A GENERAL CONVERGENCE RESULT FOR A FUNCTIONAL RELATED TO THE THEORY OF HOMOGENIZATION*

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Abstract. The convergence, as $\varepsilon \downarrow 0$, of the functional $F_{\varepsilon}(\Psi) = \int_{\mathbb{R}^N} u_{\varepsilon}(x)\Psi(x, x/\varepsilon)$ associated with a given L^2 function u_{ε} with support in a fixed compact set is studied. The test functions $\Psi(x, y)$ are continuous on $\mathbb{R}^N \times \mathbb{R}^N$ and periodic in y. A convergence theorem is proved under the weaker assumption that u_{ε} remains in a bounded subset of L^2 . Finally, the use of multiple-scale expansions in homogenization is justified, and a new approach is proposed for the mathematical analysis of homogenization problems.

Key words. partial differential equations, homogenization, convergence, functional, periodic

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1. Introduction. The mathematical analysis of homogenization problems for partial differential equations (see [1], [9]) utilizes the functionals of the type

$$F_{\varepsilon}(\Psi) = \int_{\Omega} u_{\varepsilon}(x) \Psi\left(x, \frac{x}{\varepsilon}\right) dx \ (\Omega \text{ a bounded open set in } \mathbb{R}^{N}).$$

The function u_{ε} is, say, in $L^{2}(\Omega)$ and is (or depends on) the solution of a partial differential equation on Ω with coefficients ε -periodic (i.e., periodic with period ε in each variable). The test function $\Psi(x, y)$ is continuous on $\overline{\Omega} \times \mathbb{R}^{N}$ ($\overline{\Omega}$ denotes the closure of Ω) and, for fixed x, the function $y \to \Psi(x, y)$ is periodic (with period 1 in each variable).

Let us bear in mind that for such a function, i.e., Ψ , the associated sequence $(\Psi^{\varepsilon})_{\varepsilon>0}$, with $\Psi^{\varepsilon}(x) = \Psi(x, x/\varepsilon)$ for $x \in \Omega$, converges to the function

$$x \to \tilde{\Psi}(x) = \int_{Y} \Psi(x, y) \, dy$$
 in $L^2(\Omega)$ -weak as $\varepsilon \downarrow 0$

(see, e.g., [1]), where $Y = [0, 1]^{N}$.

In view of convergence studies in the theory of homogenization two distinct situations may be considered:

(i) The sequence (u_{ε}) is assumed to contain a subsequence, still denoted by (u_{ε}) for simplicity, that converges strongly to a function u_0 in $L^2(\Omega)$ as $\varepsilon \downarrow 0$ (e.g., $u_{\varepsilon} \in H^1(\Omega)$, $\partial\Omega$ smooth, and (u_{ε}) is bounded in $H^1(\Omega)$). Hence, the corresponding sequence $(F_{\varepsilon}(\Psi))$ converges to the integral $\int_{\Omega} u_0(x)\tilde{\Psi}(x) dx$.

(ii) The more difficult situation, which we study here, is that in which the sequence (u_{ε}) only remains in a bounded subset of $L^{2}(\Omega)$. We may surely extract a weakly convergent subsequence, but we do not have any classical argument that allows us to pass to the limit in $F_{\varepsilon}(\Psi)$ for the corresponding subsequence. Indeed, for the convergence of the scalar product of two sequences in $L^{2}(\Omega)$, we classically need strong convergence for at least one of them.

Several aspects of this situation arise in homogenization. Let us point out two particularly interesting aspects:

(1) u_{ε} is some derivative of a function v_{ε} (i.e., $u_{\varepsilon} = \partial v_{\varepsilon} / \partial x_i$) that is the solution of a boundary value problem considered in the framework of homogenization, and the sequence (v_{ε}) is bounded in $H^1(\Omega)$ (see § 6). In general, this is typical of the

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so-called *regular* homogenization problems; that is, the class of the homogenization problems associated with a formal expansion (of the solution) of the type

(1.1)
$$v_{\varepsilon}(x) = v_0(x) + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 v_2\left(x, \frac{x}{\varepsilon}\right) + \cdots,$$

where the leading term v_0 , which does not depend on the local variables $y = x/\varepsilon$, "ignores" the local effects.

For the study of convergence, i.e., $\lim v_{\varepsilon} = v_0$ as $\varepsilon \downarrow 0$, which is one of the main objects in homogenization, we possess a method, the so-called Energy Method (see [1], [9]), that solves most of the problems of the above type. However, it does not exhibit the weak limit of the gradient $\partial v_{\varepsilon}/\partial x_i$, $i = 1, \dots, N$ (that is, concretely, the local behaviour of v_{ε}), which is interesting from the physical point of view.

(2) u_{ε} is the solution of a boundary value problem whose formal analysis (in the framework of homogenization) is based on an asymptotic expansion of the type

(1.2)
$$u_{\varepsilon}(x) = u_0\left(x,\frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x,\frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x,\frac{x}{\varepsilon}\right) + \cdots,$$

with a leading term depending on the local variables $y = x/\varepsilon$. The leading term is affected by the local effects and, consequently, there is no hope of extracting a strongly convergent subsequence from (u_{ε}) . Here, the Energy Method becomes inoperative and, to our knowledge, there is no systematic way of proving convergence for related homogenization problems, referred to as *singular* homogenization problems (see [4], [5], [7, Chaps. 7, 8] for typical examples of this). Although we do not consider that question in this work, we believe the study of singular homogenization problems requires an appropriate approach that should be based on an extensive analysis of functionals of the type

$$F_{\varepsilon}(\Psi) = \int_{\Omega} u_{\varepsilon}(x) \Psi\left(x, \frac{x}{\varepsilon}\right) dx.$$

Our basic result is the proof of a convergence theorem for the functional $F_{\varepsilon}(\Psi) = \int_{\mathbb{R}^{N}} u_{\varepsilon}(x)\Psi(x, x/\varepsilon) dx$ (u_{ε} having its support in a fixed compact set) under the weaker hypothesis that the sequence (u_{ε}) remains bounded in L^{2} . There is no need to assume the possibility of extracting a strongly convergent subsequence.

Next, based on the above result, we give a complete justification of the use of multiple-scale asymptotic expansions (such as (1.1) or (1.2)) in the theory of homogenization: Assuming that $u_{\varepsilon} \in L^{2}(\Omega)$, with u_{ε} bounded in the L^{2} norm, Theorem 2 gives the leading-order approximation to u_{ε} (in (1.2)). If u_{ε} lies in $H^{1}(\Omega)$ and is bounded in the H^{1} norm, Theorem 3 gives the next-order approximation to u_{ε} . Theoretically, the higher-order approximations are naturally given by similar theorems provided that $u_{\varepsilon} \in H^{2}(\Omega)$ with u_{ε} bounded, $u_{\varepsilon} \in H^{3}(\Omega)$ with u_{ε} bounded, \cdots ; however, that is quite labourious.

Finally, we propose an alternative way of proving convergence in homogenization. Our approach is carried out on a classical problem (to arrive at a correct understanding of a method, we prefer to start with a classical example). Nevertheless, we anticipate that its flexibility and its "spontaneity" make it more adaptable for unusual problems than the often very fastidious Energy Method. Indeed, the reader familiar with the so-called natural multiple-scale asymptotic method [1] will easily realize that our approach is nothing but its mathematical version. Furthermore, as we shall see in § 6, our approach exhibits the local behaviour of the solution. This is not accessible to the Energy Method, whose basic ingredient is strong convergence. This paper is organized as follows. In § 2 we present some general notation and preliminaries. Section 3 is devoted to our basic result, the case of the whole space \mathbb{R}^N . In § 4 we give a more pragmatic version (in view of the theory of homogenization) of the above result, which takes into account more realistic test functions. In § 5 we prove a convergence theorem for the gradient $\partial u_{\varepsilon}/\partial x_i$, $i = 1, \dots, N$ (i.e., for a functional $F_{\varepsilon}(\Psi)$ with $\partial u_{\varepsilon}/\partial x_i$ in place of u_{ε}). In practice, such a result furnishes the next term (i.e., $u_1(x, x/\varepsilon)$ in (1.2)) in the asymptotic expansion of the solution u_{ε} , while the leading term is given by the theorem in § 4. Thus, the use of multiple-scale asymptotic expansions of the form (1.2) (or (1.1)) is rigorously justified in homogenization.

Finally, in § 6, we present a new approach for the mathematical analysis of homogenization problems.

We will be concerned solely with vector spaces over \mathbb{R} although our result and arguments are still rigorously valid in the complex case—providing some minor modifications are made. The only measure considered in this work is the Lebesgue measure.

2. General notation and preliminaries. Let $\mathbb{R}^N (N \in \mathbb{N}, N \ge 1)$ be the *N*-dimensional Euclidean space. Points in \mathbb{R}^N are denoted by $x = (x_1, \dots, x_N)$ (the global variables) or $y = (y_1, \dots, y_N)$ (the local variables related to periodicity). The cube

$$Y = [0, 1[^{N} =]0, 1[\times \cdots \times]0, 1[(N \text{ times})]$$

is considered in the system of the local variables, with closure $\overline{Y} = [0, 1]^N$.

By a Y-periodic function we mean a function on \mathbb{R}^N that is periodic with period Y (i.e., with period 1 in each variable y_i).

Generally speaking, if E is a set (e.g., \mathbb{R}^N or any open set in \mathbb{R}^N), we denote by C(E) the space of continuous functions on E, by $\mathcal{X}(E)$ the space of those functions in C(E) with compact supports (contained in E), and by $\mathcal{D}(E)$ the subspace of $\mathcal{X}(E)$ made up of C^∞ functions.

In connection with the periodic structure, let us introduce some specific spaces.

 $C_p(\mathbb{R}^N)$ (or, for simplicity, C_p) denotes the space of functions $w \in C(\mathbb{R}^N)$, w Y-periodic.

 $L_p^2(\mathbb{R}^N)$ (or L_p^2) the space of Y-periodic functions in $L_{loc}^2(\mathbb{R}^N)$, which is a Hilbert space with the norm

$$||w||_{L^2(Y)} = \left(\int_Y |w|^2 dy\right)^{1/2}.$$

 $\mathscr{H}(\mathbb{R}^N; L_p^2)$ the space of continuous functions on \mathbb{R}^N (the Euclidean space of the variables x) with values in L_p^2 and having compact supports.

 $L^2(\mathbb{R}^N; L_p^2)$ the space of measurable functions u(x, y) on $\mathbb{R}^N \times \mathbb{R}^N$ such that for almost all x the function $y \to u(x, y)$ belongs to L_p^2 and $\int_{\mathbb{R}^N \times Y} |u(x, y)|^2 dx dy < \infty$. We endow this space with the norm

$$||u||_{L^2(\mathbb{R}^N \times Y)} = \left[\int_{\mathbb{R}^N \times Y} |u(x, y)|^2 dx dy \right]^{1/2}.$$

 $L^{2}(\mathbb{R}^{N}; L_{p}^{2})$, thus equipped, is a Hilbert space.

Finally, $\mathscr{H}(\mathbb{R}^N; C_p)$ denotes the space of continuous functions on \mathbb{R}^N with values in C_p and having compact supports. We provide the vector space $\mathscr{H}(\mathbb{R}^N; C_p)$ with its natural topology: the inductive limit topology determined by the spaces $\mathscr{H}_K(\mathbb{R}^N; C_p)$ (K ranging over the compact subsets of \mathbb{R}^N), where

$$\mathscr{H}_{K}(\mathbb{R}^{N}; C_{p}) = \{\Psi \in \mathscr{H}(\mathbb{R}^{N}; C_{p}); \operatorname{supp} \Psi \subset K\}$$

is a Banach space with the norm

$$\|\Psi\|_{K} = \sup_{x \in K} \|\Psi(x)\|_{L^{\infty}} \equiv \sup_{(x,y) \in \mathbb{R}^{N} \times \mathbb{R}^{N}} |\Psi(x, y)|$$

(note that C_p , provided with the L^{∞} norm, is a Banach space).

In § 3 we will need a very useful result from Bourbaki [2, Prop. 5, p. 46]: Let $\mathscr{H}(\mathbb{R}^N) \otimes C_p$ denote the subset of $\mathscr{H}(\mathbb{R}^N; C_p)$ consisting of all functions of the form $\sum v \otimes w$ (\otimes denotes the tensor product), v (respectively, w) ranging over a finite subset of $\mathscr{H}(\mathbb{R}^N)$ (respectively, C_p). Then $\mathscr{H}(\mathbb{R}^N) \otimes C_p$ is dense in $\mathscr{H}(\mathbb{R}^N; C_p)$.

Finally, for further needs, let us keep in mind the well-known result that asserts that $\mathscr{X}(\mathbb{R}^N; C_p)$ is dense in $L^2(\mathbb{R}^N; L_p^2)$.

In the sequel we will put, for simplicity,

$$\mathscr{H}_p \equiv \mathscr{H}(\mathbb{R}^N; C_p).$$

3. Basic result. A convergence theorem. In all that follows, ε , with $\varepsilon > 0$, denotes a real sequence destined to tend to zero, and K_0 is a fixed compact set in \mathbb{R}^N (K_0 does not depend on ε). Next, we introduce $L^2_{K_0}(\mathbb{R}^N)$, the space of all functions in $L^2(\mathbb{R}^N)$ having their (compact) supports in K_0 .

3.1. Statement of the theorem. Idea of the proof.

THEOREM 1. Let $u_{\varepsilon} \in L^{2}_{K_{0}}(\mathbb{R}^{N})$. Suppose that there exists a constant c > 0 such that

$$\|u_{\varepsilon}\|_{L^{2}} \leq c \quad \text{for any } \varepsilon.$$

Then there exist a subsequence from ε , still denoted by ε for simplicity, and a function u_0 in $L^2(\mathbb{R}^N; L_p^2)$ such that

(3.2)
$$\int_{\mathbb{R}^N} u_{\varepsilon}(x) \Psi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_{\mathbb{R}^N \times Y} u_0(x, y) \Psi(x, y) dx dy$$

as $\varepsilon \downarrow 0$, for all Ψ in \mathcal{K}_p .

Remark 1. Instead of the cube $Y = [0, 1[^N]$, if we consider a parallelepiped $Y = \prod_{i=1}^{N} [0, a_i[(a_i > 0)]$, Theorem 1 remains valid provided the right-hand side of (3.2) is multiplied by 1/|Y| (|Y| = measure of Y). \Box

We now give the idea and the main steps of the proof. The first step is to show that a subsequence (still denoted by ε for simplicity) can be extracted from ε such that for $w \in C_p$ the sequence $u_{\varepsilon}w^{\varepsilon}$ converges in L^2 -weak as $\varepsilon \downarrow 0$, where $w^{\varepsilon}(x) = w(x/\varepsilon)$. Thus, given a function w in C_p , there will exist z_w in $L^2(\mathbb{R}^N)$ such that, as $\varepsilon \downarrow 0$,

(3.3)
$$\int_{\mathbb{R}^N} u_{\varepsilon} w^{\varepsilon} v \, dx \to \int_{\mathbb{R}^N} z_{w} v \, dx \quad \text{for all } v \in \mathscr{K}(\mathbb{R}^N).$$

Next, our task is to extend (3.3) (with the same subsequence ε) to all functions in \mathscr{X}_p (see § 2 for the definition of \mathscr{X}_p). Indeed, note that the integrand on the left of (3.3) is nothing but $u_{\varepsilon}(x)\Psi(x, x/\varepsilon)$ with $\Psi(x, y) = v(x)w(y)$. It is then reasonable to hope that (3.3) could be generalized to all functions in \mathscr{X}_p . To this end, we first establish that for any Ψ in \mathscr{X}_p a real number $F_0(\Psi)$ exists such that

(3.4)
$$\int_{\mathbb{R}^N} u_{\varepsilon}(x) \Psi\left(x, \frac{x}{\varepsilon}\right) dx \to F_0(\Psi).$$

This will be obtained from (3.3), because $\mathscr{K}(\mathbb{R}^N) \otimes C_p$ is dense in \mathscr{K}_p (see § 2).

Finally, the last step is devoted to the characterization of the right of (3.4).

3.2. First convergence result. Our goal in this section is to obtain (3.3) for any w in C_p . To begin, let us establish two elementary (but fundamental) lemmas.

LEMMA 1. Let K_0 be the above compact set. Fix r > 0 and set H = $\{x \in \mathbb{R}^N; d(x, K_0) \leq r\}$, where d denotes the Euclidean metric. Then for $\varepsilon < \varepsilon_0$ (ε_0 a suitable constant) there exist a natural number n (depending on ε) and a finite family $\varepsilon(\bar{Y} + k_i), 1 \leq \varepsilon$ $i \leq n$, with $k_i \in \mathbb{Z}^N$ (\mathbb{Z} is the set of all integers) such that

(3.5)
$$K_0 \subset \bigcup_{i=1}^n \varepsilon(\bar{Y}+k_i) \subset H.$$

Proof. For arbitrarily fixed ε , we may express \mathbb{R}^N (the space of the variables x) as the union of all the $\varepsilon(\bar{Y}+k), k \in \mathbb{Z}^N$. Since K_0 is compact, a finite family $\varepsilon(\bar{Y}+k_i)$, $i = 1, \dots, n$, exists such that K_0 intersects each $\varepsilon(\bar{Y} + k_i)$ and K_0 is contained in their union.

 $i(1 \leq i \leq n)$, let $x \in \varepsilon(\bar{Y}+k_i).$ Now, for each Then $d(x, K_0) \leq$ $d(x, \varepsilon(\bar{Y}+k_i) \cap K_0) \leq \text{diam } \varepsilon(\bar{Y}+k_i) = \varepsilon \text{ diam } Y \text{ (diam denotes the diameter). Hence,}$ by putting $\varepsilon_0 = r/\text{diam } Y$ it follows that for $\varepsilon < \varepsilon_0$ the union of the sets $\varepsilon(\bar{Y} + k_i)$ is contained in H, which completes the proof.

LEMMA 2. There exists a constant $c_0 > 0$ such that for $\varepsilon < \varepsilon_0$ (ε_0 is the constant in Lemma 1) we have

$$\left|\int_{\mathbb{R}^{N}} u(x) w\left(\frac{x}{\varepsilon}\right) dx\right| \leq c_{0} \|u\|_{L^{2}} \|w\|_{L^{2}(Y)}$$

for all u in $L^2_{K_0}(\mathbb{R}^N)$ and all w in L^2_p . Proof. Let $u \in L^2_{K_0}(\mathbb{R}^N)$, $w \in L^2_p$. By Hölder's inequality we have

$$\left|\int_{\mathbb{R}^{N}} u(x) w\left(\frac{x}{\varepsilon}\right) dx\right| \leq ||u||_{L^{2}} \left[\int_{K_{0}} \left|w\left(\frac{x}{\varepsilon}\right)\right|^{2} dx\right]^{1/2}.$$

Next, by the preceding lemma, let $\varepsilon(\bar{Y}+k_i)$ $(1 \le i \le n)$ be a finite family satisfying (3.5) for $\varepsilon < \varepsilon_0$. Then

$$\int_{K_0} \left| w\left(\frac{x}{\varepsilon}\right) \right|^2 dx \leq \sum_{i=1}^n \int_{\varepsilon(Y+k_i)} \left| w\left(\frac{x}{\varepsilon}\right) \right|^2 dx.$$

By change of variable, $x = \varepsilon(y + k_i)$, and use of periodicity we have

$$\int_{\varepsilon(Y+k_i)} \left| w\left(\frac{x}{\varepsilon}\right) \right|^2 dx = \varepsilon^N \int_Y |w(y)|^2 dy.$$

It follows that

$$\int_{K_0} \left| w\left(\frac{x}{\varepsilon}\right) \right|^2 dx \leq \varepsilon^N n \|w\|_{L^2(Y)}^2.$$

But, thanks to (3.5) we have $\varepsilon^{N}n = \text{meas } \bigcup_{i=1}^{n} \varepsilon(\bar{Y} + k_i) \leq \text{meas } H$ (note that *n* depends on ε), from which the conclusion follows (with, e.g., $c_0 = (\text{meas } H)^{1/2}$).

Remark 2. For $\varepsilon < \varepsilon_0$ we have

$$\int_{K_0} \left| w\left(\frac{x}{\varepsilon}\right) \right|^2 dx \leq c_0^2 \|w\|_{L^2(Y)}^2 \quad \forall w \in L_p^2.$$

As an immediate consequence of Lemma 2, we have the following proposition, which plays an essential role throughout the rest of this section.

PROPOSITION 1. Let $f \in L^2_{K_0}(\mathbb{R}^N)$ (f may or may not depend on ε). Then for $\varepsilon < \varepsilon_0$, a unique function $f_{\varepsilon} \in L^2_p$ can be assigned to f such that

$$\int_{\mathbb{R}^{N}} f(x) w\left(\frac{x}{\varepsilon}\right) dx = \int_{Y} f_{\varepsilon}(y) w(y) dy \quad \forall w \text{ in } L^{2}_{p},$$
$$\|f_{\varepsilon}\|_{L^{2}(Y)} \leq c_{0} \|f\|_{L^{2}}.$$

Remark 3. The correspondence $f \rightarrow f_{\varepsilon}$ defined above is linear.

We are now in a position to prove the main result in this section. First, we must give some notation used frequently in the sequel.

Given w in L_p^2 we denote by w^{ε} the ε -periodic function in $L_{loc}^2(\mathbb{R}^N)$ defined by

(3.6)
$$w^{\varepsilon}(x) = w\left(\frac{x}{\varepsilon}\right).$$

Also, if $\Psi \in \mathscr{K}_p$ we put

(3.7)
$$\Psi^{\varepsilon}(x) = \Psi\left(x, \frac{x}{\varepsilon}\right).$$

It is clear that $\Psi^{\varepsilon} \in \mathcal{H}(\mathbb{R}^N)$. Moreover, if the support of Ψ is contained in K (a compact subset of \mathbb{R}^N), then the support of Ψ^{ε} lies in K for any ε .

The aim now is to prove the following proposition.

PROPOSITION 2. Under the assumptions of Theorem 1, a subsequence (still denoted by ε) can be extracted from ε such that for any w in C_p (w independent of ε), the sequence $u_{\varepsilon}w^{\varepsilon}$ converges in L^2 -weak as $\varepsilon \downarrow 0$.

Proof. (i) We begin by fixing a (nontrivial) function α in $\mathcal{D}(\mathbb{R}^N)$, α independent of ε . Next, fix x in \mathbb{R}^N and consider the function $s \to f(s) = \alpha(x-s)u_{\varepsilon}(s)$, which belongs to $L^2_{K_0}(\mathbb{R}^N)$. By Proposition 1 there exists, for $\varepsilon < \varepsilon_0$, a unique function $y \to z_{\varepsilon}(x, y)$ in L^2_p such that for any w in L^2_p we have

$$\int_{\mathbb{R}^N} \alpha(x-s) u_{\varepsilon}(s) w^{\varepsilon}(s) \ ds = \int_Y z_{\varepsilon}(x, y) w(y) \ dy,$$

that is,

(3.8)
$$[(u_{\varepsilon}w^{\varepsilon})*\alpha](x) = \int_{Y} z_{\varepsilon}(x,y)w(y) \, dy,$$

where * denotes the convolution product.

Moreover, again by Proposition 1, we have

(3.9)
$$||z_{\varepsilon}(x,\cdot)||_{L^{2}(Y)} \leq c_{0} \left[\int_{\mathbb{R}^{N}} |\alpha(x-s)u_{\varepsilon}(s)|^{2} ds \right]^{1/2}.$$

Observe that the function $(u_{\varepsilon}w^{\varepsilon}) * \alpha$ lies in $\mathscr{D}(\mathbb{R}^{N})$ and has its support in a compact set that does not depend on ε .

Thus, by (3.8) (valid for all x) we assign to u_{ε} (for $\varepsilon < \varepsilon_0$) a unique function $x \to z_{\varepsilon}(x)$ [i.e., $x \to z_{\varepsilon}(x, \cdot)$] from \mathbb{R}^N to L_p^2 , with (3.9).

(ii) For further needs we now study a few useful properties of the function z_{ε} thus constructed. To summarize, let us show that $z_{\varepsilon} \in L^{2}(\mathbb{R}^{N}; L_{p}^{2})$. It suffices to check that $z_{\varepsilon} \in \mathscr{H}(\mathbb{R}^{N}; L_{p}^{2})$ (see § 2 for notation). Clearly the function z_{ε} has compact support; then it remains to show continuity. For this, fix x in \mathbb{R}^{N} . Let $h \in \mathbb{R}^{N}$. Consider the function $s \rightarrow [\alpha(x+h-s)-\alpha(x-s)]u_{\varepsilon}(s)$, which lies in $L_{K_{0}}^{2}(\mathbb{R}^{N})$. If we replace in (i)

the function $s \rightarrow \alpha(x-s)u_{\varepsilon}(s)$ by the above function, the associated analogue of $z_{\varepsilon}(x)$ is, according to the above process, exactly $z_{\varepsilon}(x+h) - z_{\varepsilon}(x)$ (see Remark 3). Hence,

$$\|z_{\varepsilon}(x+h)-z_{\varepsilon}(x)\|_{L^{2}(Y)} \leq c_{0} \left[\int_{\mathbb{R}^{N}} |\alpha(x+h-s)-\alpha(x-s)|^{2} |u_{\varepsilon}(s)|^{2} ds \right]^{1/2},$$

which is the analogue of (3.9). Observing that the right-hand side is majorized by $cc_0 \sup_s |\alpha(x+h-s) - \alpha(x-s)|$ (c is the constant in Theorem 1) and, furthermore, α being uniformly continuous on \mathbb{R}^N , we deduce that $||z_{\varepsilon}(x+h) - z_{\varepsilon}(x)||_{L^2(Y)} \leq c|h|$, for all $h \in \mathbb{R}^N$, which shows continuity.

Thus, $z_{\epsilon} \in L^2(\mathbb{R}^N; L_p^2)$. Furthermore, by (3.9) we have

$$||z_{\varepsilon}||_{L^{2}(\mathbb{R}^{N}\times Y)} \leq c \ (c > 0) \quad \forall \varepsilon < \varepsilon_{0}$$

(where the constant c does not depend on ε).

(iii) Finally, by (3.10) we can extract a subsequence from ε , still denoted by ε , such that $z_{\varepsilon} \to z$ in $L^2(\mathbb{R}^N; L_p^2)$ -weak as $\varepsilon \downarrow 0$. Therefore, for each $v \in \mathcal{K}(\mathbb{R}^N)$ and each $w \in L_p^2$ we have

$$\int_{\mathbb{R}^{N}\times Y} z_{\varepsilon}(x, y)w(y)v(x) \, dx \, dy \to \int_{\mathbb{R}^{N}\times Y} z(x, y)w(y)v(x) \, dx \, dy,$$

so that, using (3.8) combined with Fubini's theorem, we have

(3.11)
$$\int_{\mathbb{R}^N} \left[(u_{\varepsilon} w^{\varepsilon}) * \alpha \right](x) v(x) \, dx \to \int_{\mathbb{R}^N \times Y} z(x, y) w(y) v(x) \, dx \, dy.$$

From now on, ε denotes exclusively the subsequence extracted above. By (3.11) we finally show that for each w in C_p , the sequence $u_{\varepsilon}w^{\varepsilon}$ converges weakly in $L^2(\mathbb{R}^N)$ as $\varepsilon \downarrow 0$ (that is, ε is the desired subsequence in Proposition 2). For this purpose, let w be arbitrarily fixed in C_p . Since $w^{\varepsilon} \in L^{\infty}$, we have $u_{\varepsilon}w^{\varepsilon} \in L^2$. Furthermore, we evidently have $||u_{\varepsilon}w^{\varepsilon}||_{L^2} \leq c$ (c > 0), for all ε . Therefore, we can extract ε' from ε such that

(3.12)
$$u_{\varepsilon'} w^{\varepsilon'} \to z_w \text{ in } L^2 \text{-weak as } \varepsilon' \downarrow 0,$$

so that, the transformation $v \rightarrow v * \alpha$ being continuous from L^2 into itself,

$$\int_{\mathbb{R}^{N}} \left[(u_{\varepsilon'} w^{\varepsilon'}) * \alpha \right] v \, dx \to \int_{\mathbb{R}^{N}} (z_{w} * \alpha) v \, dx$$

for all $v \in \mathcal{K}(\mathbb{R}^N)$. By comparison with (3.11) we necessarily have

(3.13)
$$(z_w * \alpha)(x) = \int_Y z(x, y) w(y) \, dy \quad \text{a.e. in } \mathbb{R}^N$$

Now, since w is the same function as in (3.12), let ε'' be another subsequence from ε such that $u_{\varepsilon''}w^{\varepsilon''} \rightarrow z'_w$ in L^2 -weak as $\varepsilon'' \downarrow 0$. Following the above process once more, we obtain

(3.14)
$$(z'_w * \alpha)(x) = \int_Y z(x, y) w(y) \, dy \quad \text{a.e. in } \mathbb{R}^N.$$

By subtracting (3.14) from (3.13) we have

$$(3.15) \qquad (z'_w-z_w)*\alpha=0$$

from which it follows that $z'_w = z_w$. Indeed the distributions (represented by the L^2 functions) α , $z'_w - z_w$ (respectively) have compact supports, i.e., they lie in $\mathscr{E}'(\mathbb{R}^N)$, the

subspace of $\mathscr{D}'(\mathbb{R}^N)$ formed of distributions having compact supports. But, since the vector space $\mathscr{E}'(\mathbb{R}^N)$ endowed with the convolution product is an algebra without zero divisor (see [8]), (3.15) implies $z'_w - z_w = 0$.

We have just established that for any subsequence ε' such that $u_{\varepsilon'}w^{\varepsilon'}$ converges weakly in L^2 , the corresponding limit does not depend on ε' . That is, the sequence $u_{\varepsilon}w^{\varepsilon}$ converges weakly in L^2 . The proof is complete.

3.3. Extension of the first convergence result. Here and throughout the rest of § 3, ε denotes the subsequence involved in Proposition 2. Then, by that proposition, a unique function $z_w \in L^2$ is assigned to each w in C_p such that (3.3) holds. In other words, if we put

(3.16)
$$\Psi(x, y) = v(x)w(y) \text{ for } v \in \mathcal{H}(\mathbb{R}^N) \text{ and } w \in C_p$$

and $F_0(\Psi) = \int_{\mathbb{R}^N} z_w v \, dx$, we have $\int_{\mathbb{R}^N} u_\varepsilon \Psi^\varepsilon \, dx \to F_0(\Psi)$ for any Ψ in \mathcal{X}_p of the form (3.16) (see (3.7) for the definition of Ψ^ε). This property is, clearly, what we call the first (or primitive) convergence result.

The aim in this section is then to extend the above property to all of \mathscr{X}_p .

LEMMA 3. Let Ψ be fixed in \mathscr{X}_p (Ψ independent of ε). Then the sequence $\varepsilon \rightarrow \int_{\mathbb{R}^N} u_{\varepsilon} \Psi^{\varepsilon} dx$ is Cauchy.

Proof. Let $\Psi \in \mathscr{X}_p$. Let $\eta > 0$. Since the set $\mathscr{H}(\mathbb{R}^N) \otimes C_p$ is dense in \mathscr{X}_p (see § 2), there exists some Ψ_η in \mathscr{X}_p , $\Psi_\eta = \sum_{i \in I} v_i \otimes w_i [v_i \in \mathscr{H}(\mathbb{R}^N), w_i \in C_p]$, with I finite, such that the supports of both Ψ and Ψ_η lie in a fixed compact set $K \subseteq \mathbb{R}^N$ that depends only on Ψ , and

(3.17)
$$\sup_{x \in \mathbb{R}^N} \|\Psi_{\eta}(x) - \Psi(x)\|_{L^{\infty}} \leq \frac{\eta}{2c}$$

where c is the constant in (3.1).

On the other hand, we evidently have for all ε

(3.18)
$$\sup_{x\in\mathbb{R}^N} |\Psi_{\eta}^{\varepsilon}(x) - \Psi^{\varepsilon}(x)| \leq \sup_{x\in\mathbb{R}^N} \|\Psi_{\eta}(x) - \Psi(x)\|_{L^{\infty}}.$$

Now, consider ε_1 , ε_2 , destined to decrease independently. By a routine technique we have

$$\begin{split} \left| \int_{\mathbb{R}^{N}} u_{e_{2}} \Psi^{e_{2}} dx - \int_{\mathbb{R}^{N}} u_{e_{1}} \Psi^{e_{1}} dx \right| \\ & \leq \left| \int_{\mathbb{R}^{N}} u_{e_{2}} (\Psi^{e_{2}} - \Psi^{e_{2}}_{\eta}) dx \right| + \left| \int_{\mathbb{R}^{N}} u_{e_{1}} (\Psi^{e_{1}}_{\eta} - \Psi^{e_{1}}) dx \right| \\ & + \left| \int_{\mathbb{R}^{N}} u_{e_{1}} \Psi^{e_{1}}_{\eta} dx - \int_{\mathbb{R}^{N}} u_{e_{2}} \Psi^{e_{2}}_{\eta} dx \right|. \end{split}$$

But (3.17) combines with (3.18) to give

$$\left|\int_{\mathbb{R}^{N}} u_{\varepsilon_{i}}(\Psi_{\eta}^{\varepsilon_{i}} - \Psi^{\varepsilon_{i}}) dx\right| \leq \frac{\eta}{2} \quad \text{for } i = 1, 2.$$

Hence

$$\left|\int_{\mathbb{R}^N} u_{\varepsilon_2} \Psi^{\varepsilon_2} \, dx - \int_{\mathbb{R}^N} u_{\varepsilon_1} \Psi^{\varepsilon_1} \, dx\right| \leq \eta + \left|\int_{\mathbb{R}^N} u_{\varepsilon_2} \Psi^{\varepsilon_2}_{\eta} \, dx - \int_{\mathbb{R}^N} u_{\varepsilon_1} \Psi^{\varepsilon_1}_{\eta} \, dx\right|.$$

Now, thanks to Proposition 2 we observe that for v in $\mathcal{X}(\mathbb{R}^N)$ and w in C_p the sequence $\varepsilon \to \int_{\mathbb{R}^N} u_\varepsilon v w^\varepsilon dx$ is Cauchy. Therefore, since $\int_{\mathbb{R}^N} u_\varepsilon \Psi_{\eta}^\varepsilon dx = \sum_{i \in I} \int_{\mathbb{R}^N} u_\varepsilon v_i w_i^\varepsilon$, the sequence $\varepsilon \to \int_{\mathbb{R}^N} u_\varepsilon \Psi_{\eta}^\varepsilon dx$ is Cauchy as a finite sum of Cauchy sequences. So we have $|\int_{\mathbb{R}^N} u_{\varepsilon_2} \Psi_{\eta}^{\varepsilon_2} dx - \int_{\mathbb{R}^N} u_{\varepsilon_1} \Psi_{\eta}^{\varepsilon_1} dx| \to 0$ as $\varepsilon_1 \downarrow 0$ and $\varepsilon_2 \downarrow 0$, and the conclusion follows from the arbitrariness of η . \Box

This brings us to one of the central preliminary convergence results in this work.

PROPOSITION 3. For any $\Psi \in \mathcal{H}_p$ (Ψ independent of ε) there exists a unique real number $F_0(\Psi)$ such that

$$\int_{\mathbb{R}^N} u_{\varepsilon} \Psi^{\varepsilon} \, dx \to F_0(\Psi) \quad as \ \varepsilon \downarrow 0.$$

3.4. End of the proof. Characterization of F_0 . The aim in this section is to show that the above transformation $\Psi \to F_0(\Psi)$ is the restriction to \mathcal{X}_p of a continuous linear form on $L^2(\mathbb{R}^N; L_p^2)$. More precisely, we must check that there exists a unique u_0 in $L^2(\mathbb{R}^N; L_p^2)$ such that

$$F_0(\Psi) = \int_{\mathbb{R}^N \times Y} u_0(x, y) \Psi(x, y) \, dx \, dy \quad \forall \Psi \text{ in } \mathscr{K}_p.$$

Since \mathcal{H}_p is dense in $L^2(\mathbb{R}^N; L_p^2)$ and the transformation $\Psi \to F_0(\Psi)$ is linear, it suffices to establish that there exists a constant c > 0 such that

(3.19)
$$|F_0(\Psi)| \le c \|\Psi\|_{L^2(\mathbb{R}^N \times Y)} \quad \forall \Psi \text{ in } \mathscr{K}_p.$$

In this connection, fix Ψ in \mathscr{X}_p (Ψ independent of ε). Then $|\int_{\mathbb{R}^N} u_{\varepsilon} \Psi^{\varepsilon} dx| \leq c (\int_{K_0} |\Psi^{\varepsilon}|^2 dx)^{1/2}$ for all ε , where c is the constant on the right of (3.1).

By Proposition 3 and the fundamental property

$$\int_{K_0} |\Psi^{\varepsilon}|^2 dx \to \int_{K_0 \times Y} |\Psi(x, y)|^2 dx dy \text{ as } \varepsilon \downarrow 0 \quad (\text{see § 1}),$$

assertion (3.19) follows immediately. The proof is complete.

Remark 4. The function u_0 has its support in the set $K_0 \times \mathbb{R}^N$ (or K_0 , if u_0 is regarded as a function from \mathbb{R}^N to L_p^2).

4. The leading-order approximation. A convergence theorem. In what follows, Ω denotes a bounded open set in the Euclidean space \mathbb{R}^N (of the variables x_1, \dots, x_N), Ω independent of ε . We denote by $\mathcal{H}(\overline{\Omega}; C_p)$ [respectively, $\mathcal{H}(\overline{\Omega})$] the set of all restrictions to Ω of functions in \mathcal{H}_p [respectively, $\mathcal{H}(\mathbb{R}^N)$]. We also introduce the space $L^2(\Omega; L_p^2)$, which is a Hilbert space with the norm

$$||u||_{L^2(\Omega \times Y)} = \left[\int_{\Omega \times Y} |u(x, y)|^2 dx dy\right]^{1/2}.$$

The aim in this section is to establish the following theorem. THEOREM 2. Let $u_{\varepsilon} \in L^2(\Omega)$. Suppose that there exists a constant c > 0 such that

$$\|u_{\varepsilon}\|_{L^{2}(\Omega)} \leq c \quad \forall \varepsilon.$$

Then a subsequence (still denoted by ε) can be extracted from ε such that, letting $\varepsilon \downarrow 0$,

(4.2)
$$\int_{\Omega} u_{\varepsilon} \Psi^{\varepsilon} dx \to \int_{\Omega \times Y} u_{0}(x, y) \Psi(x, y) dx dy \quad \forall \Psi$$

in $\mathscr{K}(\overline{\Omega}; C_p)$, where $u_0 \in L^2(\Omega; L_p^2)$. Moreover,

(4.3)
$$\int_{\Omega} u_{\varepsilon} v w^{\varepsilon} dx \to \int_{\Omega \times Y} u_{0}(x, y) v(x) w(y) dx dy \quad \forall v$$

in $\mathscr{K}(\overline{\Omega})$ and all w in L_p^2 .

Proof. Property (4.2) is straightforward by Theorem 1 and Remark 4. As for (4.3), we begin by taking in (4.2) test functions of the form $\Psi(x, y) = v(x)w(y)$ with $v \in \mathcal{X}(\overline{\Omega}), w \in C_p$. We obtain as $\varepsilon \downarrow 0$,

(4.4)
$$\int_{\Omega} u_{\varepsilon} v w^{\varepsilon} dx \to \int_{\Omega \times Y} u_0(x, y) v(x) w(y) dx dy$$

for all $v \in \mathcal{K}(\overline{\Omega})$ and all $w \in C_p$.

Next, we must extend (4.4) to all functions w in L_p^2 . Fix v in $\mathcal{X}(\overline{\Omega})$ and w in L_p^2 . Let (w_n) be a sequence from C_p (dense subspace of L_p^2) such that $w_n \to w$ in L_p^2 as $n \to \infty$. Utilizing the fact that the transformation $z \to z^e$ is continuous linear from L_p^2 to $L^2(\Omega)$ (see Remark 2), we have

(4.5)
$$\|w_n^{\varepsilon} - w^{\varepsilon}\|_{L^2(\Omega)} \leq c_0 \|w_n - w\|_{L^2(Y)} \quad \forall n, \quad \forall \varepsilon < \varepsilon_0$$

 $(c_0 \text{ and } \varepsilon_0 \text{ are the constants in Lemma 2 with } K_0 = \overline{\Omega}).$

Now we write

$$\int_{\Omega} u_{\varepsilon} v w^{\varepsilon} dx - \int_{\Omega \times Y} u_{0} v w dx dy$$

=
$$\int_{\Omega} u_{\varepsilon} v (w^{\varepsilon} - w_{n}^{\varepsilon}) dx + \int_{\Omega \times Y} u_{0} v (w_{n} - w) dx dy$$

+
$$\int_{\Omega} u_{\varepsilon} v w_{n}^{\varepsilon} dx - \int_{\Omega \times Y} u_{0} v w_{n} dx dy$$

and estimate each of the first two integrals on the right-hand side separately (use (4.5)). This yields

(4.6)
$$\left| \int_{\Omega} u_{\varepsilon} v w^{\varepsilon} dx - \int_{\Omega \times Y} u_{0} v w dx dy \right|$$
$$\leq c_{1} \| w_{n} - w \|_{L^{2}(Y)} + \left| \int_{\Omega} u_{\varepsilon} v w_{n}^{\varepsilon} dx - \int_{\Omega \times Y} u_{0} v w_{n} dx dy \right|$$

for all *n* and all $\varepsilon < \varepsilon_0$ (where c_1 is constant with respect to both ε and *n*).

Finally, let $\eta > 0$. Choose in (4.6) the natural number *n* so that $c_1 ||w_n - w||_{L^2(Y)} \leq \eta$. Then letting $\varepsilon \downarrow 0$ and using (4.4), it follows that the limit of the left-hand side of (4.6) is bounded from above by η . The desired conclusion then results from the arbitrariness of η .

Remark 5. Let u_{ε} be as in Theorem 2. First, let us observe that, by weak compactness, we may assume that in addition to (4.2) and (4.3) in Theorem 2, the subsequence ε satisfies the following property.

There exists $u \in L^2(\Omega)$ such that $u_{\varepsilon} \to u$ in $L^2(\Omega)$ -weak. Next, taking w = 1 in (4.3) we easily obtain $u(x) = \int_Y u_0(x, y) dy$ (u is the mean value of u_0). It follows that u_0 is (uniquely) expressible in the form

$$u_0(x, y) = u(x) + \bar{u}_0(x, y)$$
 with $\int_Y \bar{u}_0(x, y) \, dy = 0.$

So assume there is a subsequence from (u_{ε}) that converges strongly in $L^{2}(\Omega)$ as $\varepsilon \downarrow 0$. Then an easy computation yields $\bar{u}_{0} = 0$; that is, the leading term u_{0} in (1.2) does not depend on the local variables y. In other words, if the leading term depends on y, i.e., $\bar{u}_{0} \neq 0$, then (u_{ε}) never contains a strongly convergent subsequence (see § 1).

5. The next-order approximation. A convergence theorem.

5.1. Notation and preliminaries. We denote by C_p^{∞} the subspace of C_p formed of C^{∞} functions, H_p^1 the subspace of L_p^2 formed of functions w such that $\partial w / \partial y_i \in L_p^2$ for $i = 1, \dots, N$ (the derivatives obviously being taken in the distribution sense).

We provide H_p^1 with the norm

$$\|w\|_{H^{1}(Y)} = \left(\|w\|_{L^{2}(Y)}^{2} + \sum_{i=1}^{N} \left\|\frac{\partial w}{\partial y_{i}}\right\|_{L^{2}(Y)}^{2}\right)^{1/2},$$

which makes it a Hilbert space.

Sometimes it is more convenient to consider, instead of H_p^1 , its closed subspace

$$\frac{H_p^1}{\mathbb{R}} = \left\{ w \in H_p^1; \int_Y w \, dy = 0 \right\}$$

on which the norm

$$\|w\|_{H^{1}(Y)/\mathbb{R}} = \left(\sum_{i=1}^{N} \left\|\frac{\partial w}{\partial y_{i}}\right\|_{L^{2}(Y)}^{2}\right)^{1/2}$$

is equivalent to the above H_p^1 -norm.

We will need the following lemma.

LEMMA 4. Let $f = (f_i)$, $f_i \in L_p^2 (1 \le i \le N)$. Assume that $\sum_{i=1}^N \int_Y f_i w_i \, dy = 0$ for all $w = (w_i)$ in $(C_p^{\infty})^N$ such that div w = 0 (where div $w = \sum_{i=1}^N \partial w_i / \partial y_i$). Then there exists a unique function $q \in H_p^1/\mathbb{R}$ such that $f_i = \partial q / \partial y_i$ for $i = 1, \dots, N$.

Lemma 4 is the "periodic version" of the well-known result concerning the solvability of the equation grad q = f for f given in $(L_{loc}^2)^N$ (see, e.g., [10]). See, e.g., [6, Appendix] for the proof.

5.2. A convergence theorem (next-order approximation). We are now in a position to prove the main result in this section. In what follows, Ω denotes a smooth bounded open set in \mathbb{R}^N (Ω independent of ε). As in the preceding sections, ε ($\varepsilon > 0$) denotes a sequence tending to zero.

THEOREM 3. Let $u_{\varepsilon} \in H^{1}(\Omega)$. Suppose that there exists a constant c > 0 such that

$$\|u_{\varepsilon}\|_{H^{1}(\Omega)} \leq c \quad \forall \varepsilon.$$

Then a subsequence (still denoted by ε) can be extracted from ε such that, as $\varepsilon \downarrow 0$,

$$(5.2) u_{\varepsilon} \to u \text{ in } H^1(\Omega) \text{-weak},$$

(5.3)
$$\int_{\Omega} \frac{\partial u_{\varepsilon}}{\partial x_{i}} \Psi^{\varepsilon} v \, dx \to \int_{\Omega \times Y} \left[\frac{\partial u}{\partial x_{i}} (x) + \frac{\partial u_{1}}{\partial y_{i}} (x, y) \right] \Psi(y) v(x) \, dx \, dy,$$

 $i = 1, \dots, N$; for all Ψ in L_p^2 and all v in $\mathcal{K}(\overline{\Omega})$, where $u_1 \in L^2(\Omega; H_p^1/\mathbb{R})$.

Proof. By virtue of (5.1) we can extract a subsequence such that (5.2) holds. Moreover, by Theorem 2 there exists $z_i \in L^2(\Omega; L_p^2)$, $1 \le i \le N$, such that

(5.4)
$$\int_{\Omega} \frac{\partial u_{\varepsilon}}{\partial x_{i}} \Psi^{\varepsilon} v \, dx \to \int_{\Omega \times Y} z_{i}(x, y) \Psi(y) v(x) \, dx \, dy \quad \text{as } \varepsilon \downarrow 0$$

for all Ψ in L_p^2 and all v in $\mathcal{K}(\overline{\Omega})$.

It remains to show that there exists $u_1 \in L^2(\Omega; H_p^1/\mathbb{R})$ such that

$$z_i(x, y) = \frac{\partial u}{\partial x_i}(x) + \frac{\partial u_1}{\partial y_i}(x, y)$$
 for $i = 1, \cdots, N$.

So let $\Psi = (\Psi_i)$ be a vector function in $(C_p^{\infty})^N$ satisfying div $\Psi = 0$. Then for v in $\mathcal{D}(\Omega)$ we have

$$\sum_{i=1}^{N}\int_{\Omega}\frac{\partial u_{\varepsilon}}{\partial x_{i}}\Psi_{i}^{\varepsilon}v\,dx=-\sum_{i=1}^{N}\int_{\Omega}u_{\varepsilon}\frac{\partial}{\partial x_{i}}(\Psi_{i}^{\varepsilon}v)\,dx.$$

By Leibniz's formula and the fact that div $\Psi = 0$ (note that $\sum_{i=1}^{N} \partial \Psi_{i}^{\varepsilon} / \partial x_{i} = 1/\varepsilon (\operatorname{div} \Psi)^{\varepsilon}$) it follows that

$$\sum_{i=1}^{N}\int_{\Omega}\frac{\partial u_{\varepsilon}}{\partial x_{i}}\Psi_{i}^{\varepsilon}v\,dx=-\sum_{i=1}^{N}\int_{\Omega}u_{\varepsilon}\Psi_{i}^{\varepsilon}\frac{\partial v}{\partial x_{i}}\,dx.$$

By the Rellich theorem we may assume that the above subsequence ε satisfies the further property

$$u_{\varepsilon} \rightarrow u$$
 in $L^{2}(\Omega)$ -strong,

so that, letting $\varepsilon \downarrow 0$ and recalling (5.4), we obtain

$$\sum_{i=1}^{N} \int_{\Omega \times Y} \left[z_i(x, y) - \frac{\partial u}{\partial x_i}(x) \right] \Psi_i(y) v(x) \, dx \, dy = 0$$

for all $\Psi \in (C_p^{\infty})^N$, div $\Psi = 0$, and all $v \in \mathcal{D}(\Omega)$. Hence we have for almost all $x \in \Omega$

$$\sum_{i=1}^{N} \int_{Y} \left[z_i(x, y) - \frac{\partial u}{\partial x_i}(x) \right] \Psi_i(y) \, dy = 0 \quad \text{for } \Psi \in (C_p^{\infty})^N, \text{ div } \Psi = 0$$

It follows by Lemma 4 that there exists a function u_1 from Ω to H_p^1/\mathbb{R} such that

(5.5)
$$z_i(x, \cdot) - \frac{\partial u}{\partial x_i}(x) = \frac{\partial u_1(x)}{\partial y_i}$$
 a.e. in $\Omega(i = 1, \cdots, N)$.

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Finally, from (5.5) we can easily show (e.g., by Lusin's characterization [3]) that u_1 is a measurable function from Ω to H^1_p/\mathbb{R} (obtained from appropriate norm defined in § 5.1). Furthermore, again by (5.5) we have

$$\Big|_{\Omega} \|u_1(x)\|_{H^1(Y)/\mathbb{R}}^2 dx < \infty$$

and the conclusion follows.

6. A new approach in the theory of homogenization. Classically, the mathematical analysis of homogenization problems proceeds in two steps [1]. The first step, which is formal, derives, for example, from two-scale asymptotic expansions of the form

$$u_{\varepsilon}(x) = u_0(x, y) + \varepsilon u_1(x, y) + \cdots, \quad y = \frac{x}{\varepsilon},$$

 u_0, u_1, \cdots, Y -periodic in y.

(6.1)

More precisely, we postulate that the solution
$$u_{\varepsilon}$$
 of a given problem (associated with a partial differential equation with coefficients ε -periodic) is similar to (6.1). Next, introducing (6.1) into the given problem yields a sequence of problems that determine u_0, u_1, \cdots .

The second step consists of rigorously proving the convergence of the preceding homogenization process, i.e., we must find some suitable topology in order that $\lim u_{\varepsilon} = u_0$ as $\varepsilon \to 0$. This validates the above formal calculations.

In this last section we propose an alternative approach. More precisely, we introduce a new asymptotic method for the mathematical analysis of homogenization problems. The method is quite straightforward. There is no need to postulate the existence of the functions u_0, u_1 in (6.1), since by Theorems 2 (or 1) and 3 such functions are available for a suitable subsequence from ε .

Our approach is illustrated by a regular homogenization problem. Nevertheless, the basic ideas can easily be extended to problems of the singular type.

6.1. Setting of the problem. In all that follows, unless otherwise specified, the summation convention is used.

Let Ω be a smooth bounded open set in \mathbb{R}^N (the space of the variables x_1, \dots, x_N) with boundary $\partial \Omega$. Let a_{ij} $(1 \le i, j \le N)$ be given functions defined on \mathbb{R}^N (the space of the variables y_1, \dots, y_N) and subject to the following conditions:

(6.2)
$$a_{ij} \in L^{\infty}, \quad a_{ij} \text{ Y-periodic}, \quad a_{ij} = a_{ji}.$$

There exists $\alpha > 0$ such that the following holds for almost all y:

(6.3)
$$a_{ij}(y)\xi_i\xi_j \ge \alpha |\xi|^2 \quad \forall \xi = (\xi_i) \in \mathbb{R}^N$$

(where the summation convention is utilized) with $|\xi|^2 = \sum_{i=1}^{N} \xi_i^2$.

Finally, let $f \in L^2(\Omega)$, and for each $\varepsilon > 0$ let u_{ε} be defined by

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(6.4)
$$u_{\varepsilon} \in H^{*}(\Omega),$$
$$-\frac{\partial}{\partial x_{i}} \left(a_{ij}^{\varepsilon} \frac{\partial u_{\varepsilon}}{\partial x_{j}} \right) = f \text{ in } \Omega,$$
$$u_{\varepsilon} = 0 \text{ on } \partial \Omega$$

where $a_{ij}^{\varepsilon}(x) = a_{ij}(x/\varepsilon)$ (see (3.6)).

Clearly, from (6.2) (the first assumption) and (6.3), we see (6.4) uniquely determines u_{e} .

Our aim is to find lim u_{ε} as $\varepsilon \downarrow 0$. In other words, we must study the homogenization problem associated with (6.4). Note that this problem has been solved in [1], where the results of the formal analysis were made rigorous by applying the Energy Method. As mentioned in § 1, we propose an alternative approach that should be more flexible and thus more adaptable for the study of unusual problems.

6.2. Description of the method. First, observe that u_{ε} , the solution of (6.4), satisfies

(6.5)
$$u_{\varepsilon} \in H_{0}^{1}(\Omega),$$
$$\int_{\Omega} a_{ij}^{\varepsilon} \frac{\partial u_{\varepsilon}}{\partial x_{j}} \frac{\partial v}{\partial x_{i}} dx = \int_{\Omega} f v \, dx \quad \forall v \in H_{0}^{1}(\Omega).$$

Next we estimate $||u_{\varepsilon}||_{H_0^1(\Omega)}$. Taking the particular test function $v = u_{\varepsilon}$ and using the boundedness and the coerciveness (from (6.3)) of the bilinear form in (6.5), we obtain

$$\|u_{\varepsilon}\|_{H^{1}(\Omega)} \leq c \ (c > 0) \quad \forall \varepsilon.$$

Hence, the hypotheses of Theorem 3 are fulfilled. We can extract a subsequence still denoted by ε for simplicity such that

(6.6)
$$u_{\varepsilon} \rightarrow u \text{ in } H_0^1(\Omega) \text{-weak as } \varepsilon \downarrow 0$$

and, for all $\Psi \in L_p^2$, $v \in \mathcal{K}(\overline{\Omega})$,

(6.7)
$$\int_{\Omega} \frac{\partial u_{\varepsilon}}{\partial x_{j}} \Psi^{\varepsilon} v \, dx \to \int_{\Omega \times Y} \left[\frac{\partial u}{\partial x_{j}} (x) + \frac{\partial u_{1}}{\partial y_{j}} (x, y) \right] \Psi(y) v(x) \, dx \, dy,$$
$$i = 1, \cdots, N,$$

where $u_1 \in L^2(\Omega; H_p^1/\mathbb{R})$.

Derivation of the local problem. In (6.5) we take test functions of the form $v = \varepsilon w^{\varepsilon} \phi$ with $w \in H_p^1$, $\phi \in \mathcal{D}(\Omega)$. Then, noting that $\partial w^{\varepsilon} / \partial x_i = 1/\varepsilon (\partial w / \partial y_i)^{\varepsilon}$, where of course $(\partial w / \partial y_i)^{\varepsilon}(x) = \partial w / \partial y_i(x/\varepsilon)$, we are led to

$$\int_{\Omega} a_{ij}^{\varepsilon} \frac{\partial u_{\varepsilon}}{\partial x_{j}} \left(\frac{\partial w}{\partial y_{i}} \right)^{\varepsilon} \phi \, dx + \varepsilon \int_{\Omega} a_{ij}^{\varepsilon} \frac{\partial u_{\varepsilon}}{\partial x_{j}} \, w^{\varepsilon} \frac{\partial \phi}{\partial x_{i}} \, dx = \varepsilon \int_{\Omega} f w^{\varepsilon} \phi \, dx$$

Now we propose passing to the limit as $\varepsilon \downarrow 0$. It is easy to check that both the second term on the left and the term on the right tend to zero. Hence,

$$\int_{\Omega} a_{ij}^{\varepsilon} \frac{\partial u_{\varepsilon}}{\partial x_j} \left(\frac{\partial w}{\partial y_i} \right)^{\varepsilon} \phi \, dx \to 0.$$

On the other hand, choose in (6.7) $\Psi = a_{ij}(\partial w/\partial y_i)$ (summation) with $w \in H_p^1$. By the above result we are finally led to

$$\int_{\Omega \times Y} a_{ij}(y) \left[\frac{\partial u}{\partial x_j}(x) + \frac{\partial u_1}{\partial y_j}(x, y) \right] \frac{\partial w}{\partial y_i}(y) \phi(x) \, dx \, dy = 0$$

for all $w \in H_p^1$ and all $\phi \in \mathcal{D}(\Omega)$. Hence the following holds for almost every x in Ω :

(6.8)
$$\int_{Y} a_{ij}(y) \left[\frac{\partial u}{\partial x_j}(x) + \frac{\partial u_1}{\partial y_j}(x, y) \right] \frac{\partial w}{\partial y_i}(y) \, dy = 0 \quad \forall w \in H_p^1.$$

Equation (6.8) is exactly that obtained by the formal method using multiple-scale asymptotic expansions (see [1]). It associates with the relation $u_1(x, \cdot) \in H_p^1/\mathbb{R}$ to give the so-called local problem, which permits us to express u_1 in terms of u. Evidently u_1 satisfies, for fixed x,

(6.9)
$$u_{1}(x, \cdot) \in \frac{H_{p}^{1}}{\mathbb{R}},$$
$$\int_{Y} a_{ij} \frac{\partial u_{1}}{\partial y_{j}}(x, \cdot) \frac{\partial w}{\partial y_{i}} dy = -\frac{\partial u}{\partial x_{j}}(x) \int_{Y} a_{ij} \frac{\partial w}{\partial y_{i}} dy \quad \forall w \in \frac{H_{p}^{1}}{\mathbb{R}}$$

which is an elliptic variational problem for $u_1(x, \cdot)$, admitting one and only one solution.

We should stress that, contrary to the classical method, the resolution of the above problem does not concern us since u_1 has been constructed in Theorem 3. We only observe that u_1 is unique (i.e., independent of the subsequence extracted above) as soon as u is well determined. Now we calculate u_1 in terms of u. Following [1], let χ^j $(j = 1, \dots, N)$ be defined by

$$\chi^j \in rac{H_p^1}{\mathbb{R}},$$

$$\int_{Y} a_{kh} \frac{\partial \chi^{j}}{\partial y_{h}} \frac{\partial w}{\partial y_{k}} \, dy = \int_{Y} a_{kj} \frac{\partial w}{\partial y_{k}} \, dy \quad \forall w \in \frac{H_{p}^{1}}{\mathbb{R}}.$$

Then, from the preceding remarks, we see that u_1 is given by

(6.11)
$$u_1(x, y) = -\frac{\partial u}{\partial x_j}(x)\chi^j(y).$$

Indeed, the function on the right-hand side is the solution of (6.9).

Remark 6. It is not difficult to verify that the equation in (6.10) can be written under the form

(6.12)
$$a(\chi^j - y_i, w) = 0 \quad \forall w \in H^1_p / \mathbb{R}$$

where $a(\cdot, \cdot)$ is the bilinear form that figures in (6.10), and $y_j(1 \le j \le N)$ are the coordinate functions. \Box

Derivation of the global (or limit) problem. The point now is to derive the boundary value problem satisfied by the global limit u. This is straightforward. Choose in (6.7) the particular function $\Psi = a_{ij}$, and in place of the v's consider the derivatives $\partial v/\partial x_i$, with $v \in \mathcal{D}(\Omega)$. Hence, summing over i, j on both sides of (6.7) and using (6.5) yields

$$\int_{\Omega \times Y} a_{ij}(y) \left[\frac{\partial u}{\partial x_j}(x) + \frac{\partial u_1}{\partial y_j}(x, y) \right] \frac{\partial v}{\partial x_i}(x) \, dx \, dy = \int_{\Omega} f v \, dx,$$

which by (6.11) becomes

$$\int_{\Omega} \left(\tilde{a}_{ij} - \int_{Y} a_{ih} \frac{\partial \chi^{j}}{\partial y_{h}} dy \right) \frac{\partial u}{\partial x_{j}} \frac{\partial v}{\partial x_{i}} dx = \int_{\Omega} f v \, dx$$

where $\tilde{a}_{ij} = \int_{Y} a_{ij}(y) dy$. But it is easy to check that

$$\tilde{a}_{ij} - \int_{Y} a_{ih} \frac{\partial \chi^{j}}{\partial y_{h}} dy = -a(y_{i}, \chi^{j} - y_{j})$$

On the other hand, by (6.12) we have easily that $-a(y_i, \chi^j - y_j) = a(\chi^i - y_i, \chi^j - y_j)$ (note that the form $a(\cdot, \cdot)$ is symmetric). From all that we deduce the problem for u:

(6.13)
$$u \in H_0^1(\Omega),$$
$$\int_{\Omega} q_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx = \int_{\Omega} f v \, dx \quad \forall v \in H_0^1(\Omega)$$

where

(6.14)
$$q_{ij} = a(\chi^i - y_i, \chi^j - y_j).$$

The constants q_{ij} are the so-called homogenized coefficients. They satisfy the ellipticity condition

$$q_{ij}\xi_i\xi_j \ge c|\xi|^2(c>0) \quad \forall \xi \in \mathbb{R}^N \quad (\text{see } [1])$$

(6.10)

so that u is uniquely determined by (6.13). Consequently, the subsequence ε in (6.6) and (6.7) may be replaced by the whole sequence from which it was extracted.

Thus, we have proved the following homogenization theorem.

THEOREM 4. For each $\varepsilon > 0$ let u_{ε} be the solution of the boundary value problem (6.4). Then, as $\varepsilon \downarrow 0$,

(6.15)
$$u_{\varepsilon} \to u \quad in \ H^{1}_{0}(\Omega) - weak,$$
$$\int_{\Omega} \frac{\partial u_{\varepsilon}}{\partial x_{j}} \Psi^{\varepsilon} v \ dx \to \left(\int_{\Omega} \frac{\partial u}{\partial x_{k}} v \ dx \right) \int_{Y} \Psi \frac{\partial}{\partial y_{j}} (y_{k} - \chi^{k}) \ dy,$$

 $j = 1, \dots, N$, for all $\Psi \in L_p^2$ and all $v \in \mathcal{K}(\overline{\Omega})$, where u is the solution of the boundary value problem

$$-\frac{\partial}{\partial x_i} \left(q_{ij} \frac{\partial u}{\partial x_j} \right) = f \quad in \ \Omega$$
$$u = 0 \quad on \ \partial\Omega,$$

 q_{ii} given by (6.14), and $\chi^{k}(k=1, \dots, N)$ given by (6.10).

Remark 7. The usual homogenization theorem [1], [9] does not involve (6.15). This property obviously follows from (6.7) and (6.11).

6.3. Concluding remarks. We have just proposed a new asymptotic method for the mathematical analysis of homogenization problems. The method is straightforward and quite natural. Note that, although we have chosen a problem of the regular type for a detailed analysis, our approach is essentially based on Theorem 1, which requires only the weaker assumption that u_{ε} remains bounded in L^2 . So it is reasonable to assume the above idea can be successfully extended to a more general situation involving problems of the singular type.

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REFERENCES

- [1] A. BENSOUSSAN, J. L. LIONS, AND G. PAPANICOLAOU, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.
- [2] N. BOURBAKI, Eléments de Mathématiques, Vol. XIII, Intégration, Hermann, Paris, 1965.
- [3] R. E. EDWARDS, Functional Analysis, Holt, Rinehart and Winston, New York, 1965.
- [4] F. FLEURY, Propagation des ondes dans une suspension de particules solides, Compt. Rend. Acad. Sci. Paris, sér. A, 288 (1979), pp. 77-80.
- [5] T. LEVY, Propagation of waves in a fluid-saturated porous elastic solid, Internat. J. Engrg. Sci., 17 (1979), pp. 1005-1014.
- [6] G. NGUETSENG, Sur quelques problèmes de perturbations dans des ouverts périodiques et applications à la mécanique des composites, Thèse d'Etat, Université Paris 6, 1984.
- [7] E. SANCHEZ-PALENCIA, Non-Homogeneous Media and Vibration Theory, Springer-Verlag, Berlin, Heidelberg, New York, 1980.
- [8] L. SCHWARTZ, Théories des distributions, Hermann, Paris, 1966.
- [9] L. TARTAR, Problèmes d'homogenization dans les équations aux dérivées partielles, Cours Peccot (rédigé par F. Murat), Collège de France, 1977.
- [10] R. TEMAM, Navier-Stokes Equations, North-Holland, Amsterdam, 1977.