1}} dW_{s_k}^{i_k} g(X_{s_k})$$

with some $k \in \{1, 2, \dots\}$. Here the characteristic index of the integral $\boldsymbol{i} = (i_1, i_2, \dots, i_k)$ and $i_j \in \{0, 1\}$ for $j = 1, 2, \dots, k$. The integrand g is the action of some compositions of operators L_1 and L_2 on function b or σ. We take the convention $W_t^0 := t$ and $W_t^1 := W_t$ for the ease of notation. This set-up can be extended to the system driven by multidimensional Brownian easily.

Now similar with solving deterministic ODEs, we truncate the Ito-Taylor series to different orders to obtain different schemes. For example, if we only keep terms until (1.3), then we have

(1) Euler-Maruyama scheme

$$X_{n+1} = X_n + b(X_n)\delta t_n + \sigma(X_n)\delta W_n, \tag{1.7}$$

where $\delta W_n \sim N(0, \delta t_n)$. The Euler-Maruyama scheme is the most commonly used numerical scheme for its simplicity.

From the basic intuition $dW_t \sim \sqrt{dt}$, we have that roughly (1.8)$\sim O(\delta t)$, (1.5)$\sim O(\delta t^{3/2})$ and (1.6)$\sim O(\delta t^2)$. By extracting the leading order term (1.8), we obtain

$$\begin{aligned}\int_{t_n}^{t_{n+1}} dW_s \int_{t_n}^{s} (L_2\sigma)(X_\tau) dW_\tau &\approx (L_2\sigma)(X_{t_n}) \int_{t_n}^{t_{n+1}} dW_s \int_{t_n}^{s} dW_\tau \\ &= \frac{1}{2}(L_2\sigma)(X_{t_n})[(\delta W_n)^2 - \delta t_n].\end{aligned}$$

Substitute this into the Ito-Taylor expansion we obtain the well-known Milstein scheme.

(2) Milstein scheme

$$X_{n+1} = X_n + b(X_n)\delta t_n + \sigma(X_n)\delta W_n + \frac{1}{2}(\sigma\sigma')(X_n)[(\delta W_n)^2 - \delta t_n]. \tag{1.8}$$

We should remark that although Milstein scheme is more accurate than the Euler-Maruyama scheme in some sense, it is only practical for the SDEs driven by single Wiener process. That is because the explicit characterization

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_s dW_\tau = \frac{1}{2}[(\delta W_n)^2 - \delta t_n]$$

is only valid in one dimensional case. In multi-dimensions, when $i \neq j$ it is impossible to get an explicit sampling form of

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_s^i dW_\tau^j,$$

where W_t^i, W_t^j are independent Wiener processes. Though some strategies are proposed to approximate the above random variables, they are not so common in practical applications. The readers may be refereed to [1] for more details.

Although Milstein scheme is more accurate, it is not so popular in practice for the term $\sigma'(X_t)$ may be too complicate to compute even in 1D case. To overcome this issue, one can take the following type of schemes by borrowing the idea from Runge-Kutta method for solving ODEs.

(3) Runge-Kutta scheme

$$\begin{aligned} \hat{X}_n =& X_n + \sigma(X_n)\sqrt{\delta t_n}, \\ X_{n+1} =& X_n + b(X_n)\delta t_n + \sigma(X_n)\delta W_n \\ &+ \frac{1}{2}\frac{1}{\sqrt{\delta t_n}}[\sigma(\hat{X}_n) - \sigma(X_n)][(\delta W_n)^2 - \delta t_n]. \end{aligned} \tag{1.9}$$

If we formally take higher order Itô-Taylor expansion, specifically applying formula (1.2) to

$$(L_1 b)(X_\tau),\ (L_2 b)(X_\tau),\ (L_1\sigma)(X_\tau),\ (L_2\sigma)(X_\tau)$$

and dropping higher order terms, we have the following higher order scheme.

(4) Higher order scheme

$$\begin{aligned} X_{n+1} =& X_n + b\delta t_n + \sigma\delta W_{t_n} + \frac{1}{2}\sigma\sigma'\{(\delta W_n)^2 - \delta t_n\} \\ &+ \sigma b'\Delta Z_n + \frac{1}{2}(bb' + \frac{1}{2}\sigma^2 b'')\delta t_n^2 \\ &+ (b\sigma' + \frac{1}{2}\sigma^2\sigma'')(\delta W_n \delta t_n - \delta Z_n) \\ &+ \frac{1}{2}\sigma\left[\sigma\sigma'' + (\sigma')^2\right]\left[\frac{1}{3}(\delta W_n)^2 - \delta t_n\right]\delta W_n, \end{aligned} \tag{1.10}$$

where

$$\Delta Z_n := \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_\tau ds$$

is a Gaussian R.V. satisfying $\mathbb{E}(\Delta Z_n) = 0,\ \mathbb{E}((\Delta Z_n)^2) = \delta t_n^3/3, \mathbb{E}(\delta Z_n \delta W_n) = \delta t_n^2/2$.

The convergence of the discretized solution of SDEs has two senses according to the needs of realistic applications. They are called *strong convergence* and *weak convergence*, respectively. Define $\{X_t^{\delta t}\}$ a numerical solution of SDEs with maximal stepsize δt, X_t is the exact solution, then we have the following definition.

Definition 1.1 (Convergence of numerical solutions)**.** *We have the following two typical concepts of convergence for the numerical solution of SDEs.*

(1) Strong convergence (mean-square convergence) *If*

$$\max_{0\le t\le T} \mathbb{E}|X_t^{\delta t} - X_t|^2 \le C(\Delta t)^{2\alpha},$$

where C is a constant independent of δt, then we call $\{X_t^{\Delta t}\}$ strongly converges, or converges in the mean-square sense, to X_t with order α.

(2) Weak convergence (convergence w.r.t. expectation) *If*

$$\max_{0\le t\le T} |\mathbb{E}f(X_t^{\delta t}) - \mathbb{E}f(X_t)| \le C_f(\delta t)^{\beta},$$

for any $f \in C_b^\infty(\mathbb{R}^n)$, where C_f is a constant independent of δt but may depend on f, then we call $\{X_t^{\delta t}\}$ weakly converges to X_t with order β.

A straightforward result about the convergence order is below.

Proposition 1.2. *When the considered function f in the weak convergence has the property $\|f'\|_\infty \le K$, we have $\beta \ge \alpha$.*

Proof. By the mean value theorem and the Hölder's inequality, we obtain

$$|\mathbb{E}f(X_t^{\delta t}) - \mathbb{E}f(X_t)| \le \mathbb{E}|f(X_t^{\delta t}) - f(X_t)| \le K\mathbb{E}|X_t^{\delta t} - X_t| \le K(\mathbb{E}|X_t^{\delta t} - X_t|^2)^{\frac{1}{2}}.$$

The above proposition gives a rationale why the former is called *strong convergence* compared with the other one in some sense. Before introducing the convergence analysis, let us state the main theorem about the convergence of numerical schemes

Theorem 1.3 (Convergence order)**.** *Define the length of the multi-index $\boldsymbol{i} = (i_1, i_2, \dots, i_k)$ as*

$$l(\boldsymbol{i}) := k, \quad n(\boldsymbol{i}) := \{\textit{the number of zeros in } \boldsymbol{i}\}.$$

and the set of indices

$$\mathcal{S}_\alpha = \left\{\boldsymbol{i}|\ l(\boldsymbol{i}) + n(\boldsymbol{i}) \le 2\alpha \quad \textit{or} \quad l(\boldsymbol{i}) = n(\boldsymbol{i}) = \alpha + \frac{1}{2}\right\} \quad \textit{for} \quad \alpha \in \left\{\frac{1}{2}, 1, \frac{3}{2}, \cdots\right\},$$

$$\mathcal{W}_\beta = \{\boldsymbol{i}|l(\boldsymbol{i}) \le \beta\} \qquad \textit{for} \quad \beta \in \{1, 2, 3, \cdots\}.$$

Then with mild smoothness conditions on b, σ and the function f in weak approximation, the scheme derived by truncating the Ito-Taylor expansion up to all indices with $\boldsymbol{i} \in \mathcal{S}_\alpha$ has strong order α; the scheme derived by truncating the Ito-Taylor expansion up to terms with $\boldsymbol{i} \in \mathcal{W}_\beta$ has weak order β.

The proof and detailed requirements about the smoothness conditions on b, σ and f may be found in [1] (Theorems 10.6.3 and 14.5.1). Applying this theorem to the constructed schemes in this section, we have Table 1.

	Strong order	Weak order
Euler-Maruyama	1/2	1
Milstein	1	1
Scheme (1.9)	1	1
Scheme (1.10)	2	2

Table 1: The convergence order of some numerical schemes for SDE.

2 Strong convergence

We will analyze the mean-square convergence of the Euler-Maruyama scheme under the assumption that $b(x)$ satisfies global Lipschitz and linear grow condition with constant L and $\sigma = 1$, i.e. the additive noise case.

Now suppose the SDE takes the form

$$dX_t = b(X_t)dt + dW_t \tag{2.1}$$

with the Euler-Maruyama discretization

$$X_{n+1} = X_n + b(X_n)\delta t_n + \delta W_n. \tag{2.2}$$

Introduce the "linear stochastic" interpolation of X_n as

$$d\bar{X}_t = b(X_n)dt + dW_t, \quad t \in [t_n, t_{n+1}).$$

where the driving term W_t is assumed to be the same as that in continuous form. Then $\bar{X}_{t_n} = X_n$ and we have the so called "discrete Ito formula" for $f \in C^2(\mathbb{R})$

$$df(\bar{X}_t) = f'(\bar{X}_t)d\bar{X}_t + \frac{1}{2}f''(\bar{X}_t)(d\bar{X}_t)^2,$$

i.e.

$$f(\bar{X}_t) = f(X_n) + \int_{t_n}^t \Big[f'(\bar{X}_s)b(X_n) + \frac{1}{2}f''(\bar{X}_s)\Big]ds + \int_{t_n}^t f'(\bar{X}_s)dW_s, \quad t \in [t_n, t_{n+1}).$$

Lemma 2.1. *Let $\delta t = \max_n \delta t_n$. We have the following bounds for X_t*

$$\sup_{t\le T} \mathbb{E}|X_t|^2 \le K_1(T), \qquad \sup_{t\in[t_n,t_{n+1})} \mathbb{E}|X_t - X_{t_n}|^2 \le K_2(T)\delta t,$$

where the constant $K_1(T)$ depends on T, L and $\mathbb{E}|X_0|^2$, and $K_2(T)$ depends on $L, \delta t$ and $K_1(T)$.

Proof. Applying Ito formula to $|X_t|^2$, we have

$$d|X_t|^2 = 2X_t \cdot (b(X_t) + dW_t) + dt.$$

Integrating from 0 to t and taking expectation we have

$$\mathbb{E}|X_t|^2 = \mathbb{E}|X_0|^2 + 2\mathbb{E}\int_0^t X_s \cdot b(X_s)ds + 2\mathbb{E}\int_0^t X_s dW_s + t.$$

Taking advantage of (**??**) for the Ito integral and the inequality $2ab \leq a^2 + b^2$, we obtain

$$\mathbb{E}|X_t|^2 \leq \mathbb{E}|X_0|^2 + T + \int_0^t \mathbb{E}|X_s|^2 ds + L\int_0^t (1 + \mathbb{E}|X_s|^2)ds.$$

The Gronwall inequality gives

$$\sup_{t\leq T} \mathbb{E}|X_t|^2 \leq (\mathbb{E}|X_0|^2 + T + LT)\exp((L+1)T).$$

For the second inequality, we have from SDE

$$X_t - X_{t_n} = \int_{t_n}^t b(X_s)ds + (W_t - W_{t_n}).$$

Squaring both sides and taking expectation we get

$$\mathbb{E}|X_t - X_{t_n}|^2 \leq 2\mathbb{E}\left(\int_{t_n}^t b(X_s)ds\right)^2 + 2\delta t.$$

From Hölder's inequality we obtain

$$\mathbb{E}|X_t - X_{t_n}|^2 \leq 2L\delta t\int_{t_n}^t (1 + \mathbb{E}|X_s|^2)ds + 2\delta t \leq 2L\delta t^2(1 + K_1(T)) + 2\delta t, \quad t \in [t_n, t_{n+1}).$$

Proposition 2.2 (Half order mean-square convergence)**.** *The Euler-Maruyama scheme is of strong order 1/2.*

Proof. From (2.1) we have

$$X_{t_{n+1}} = X_{t_n} + \int_{t_n}^{t_{n+1}} b(X_t)dt + \delta W_n,$$

and the Equation (2.2) can be rewritten as

$$X_{n+1} = X_n + \int_{t_n}^{t_{n+1}} b(X_n)dt + \delta W_n.$$

Define the error $e_{n+1} = X_{t_{n+1}} - X_{n+1}$, then

$$e_{n+1} = e_n + \int_{t_n}^{t_{n+1}} (b(X_t) - b(X_n))dt.$$

Squaring both sides and from the inequality $2ab \le a^2\delta t + b^2/\delta t$, we obtain

$$\begin{aligned}
|e_{n+1}|^2 &= |e_n|^2 + \Big[\int_{t_n}^{t_{n+1}} (b(X_t) - b(X_n))dt\Big]^2 + 2e_n \cdot \Big[\int_{t_n}^{t_{n+1}} (b(X_t) - b(X_n))dt\Big] \\
&\le |e_n|^2(1+\delta t) + \left(1 + \frac{1}{\delta t}\right)\left[\int_{t_n}^{t_{n+1}} (b(X_t) - b(X_n))dt\right]^2 \\
&\le |e_n|^2(1+\delta t) + L^2(1+\delta t)\int_{t_n}^{t_{n+1}} |X_t - X_n|^2 dt, \qquad (2.3)
\end{aligned}$$

where the last inequality is from Hölder's inequality and Lipschitz condition.

From the inequality $|X_t - X_n|^2 \le 2|X_t - X_{t_n}|^2 + 2|X_{t_n} - X_n|^2$ we have

$$\mathbb{E}|e_{n+1}|^2 \le \mathbb{E}|e_n|^2(1 + L_1\delta t) + L_2\delta t^2,$$

where $L_1 = 1 + 2L^2(1+\delta t)$ and $L_2 = 2L^2(1+\delta t)K_2(T)$ can be bounded by positive constants independent of δt if δt is small.

The discrete Gronwall's inequality then guarantees

$$\mathbb{E}|e_n|^2 \le \mathbb{E}|e_0|^2(1 + L_1\delta t)^n + L_2\delta t^2 \frac{(1 + L_1\delta t)^n - 1}{L_1\delta t} \le \frac{L_2}{L_1}(e^{L_1 T} - 1)\delta t$$

if we assume $e_0 = 0$. The proof is complete.

We want to remark here that in the considered additive noise case, the Euler-Maruyama scheme is exactly the Milstein scheme since $\sigma'\sigma = 0$ and thus the last term in (1.8) diminishes! From Theorem 1.3, we can prove it is of strong order 1 in principle. It is indeed true but the proof will be more tedious with higher smoothness condition on b. We leave the proof as an exercise to the reader.

3 Weak Convergence

Now let us consider the weak convergence of the Euler-Maruyama scheme with the tools from PDEs. We will only consider the 1D case with $\sigma = 1$ for simplicity. But the essential part of the proof is the same for high dimensional case. The weak convergence is to analyze the error

$$e_n = \mathbb{E}f(X_n) - \mathbb{E}f(X_{t_n})$$

for smooth function f. From the stated result in Theorem 1.3, we know that the Euler-Maruyama scheme is of weak order 1. Before we go to the rigorous proof, let us give a more transparent observation on this point by elementary deductions.

Suppose $X_0 = x$, f is a smooth enough function. Formally in order to consider the weak convergence of a numerical scheme to approximate Markov process X_t, we start from the weak Ito-Taylor expansion

$$\mathbb{E}^x f(X_h) = f(x) + \int_0^h \mathcal{A} f(X_t)dt \sim \sum_{n=0} \frac{\mathcal{A}^n f(x)}{n!} h^n, \tag{3.1}$$

where $\mathcal{A}$ is the infinitesimal generator of X_t and h is the time stepsize. Correspondingly for the numerical solution X_t^N, we have

$$\mathbb{E}^x f(X_h^N) = f(x) + \int_0^h \mathcal{A} f(X_t^N)dt \sim \sum_{n=0} \frac{\mathcal{A}_N^n f(x)}{n!} h^n, \tag{3.2}$$

where $\mathcal{A}_N$ is the infinitesimal generator of X_t^N. To have an idea about the global weak convergence order, we need to figure out the local weak truncation order at first.

Take the diffusion process as a specific example. Now

$$dX_t = b(X_t)dt + dW_t$$

and the Euler-Maruyama scheme reads

$$X_{n+1} = X_n + b(X_n)\Delta t + \Delta W_n.$$

Define the continuous extension of the numerical solution as

$$dX_t^N = b(X_n)dt + dW_t, \quad t \in [t_n.t_{n+1}).$$

We have the infinitesimal generator

$$\mathcal{A} f(y) = b(y)f'(y) + \frac{1}{2} f''(y)$$

and

$$\mathcal{A}^2 f(y) = b(y)\Big[b(y)f'(y) + \frac{1}{2}f''(y)\Big] + \frac{1}{2}\Big[b(y)f'(y) + \frac{1}{2}f''(y)\Big]''.$$

Correspondingly for the numerical solution X_t^N we have

$$\mathcal{A}_N f(y) = b(x)f'(y) + \frac{1}{2} f''(y)$$

and

$$\mathcal{A}_N^2 f(y) = b(x)\Big[b(x)f'(y) + \frac{1}{2}f''(y)\Big] + \frac{1}{2}\Big[b(x)f'(y) + \frac{1}{2}f''(y)\Big]'',$$

where x is the initial condition. Now it is obvious that

$$\mathbb{E}^x f(X_h^N) - \mathbb{E}^x f(X_h) = O(h^2) \tag{3.3}$$

and thus the weak local truncation error is of second order and we can expect that the Euler-Maruyama scheme is of weak order 1. In fact if one can bound the expansion terms in

(3.1) and (3.2) up to corresponding order, the above formal derivations based on the local error analysis can be made rigorous.

It will become clear soon that the weak convergence analysis essentially relies on some estimates about the solution of the backward equation. Let us first consider the partial differential equation

$$\partial_t u(x,t) = \mathcal{L}u(x,t) = b(x)\partial_x u + \frac{1}{2}\partial_{xx}u, \quad u(x,0) = f(x). \tag{3.4}$$

Define the notation $C_P^m(\mathbb{R}^d, \mathbb{R})$ the space of functions $w \in C^m(\mathbb{R}^d, \mathbb{R})$ for which all partial derivatives up to order m have polynomial growth. More concretely, there exist a constant $K > 0$, and $m, p \in \mathbb{N}$ such that

$$|\partial_{\boldsymbol{x}}^{\boldsymbol{j}} w(\boldsymbol{x})| \leq K(1 + |\boldsymbol{x}|^{2p}), \quad \forall\ |\boldsymbol{j}| < m$$

for any $\boldsymbol{x} \in \mathbb{R}^d$, where $\boldsymbol{j}$ is a d-multi-index. Here we have $d = 1$ and we will simply denote $C_P^m(\mathbb{R}^d, \mathbb{R})$ as C_P^m in later texts.

The following important lemma can be found in [1] (Theorem 4.8.6, pp. 153).

Lemma 3.1. *Suppose that $f \in C_P^{2\beta}$ for some $\beta \in \{2, 3, \ldots\}$, $\boldsymbol{X}_t$ is time-homogeneous and $b \in C_P^{2\beta}$ with uniformly bounded derivatives. Then $\partial u/\partial t$ is continuous and*

$$u(\cdot, t) \in C_P^{2\beta}, \quad t \leq T$$

for any fixed $T < \infty$.

Theorem 3.2 (Weak convergence)**.** *Assume that b is Lipschitz and the conditions in Lemma 3.1 also hold for b and f, then the Euler-Maruyama scheme is of weak order 1 .*

Proof. Define the backward operator

$$\tilde{\mathcal{L}} = \partial_t + b(x)\partial_x + \frac{1}{2}\partial_{xx},$$

and denote by v the solution of

$$\tilde{\mathcal{L}}v = 0, \quad t \in (0, t_n) \tag{3.5}$$

with the final condition $v(x, t_n) = f(x)$. It is straightforward that $v(x,t) = u(x, t_n - t)$ for the solution u of (3.4).

By Itô's formula we have

$$\mathbb{E}v(X_0, 0) = \mathbb{E}v(X_{t_n}, t_n) = \mathbb{E}f(X_{t_n}).$$

Hence

$$\begin{aligned}|e_n| &= |\mathbb{E}f(X_n) - \mathbb{E}f(X_{t_n})| \\ &= |\mathbb{E}v(X_n, t_n) - \mathbb{E}v(X_0, 0)| \\ &= \Big|\mathbb{E}\Big(\int_0^{t_n} \big(\partial_t v(\bar{X}_s, s) + b(X_{n_s})\partial_x v(\bar{X}_s, s) + \frac{1}{2}\partial_{xx} v(\bar{X}_s, s) - \tilde{\mathcal{L}} v(\bar{X}_s, s)\big) ds\Big)\Big|,\end{aligned}$$

where $n_s := \{m | t_m \le s < t_{m+1}\}$, and $\bar{X}_s$ is the continuous extension of X_n defined as

$$d\bar{X}_s = b(X_{n_s})ds + dW_s, \quad x \in [t_m, t_{m+1}).$$

With this definition, we obtain

$$\begin{aligned}|e_n| &= \Big|\mathbb{E}\Big(\int_0^{t_n} \big(b(X_{n_s})\partial_x v(\bar{X}_s, s) - b(\bar{X}_s)\partial_x v(s, \bar{X}_s)\big) ds\Big)\Big| \\ &\le \Big|\mathbb{E}\Big(\int_0^{t_n} \big(b(X_{n_s})\partial_x v(X_{n_s}, t_{n_s}) - b(\bar{X}_s)\partial_x v(\bar{X}_s, s)\big) ds\Big)\Big| \\ &+ \Big|\mathbb{E}\Big(\int_0^{t_n} b(X_{n_s})\big(\partial_x v(\bar{X}_s, s) - \partial_x v(X_{n_s}, t_{n_s})\big) ds\Big)\Big| \\ &= \Big|\mathbb{E}\sum_m \int_{t_m}^{t_{m+1}} \Big(b(X_m)\partial_x v(X_m, t_m) - b(\bar{X}_s)\partial_x v(\bar{X}_s, s)\Big) ds\Big| \\ &+ \Big|\mathbb{E}\sum_m \int_{t_m}^{t_{m+1}} b(X_m)\big(\partial_x v(\bar{X}_s, s) - \partial_x v(X_m, t_m)\big) ds\Big|. \end{aligned} \tag{3.6}$$

Using Itô's formula again, we have for any function $g(x,t)$

$$\begin{aligned}g(\bar{X}_t, t) - g(X_m, t_m) &= \int_{t_m}^{t} \Big[\partial_t g(\bar{X}_s, s) + b(X_m)\partial_x g(\bar{X}_s, s) + \frac{1}{2}\partial_{xx} g(\bar{X}_s, s)\Big] ds \\ &+ \int_{t_m}^{t} \partial_x g(\bar{X}_s, s) dW_s, \quad t \in [t_m, t_{m+1}).\end{aligned}$$

Using this with $g = b\partial_x v$ and $g = \partial_x v$ in (3.6), we have the highest derivatives $\partial_{xxx} v, \partial_{xx} b \in C_P^{2\beta}$ as long as $\beta \ge 2$. Notice that $b(X_m)$ is independent of $\int_{t_m}^{t} \partial_x g(\bar{X}_s, s) dW_s$ conditional on X_m, together with the fact that $\mathbb{E}|X_m|^{2r}$ and $\mathbb{E}|\bar{X}_t|^{2r} \le C$ for any $r \in \mathbb{N}$ (Exercise 3), we get

$$|e_n| \le C \sum_m \Delta t^2 \le C\Delta t,$$

which is the desired estimate. □

Example 3.3 (Weak approximation). *For the SDE*

$$dX_t = -\frac{1}{2}X_t dt + dW_t, \quad X_0 = 0,$$

compute $u = \mathbb{E}X_t^2|_{t=1}$ *with the Euler-Maruyama scheme.*

Solution. The exact solution of u is

$$u = \mathbb{E}X_t^2|_{t=1} = 1 - e^{-1} \approx 0.632.$$

In order to compute the expectation numerically, we take the Euler-Maruyama scheme

$$X_{n+1,k} = (1 - \frac{\Delta t}{2})X_{n,k} + \sqrt{\Delta t} \cdot R_{n,k}, \quad k = 1, 2, \ldots, N,$$

where $\Delta t = 1/M$, $n = 0, 1, \ldots, M-1$ and $R_{n,k}$ are i.i.d. $N(0,1)$ random variables. So the approximate solution

$$u_{N,\Delta t} = \frac{1}{N}\sum_{k=1}^{N}(X_{M,k})^2.$$

We take $M = 2000$, and compute u with different sample size N as follows.

N	100	200	300	400	500	600
u	0.6586	0.6563	0.6785	0.6234	0.6407	0.6320
Error	0.0265	0.0242	0.0464	0.0087	0.0086	0.0001

Table 2: Weak approximation with Euler-Maruyama scheme

Homeworks

1. Give a sampling method for the random variables

$$\Delta Z_1 := \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_\tau ds, \quad \Delta Z_2 := \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\tau dW_s.$$

   and ΔW_n.

2. Prove the Euler-Maruyama scheme is of strong order 1 for the SDE (2.1) with additive noise and higher smoothness condition on b.

3. Prove that for the Euler-Maruyama scheme

$$\mathbb{E}|X_n|^{2r},\ \mathbb{E}|X_t|^{2r},\ \mathbb{E}|\bar{X}_t|^{2r} \leq C$$

   for $t \leq T$, $n \leq N$ and any $r \in \mathbb{N}$.

References

[1] P.E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations.* Springer-Verlag, 1992.