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GRADUATE SCHOOL

ON THE CONNECTION BETWEEN SOME GRAPH INVARIANTS AND THE SYNCHRONIZABILITY OF NETWORKS

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ABSTRACT

While there are numerous interesting and important problems in the synchronization theory of networks, we select two from the large problem set, namely, the connection between the diameter and synchronizability of networks, and finding optimal three-oscillator networks, to present in this thesis.

The first part of this thesis aims to connect the diameter of a network with synchronizability. We first derive analytic estimates on the Laplacian eigenvalues that characterizes synchronizability in terms of the diameter. Then we construct two classes of networks preserving the diameter, either of which represents an extreme case of synchronizability. Motivated by these results and related research on the degree sequence, we give the definition of a weak indicator of synchronizability, and conclude that the diameter and degree sequence are both weak indicators of synchronizability.

Finding the networks maximizing synchronizability is of great practical concern. However, finding optimal networks in the general case is really hard and only a narrow class of solutions has been found. In the second part of this thesis, we focus on finding optimal networks in a lower-dimensional case, namely, three-oscillator networks. This case is analytically accessible and we are able to describe all the solutions and design algorithms to produce many of them.

CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Graph Theory and the Study of Networks

The concept "network" has more than a few meanings in different contexts. In mathematics, it is a term in graph theory, which is closely related to the concept "network flow". In this context, a network refers to a directed graph with weighted edges. Usually, the graph is assumed to be simple (no loops or multiple edges) and connected, and the weights are assumed to be nonnegative integers. By defining the sources (vertices with indegree 0) and sinks (vertices with outdegree 0), this kind of graphs is especially useful for analyzing transportation networks by which commodities are shipped from their production centers to their markets [Bondy & Murty, 1976; Ahuja *et al.*, 1993].

However, in a general context, "network" is more often used as a synonym of "graph", emphasizing its applied aspects. Correspondingly, the terms "vertex" and "edge" are replaced by "node" and "link" (or "connection"), respectively. Thus, network theory is the counterpart of graph theory in applied areas, such as physics, biology, engineering, and sociology. The scope of network theory has been expanding very fast in recent years and more and more fields are involved in it. Thanks to various concerns and goals in real-world applications, network theory has many characteristics that distinguishes itself from classical graph theory. To start with, let us discuss these characteristics briefly.

Graph theory has a long history and one of the earliest results in graph theory appeared in Euler's celebrated paper on Seven Bridges of Königsberg, published in 1736. Since then, the subject has mostly been concerned with combinatorial and topological properties of graphs and has very little connection with probability, stochastic processes, dynamical systems, and other branches of modern mathematics. No changes to this situation occurred until Erdős and Rényi [1959, 1960] introduced the concept and method of random graphs into the subject. Also, starting from the 1980's, a significant effort has been devoted to relating properties of graphs to random walks and other diffusion processes [Lovász, 1996]. The efforts attempting to bridge graph theory and other branches of mathematics are also reflected in the study of networks. Two of the characteristics of network theory which are relevant to this thesis are:

(1) Networks are usually associated with information or energy transmissions and the states of nodes evolve over time. Typical examples in real-world networks include the Internet, World Wide Web, electric power grids, neural networks, food webs, and many others. Thus, dynamics on networks is essential to the study of networks. The synchronization theory of networks is based on assigning identical or nearly identical dynamical systems to individual nodes, which are coupled through links. How the topology of a network affects synchronization in it is an intriguing topic and has attracted much attention.

(2) Networks may consist of a huge number of nodes and links so that they are almost impossible to be studied by enumeration or other deterministic methods. In this case, probabilistic methods, especially the theory of random graphs founded by Erdős and Rényi, become an essential and powerful tool. In this thesis, we shall use random graphs to construct examples of networks with good synchronizability and preserving the diameter.

Although network theory already exists for a long time, it was the recent introduction of small-world networks and scale-free networks that greatly encouraged related research and brought it into a rapidly developing area [Watts & Strogatz, 1998; Barabási & Albert, 1999]. The emerging theory of complex networks concerns itself with the networks lying between order and randomness, including most networks in the real world; for surveys, see [Strogatz, 2001; Newman, 2003; Boccaletti *et al.*, 2006].

The model of small-world networks proposed by Watts and Strogatz was formed by rewiring a small fraction of links in a regular lattice. By adding such short cuts, the average distance in the network is remarkably reduced as in a random network, but the nodes are still locally clustered as in a regular lattice. A scale-free network is a network in which the node degree distribution satisfies a power law. This property is different from random networks, in which the degrees of nodes have a Poisson distribution, but is consistent with most of real-world networks such as the World Wide Web, in which some nodes are highly connected whereas most nodes have only a few links. The Barabási–Albert model achieves this property by growing a network with preferential attachments, namely, highly connected nodes receive more priority when connecting a new node to an existing one. These two types of complex networks have properties which do not appear in classical models in graph theory, and hence challenged the study of networks.

1.2 Synchronization in Networks

The concept of synchronization, like networks, also has many stories that can be told in various contexts. In particular, it is widely used in physics, engineering and biological sciences [Pikovsky *et al.*, 2001]. Without formulae, synchronization can be understood as the adjustment of states of oscillating objects toward consistency. One can cite examples of oscillating objects either in man-made systems, from pendulum clocks to musical instruments, electronic generators, and lasers, or in natural systems, from fireflies emitting light pulses to chirping crickets, birds flapping their wings, and beating human hearts. A general class of such phenomena can be modeled by a network of identical or nearly identical dynamical systems, which are coupled according to a particular topology; synchronization can then be defined to be that the difference between states of any two individual systems approaches zero as time goes to infinity.

Within this framework, many different models can be chosen for describing synchronization in networks, which differ from each other in the type of individual dynamical systems and/or the type of coupling. Specifically, the dynamical systems can be continuous, discrete, or impulsive, and the coupling can be linear or nonlinear. A widely studied model with continuous dynamical systems and linear coupling is described by the differential equations

$$\frac{dx^i(t)}{dt} = f(x^i(t)) + \sigma B \sum_{j=1}^{n} a_{ij} \left[x^j(t) - x^i(t)\right], \quad i = 1, 2, \ldots, n, \tag{1.1}$$

where $x^i(t) = (x_1^i(t), x_2^i(t), \ldots, x_m^i(t))^{\mathrm{T}} \in \mathbb{R}^m$ is the state of the ith system, $\sigma > 0$ is the coupling strength, $B \in \mathbb{R}^{m \times m}$ is the configuration matrix, depending on which a subset of components is coupled, and a_{ij} are the entries of the adjacency matrix A, defined by

$$a_{ij} = \begin{cases} 1, & \text{if } i \neq j \text{ and nodes } i \text{ and } j \text{ are connected,} \\ 0, & \text{otherwise.} \end{cases}$$

The initial conditions are $x^i(0) = x_0^i \in \mathbb{R}^m$ for $i = 1, 2, \ldots, n$.

Note that (1.1) can be rewritten in the form

$$\frac{dx^i(t)}{dt} = f(x^i(t)) - \sigma B \left[\left(\sum_{j=1,\, j \neq i}^{n} a_{ij} \right) x^i(t) - \sum_{j=1,\, j \neq i}^{n} a_{ij} x^j(t) \right], \quad i = 1, 2, \ldots, n. \tag{1.2}$$

Let $d_i = \sum_{j=1,\, j \neq i}^{n} a_{ij}$ be the degree of node i, $\Delta = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, and l_{ij} be the entries of the Laplacian matrix $L = \Delta - A$, i.e.

$$l_{ij} = \begin{cases} d_i, & \text{if } i = j, \\ -1, & \text{if nodes } i \text{ and } j \text{ are connected,} \\ 0, & \text{otherwise.} \end{cases}$$

Then (1.2) becomes

$$\frac{dx^i(t)}{dt} = f(x^i(t)) - \sigma B \sum_{j=1}^{n} l_{ij} x^j(t), \quad i = 1, 2, \ldots, n, \tag{1.3}$$

and we have the following definition.

Definition 1.1. *System (1.3) is said to synchronize if, starting from an open set of initial conditions,* $\lim_{t\to\infty} \|x^i(t) - x^j(t)\| = 0$ *for all* $i \neq j$.

The Laplacian matrix of a graph also arises in many other applications [Chung, 1997], one of which that has received much attention is finding the fastest mixing Markov process on a graph [Boyd *et al.*, 2004; Sun *et al.*, 2006]. In most cases, the eigenvalues of the Laplacian matrix turn out to be essential. Thus, it is not too surprising to see that the synchronization of system (1.3) depends on the eigenvalues of L.

Analysis of system (1.3) is based on the decomposition of each state into a component on the synchronization manifold and a component in the transverse subspace, namely, $x^i(t) = s(t) + z^i(t)$, where $s(t)$ is assumed to be on the synchronization manifold [Pecora & Carroll, 1998; Lu & Chen, 2006; Stilwell *et al.*, 2006]. Linearizing (1.3) about $s(t)$ gives

$$\frac{dz^i(t)}{dt} = F(t)z^i(t) - \sigma B \sum_{j=1}^{n} l_{ij} z^j(t), \quad i = 1, 2, \ldots, n, \tag{1.4}$$

where $F(t) = Df(s(t))$ is the Jacobian of f at $s(t)$.

To keep the notation concise, we turn to the Kronecker product. Let $z(t) = [z^1(t)^{\mathrm{T}}, z^2(t)^{\mathrm{T}}, \ldots, z^n(t)^{\mathrm{T}}]^{\mathrm{T}}$. Then (1.4) can be rewritten as

$$\frac{dz(t)}{dt} = (I_n \otimes F(t) - \sigma L \otimes B)\, z(t), \tag{1.5}$$

where $\otimes$ is the Kronecker product. Also, it is helpful to recall the following properties of the Kronecker product:

(1) For conformable matrices A, B, C, and D, $(A \otimes B)(C \otimes D) = AC \otimes BD$.

(2) For invertible matrices A and B, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Now let $L = PJP^{-1}$ be the Jordan decomposition of L (an alternate method is to use the Schur factorization $L = QTQ^{\mathrm{T}}$, where Q is unitary and T is upper-triangular). Substituting $\xi(t) = (P \otimes I_m)^{-1} z(t)$ in (1.5) yields

$$\begin{aligned} \frac{d\xi(t)}{dt} &= (P \otimes I_m)^{-1} (I_n \otimes F(t) - \sigma L \otimes B) (P \otimes I_m) \xi(t) \\ &= \left(I_n \otimes F(t) - \sigma P^{-1} L P \otimes B\right) \xi(t) \\ &= (I_n \otimes F(t) - \sigma J \otimes B) \xi(t). \end{aligned} \tag{1.6}$$

Noticing the special structure of J, it is easily seen that the stability of system (1.6) is equivalent to the stability of the uncoupled subsystems

$$\frac{d\xi^i(t)}{dt} = (F(t) - \sigma \lambda_i B) \xi^i(t), \quad i = 1, 2, \ldots, n, \tag{1.7}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of L.

Since all the row sums of L are zero, L has an eigenvalue 0 and corresponding eigenvector $\left[\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right]^{\mathrm{T}}$. Since L is symmetric, all its eigenvalues are real. Moreover, by Gerschgorin's theorem, all the eigenvalues are nonnegative. Without loss of generality, we order them as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then the first subsystem in (1.7)

$$\frac{d\xi^1(t)}{dt} = F(t)\xi^1(t)$$

evolves along the synchronization manifold, while the other $n-1$ subsystems evolve in the transverse subspace. By computing the Lyapunov exponents, the stability conditions of the $n-1$ subsystems can be given by $0 < \alpha_1 < \sigma\lambda_i < \alpha_2$, $i = 2, 3, \ldots, n$, or $\alpha_1/\lambda_2 < \sigma < \alpha_2/\lambda_n$, where α_1 and α_2 are given by the master stability function. To guarantee the existence of such σ, we need $\alpha_1/\lambda_2 < \alpha_2/\lambda_n$, or $\lambda_2/\lambda_n > \alpha_1/\alpha_2$. It follows that the synchronizability of system (1.3) can be characterized by the ratio λ_2/λ_n, since a larger λ_2/λ_n enables more systems to synchronize.

Two remarks are in order:

(1) Most part of the analysis above can be generalized to the case of directed weighted networks. Let $w_{ij} \geq 0$ be the weight of the link from node i to j, and $d_i = \sum_{j=1, j\neq i}^{n} w_{ij}$ be the indegree of node i. Then the Laplacian matrix L can be

defined by

$$l_{ij} = \begin{cases} d_i, & \text{if } i = j, \\ -w_{ij}, & \text{if node } i \text{ is connected to node } j, \\ 0, & \text{otherwise.} \end{cases}$$

Note that since the row sums of L are still zero, L also has eigenvalue 0, but the other eigenvalues need not be real. In this case, we can order the eigenvalues such that $0 = \lambda_1 \leq \mathrm{Re}\,\lambda_2 \leq \cdots \leq \mathrm{Re}\,\lambda_n$. By the same arguments, the synchronization of system (1.3) is also equivalent to the stability of the last $n-1$ subsystems in (1.7).

(2) It is possible that λ_2 instead of λ_2/λ_n affects the synchronization when a specific model is analyzed. It is coincident that λ_2 also plays a critical role in finding the fastest mixing Markov process on a graph. In the literature, λ_2 is often called the algebraic connectivity of a graph, or the spectral gap of a graph.

1.3 Outline of the Thesis

Although there are numerous interesting and important problems in the synchronization theory of networks, we only select two from the large problem set, namely, the connection between the diameter and synchronizability of networks, and finding optimal three-oscillator networks, to present in this thesis.

From the previous section, we have already known that the synchronizability of networks can be characterized by the ratio λ_2/λ_n. However, it is not always easy to compute the value of this ratio, especially for large networks. Moreover, the eigenvalues of the Laplacian matrix do not provide explicit information on the network topology. On the other hand, many other graph invariants, such as the degree sequence, average distance, and diameter, often arise in constraints of network topology. Hence, it is important and useful to explore the connection between such graph invariants and the synchronizability of networks. This can be done by relating the graph invariants with λ_2/λ_n.

The first part of this thesis will be devoted to connecting the diameter of a network with synchronizability. We first derive analytic estimates on λ_2 and λ_2/λ_n

in terms of the diameter, and then construct two classes of networks preserving the diameter, either of which represents an extreme case of synchronizability. Motivated by these results and related research on the degree sequence, we give the definition of a weak indicator of synchronizability, and conclude that the diameter and degree sequence are both weak indicators of synchronizability. This leads to the question: Is there a graph invariant which is not a weak indicator of synchronizability? The answer is probably yes, but it will be very hard to find a simple one.

Finding the networks maximizing synchronizability is of great practical concern. For directed weighted networks, it has been proved that the synchronizability is maximized if all eigenvalues are real and $0 = \lambda_1 < \lambda_2 = \cdots = \lambda_n$, provided that the master stability function has a convex stability region [Nishikawa & Motter, 2006]. Note that finding optimal networks in the general case is really hard and only a narrow class of solutions has been found. In the second part of this thesis, we shall focus on finding optimal networks in a lower-dimensional case, namely, three-oscillator networks. This case is analytically accessible and we are able to describe all the solutions and design algorithms to produce many of them. At the time of writing this thesis, I am referred to a series of papers [Boyd *et al.*, 2004; Sun *et al.*, 2006; Boyd, 2006], which developed a convex optimization method for maximizing or minimizing some function of the Laplacian eigenvalues of an undirected weighted graph, subject to some constraints on the weights. Hence, the problem of finding optimal networks can be effectively solved numerically.

CHAPTER 2

DIAMETER AND SYNCHRONIZABILITY OF NETWORKS

2.1 Analytic Estimation of Synchronizability

The diameter, defined as the maximum distance between two nodes, is one of the most important graph invariants. A small diameter is one of the characteristics of random graphs and small-world networks. The question we are concerned with here is: How good and how bad can the synchronizability be for a network with specified diameter? To answer this question, the first step could be to derive analytic estimates on λ_2 and λ_2/λ_n in terms of diameter, i.e., give a possible range of synchronizability for such networks.

The estimates are based on several inequalities established in spectral graph theory. The first lemma gives a lower bound on the diameter in terms of λ_2 [Mohar, 1991, Theorem 4.2].

Lemma 2.1. *For a graph of order n, λ_2 imposes a lower bound on the diameter D,*

$$D \geq \frac{4}{n\lambda_2}. \tag{2.1}$$

An upper bound on λ_2/λ_n is provided by the following lemma.

Lemma 2.2. *For a graph of order n, the diameter D imposes an upper bound on*

λ_2/λ_n,

$$\frac{\lambda_2}{\lambda_n} \leq \frac{\cosh\left(\frac{\cosh^{-1}(n-1)}{D-1}\right) - 1}{\cosh\left(\frac{\cosh^{-1}(n-1)}{D-1}\right) + 1}. \tag{2.2}$$

Proof. Chung *et al.* [1994] proved an upper bound on the diameter D in terms of λ_2 and λ_n,

$$D \leq \left\lfloor \frac{\cosh^{-1}(n-1)}{\cosh^{-1}\left(\frac{\lambda_n+\lambda_2}{\lambda_n-\lambda_2}\right)} \right\rfloor + 1, \tag{2.3}$$

which implies that

$$D - 1 \leq \frac{\cosh^{-1}(n-1)}{\cosh^{-1}\left(\frac{1+\lambda_2/\lambda_n}{1-\lambda_2/\lambda_n}\right)},$$

or

$$\cosh^{-1}\left(\frac{1+\lambda_2/\lambda_n}{1-\lambda_2/\lambda_n}\right) \leq \frac{\cosh^{-1}(n-1)}{D-1}.$$

Noticing that the hyperbolic cosine function is increasing on $[0, \infty)$, we have

$$\frac{1+\lambda_2/\lambda_n}{1-\lambda_2/\lambda_n} \leq \cosh\left(\frac{\cosh^{-1}(n-1)}{D-1}\right).$$

Solving this inequality for λ_2/λ_n yields the desired bound. □

Remark. By (2.3), if

$$\frac{\cosh^{-1}(n-1)}{\cosh^{-1}\left(\frac{1+\lambda_2/\lambda_n}{1-\lambda_2/\lambda_n}\right)} < 1, \tag{2.4}$$

then $D = 1$, i.e., the graph is a complete graph. Note that (2.4) is equivalent to that

$$\frac{1+\lambda_2/\lambda_n}{1-\lambda_2/\lambda_n} > n - 1,$$

or

$$\frac{\lambda_2}{\lambda_n} > \frac{n-2}{n}.$$

In particular, a graph with the best possible synchronizability, namely, a graph with

$$0 = \lambda_1 < \lambda_2 = \cdots = \lambda_n \tag{2.5}$$

must be a complete graph. However, in the next chapter, we shall see that a directed weighted network satisfying (2.5) need not be complete.

The following lemma gives a simple upper bound on λ_n [Kel'mans, 1967].

Lemma 2.3. *If G is a simple graph, then*

$$\lambda_n \leq n, \tag{2.6}$$

with equality if and only if the complement of G is not connected.

Combining the above lemmas, we obtain the following result.

Theorem 2.4. *The diameter D imposes lower and upper bounds on λ_2 and λ_2/λ_n,*

$$\frac{4}{nD} \leq \lambda_2 \leq n \frac{\cosh\left(\frac{\cosh^{-1}(n-1)}{D-1}\right) - 1}{\cosh\left(\frac{\cosh^{-1}(n-1)}{D-1}\right) + 1},$$

$$\frac{4}{n^2 D} \leq \frac{\lambda_2}{\lambda_n} \leq \frac{\cosh\left(\frac{\cosh^{-1}(n-1)}{D-1}\right) - 1}{\cosh\left(\frac{\cosh^{-1}(n-1)}{D-1}\right) + 1}.$$

Proof. Lemma 2.1 and Lemma 2.2 give the lower bound on λ_2 and the upper bound on λ_2/λ_n, respectively. The upper bound on λ_2 follows from (2.2) and (2.6), and the lower bound on λ_2/λ_n follows from (2.1) and (2.6). □

To make the upper bound on λ_2/λ_n more readable, it is easily calculated that

$$\begin{aligned}
\frac{1+\lambda_2/\lambda_n}{1-\lambda_2/\lambda_n} &\leq \cosh\left(\frac{\cosh^{-1}(n-1)}{D-1}\right) \\
&= \cosh\left(\frac{\ln(n-1+\sqrt{n^2-2n})}{D-1}\right) \\
&= \frac{(n-1+\sqrt{n^2-2n})^{\frac{1}{D-1}} + (n-1-\sqrt{n^2-2n})^{\frac{1}{D-1}}}{2} \\
&\sim \frac{(2n)^{\frac{1}{D-1}}}{2}.
\end{aligned} \tag{2.7}$$

It is clear that the estimate on λ_2/λ_n is trivial when n is large, since the lower and upper bounds tend to 0 and 1, respectively, as $n \to \infty$. Also, the lower and upper bounds on λ_2 tend to 0 and ∞, respectively. However, in the next section, we shall show that these bounds are, in some sense, best possible. Hence, the synchronizability is really unpredictable when the network is large, if only the diameter is known.

On the other hand, when D increases, all the bounds in Theorem 2.4 decrease. This observation is consistent with the intuition that the diameter has a negative effect on synchronizability, if the other factors are comparable.

2.2 Construction of Two Classes of Networks

To show that the estimates obtained in previous section are, in some sense, best possible, we now construct two classes of networks, either of which represents an extreme case of synchronizability.

First, let us consider how bad the synchronizability of a network can be with a specified diameter. Our construction is based on the idea of dividing the nodes into two major groups, maximizing intra-group connections, and minimizing inter-group connections. As a standard notation, a complete graph of order n is denoted by K_n. The construction is as follows.

Construction 2.5 (Polarized Networks). Assume that the number of nodes n and the diameter D have been given. A polarized network $P(n, D)$ is formed by joining two complete graphs $K_{\lfloor \frac{n-D+3}{2} \rfloor}$ and $K_{\lceil \frac{n-D+3}{2} \rceil}$ by a path of length $D-2$. It is clearly seen that $P(n, D)$ has n nodes and diameter D. An example $P(26, 5)$ is shown in Fig. 2.1.

To derive estimates on λ_2 and λ_2/λ_n for the polarized network $P(n, D)$, we need the following lemmas, proved by Alon and Milman [1985] and Fiedler [1973], respectively.

Lemma 2.6. *Let A and B be two sets of vertices at distance ρ (the minimum distance between a vertex in A and a vertex in B), and F the set of edges with at*

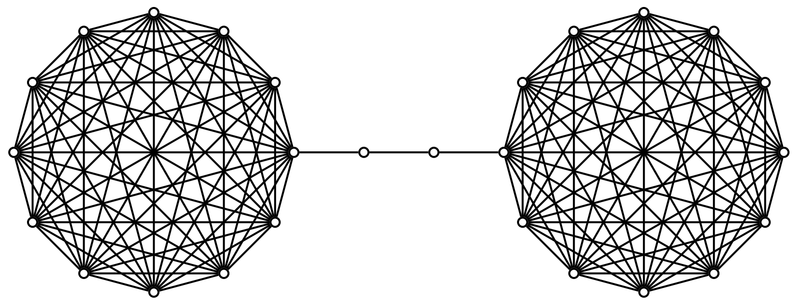

Figure 2.1. An example of polarized networks, $P(26, 5)$.

least one vertex not in A or B. The following inequality holds:

$$|F| \geq \rho^2 \lambda_2 \frac{|A||B|}{|A| + |B|},$$

where $|\cdot|$ denotes the number of vertices or edges in the set.

Lemma 2.7. *Let $d_{\max}$ be the maximum degree of vertices in a graph. $d_{\max}$ imposes a lower bound on λ_n,*

$$\lambda_n \geq \frac{n}{n-1} d_{\max}.$$

We have the following results regarding λ_2 and λ_2/λ_n of $P(n, D)$.

Theorem 2.8. *λ_2 and λ_2/λ_n for the polarized network $P(n, D)$ in Construction 2.5 satisfy*

$$\lambda_2 \leq \frac{4}{(D-2)(n-D+3)}, \tag{2.8}$$

$$\frac{\lambda_2}{\lambda_n} \leq \frac{8(n-1)}{n(D-2)(n-D+3)^2}. \tag{2.9}$$

Proof. Let A and B be the vertex sets of the two complete graphs $K_{\lfloor \frac{n-D+3}{2} \rfloor}$ and $K_{\lceil \frac{n-D+3}{2} \rceil}$ in Construction 2.5. Then we have $|A| = \lfloor \frac{n-D+3}{2} \rfloor$, $|B| = \lceil \frac{n-D+3}{2} \rceil$, and

$|F| = \rho = D - 2$. It follows from Lemma 2.6 that

$$\begin{aligned}\lambda_2 &\leq \frac{|F|(|A|+|B|)}{\rho^2|A||B|} = \frac{n-D+3}{(D-2)\left\lfloor\frac{n-D+3}{2}\right\rfloor\left\lceil\frac{n-D+3}{2}\right\rceil} \\ &= \begin{cases} \dfrac{4(n-D+3)}{(D-2)(n-D+2)(n-D+4)}, & \text{if } n-D \text{ even,} \\ \dfrac{4}{(D-2)(n-D+3)}, & \text{if } n-D \text{ odd} \end{cases} \\ &\leq \frac{4}{(D-2)(n-D+3)}.\end{aligned} \tag{2.10}$$

Also, note that the maximum degree $d_{\max} = \left\lceil\frac{n-D+3}{2}\right\rceil$. By Lemma 2.7, we have

$$\lambda_n \geq \frac{n}{n-1}\left\lceil\frac{n-D+3}{2}\right\rceil \geq \frac{n(n-D+3)}{2(n-1)}. \tag{2.11}$$

Combining (2.10) and (2.11) yields

$$\frac{\lambda_2}{\lambda_n} \leq \frac{8(n-1)}{n(D-2)(n-D+3)^2}. \qquad \square$$

Remark. Note that when $n \to \infty$, $D \to \infty$, and $D \ll n$, the upper bound on λ_2 in (2.8) is in the order of $4/(nD)$, the same as the lower bound on λ_2 in Theorem 2.4; also, the upper bound on λ_2/λ_n in (2.9) is in the order of $8/(n^2D)$, different from the lower bound on λ_2/λ_n in Theorem 2.4 only by a factor 2. Hence, Construction 2.5 shows that the lower bound estimates in Theorem 2.4 is, in some sense, best possible.

We now turn to our second construction, a class of networks that has the same diameter as the polarized networks in Construction 2.5 but has good synchronizability. Instead of constructing a determinate network explicitly, we adopt the random graph method and construct a class of random networks that preserves the diameter and demonstrates good synchronizability almost surely.

An Erdős–Rényi random graph $G(n,p)$ has n vertices and its edges are chosen independently with probability p. We have the following construction.

Construction 2.9 (Diameter-Preserved Random Networks). A diameter--preserved random network $H(n, D)$ is formed from a random graph $G(n, p)$ by letting

$$p = \alpha n^{\frac{1}{D-\varepsilon}-1}(\ln n)^{\beta}, \tag{2.12}$$

where $D \geq 2$, $0 < \varepsilon < 1$, $\alpha > 0$, and $\beta \geq 1 - \frac{1}{D-\varepsilon}$ are constants.

Remark. Note that since $\frac{1}{D-\varepsilon} - 1 < 0$, $p \to 0$ as $n \to \infty$. However, we have $np = \alpha n^{\frac{1}{D-\varepsilon}}(\ln n)^{\beta} \to \infty$ as $n \to \infty$.

To show that $H(n, D)$ really preserves the diameter and estimate its λ_2 and λ_2/λ_n, we need several lemmas regarding the properties of a random graph $G(n, p)$. The following lemma [Bollabás, 2001, Corollary 10.12] provides a general approach to construct random graphs with specified diameter. Here, "almost surely" means that the probability that a random graph has the given property tends to 1 as $n \to \infty$.

Lemma 2.10. *Assume that functions $D = D(n) \geq 3$ and $0 < p = p(n) < 1$ satisfy*

$$\frac{\ln n}{D} - 3 \ln \ln n \to \infty, \tag{2.13}$$

$$p^{D} n^{D-1} - 2 \ln n \to \infty, \tag{2.14}$$

$$p^{D-1} n^{D-2} - 2 \ln n \to -\infty. \tag{2.15}$$

Then $G(n, p)$ has diameter D almost surely.

The following lemma shows that $G(n, p)$ has a very narrow degree distribution [Krivelevich & Sudakov, 2005].

Lemma 2.11. *Assume that the function $p = p(n)$ satisfies*

$$\frac{pn}{\ln n} \to \infty, \tag{2.16}$$

$$(1 - p) n \ln n \to \infty. \tag{2.17}$$

Then all the degrees of $G(n, p)$ are equal to $(1 + o(1))np$ almost surely.

It is well known that the eigenvalues of the adjacency matrix A of $G(n,p)$ follow a semicircle law [Wigner, 1955, 1958]. Let $\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_n$ be the eigenvalues of A. By Lemma 2.11, it is easily seen that $\bar{\lambda}_1$ lies around np almost surely. The following lemma shows that all the other eigenvalues of A stay far from $\bar{\lambda}_1$ [Füredi & Komlós, 1981; Krivelevich & Sudakov, 2005].

Lemma 2.12. *Assume that the function $p = p(n)$ is greater than or equal to a polynomial of $\ln n/n$. Then*

$$\max_{2\leq i\leq n} |\bar{\lambda}_i| = O(\sqrt{np}).$$

Now we are ready to establish the following result regarding properties of $H(n,D)$ in Construction 2.9.

Theorem 2.13. *A diameter-preserved random network $H(n,D)$ in Construction 2.9 almost surely has diameter D, and has λ_2 and λ_2/λ_n with $\lambda_2 \to \infty$ and $\lambda_2/\lambda_n \to 1$ as $n \to \infty$.*

Proof. One can verify (2.14) and (2.15) by substituting (2.12) for p,

$$\begin{aligned} p^D n^{D-1} - 2\ln n &= \frac{(pn)^D}{n} - 2\ln n = \frac{\left[\alpha n^{\frac{1}{D-\varepsilon}} (\ln n)^{\beta}\right]^D}{n} - 2\ln n \\ &= \alpha^D n^{\frac{\varepsilon}{D-\varepsilon}} (\ln n)^{\beta D} - 2\ln n \to \infty, \\ p^{D-1} n^{D-2} - 2\ln n &= \frac{(pn)^{D-1}}{n} - 2\ln n = \frac{\left[\alpha n^{\frac{1}{D-\varepsilon}} (\ln n)^{\beta}\right]^{D-1}}{n} - 2\ln n \\ &= \alpha^{D-1} n^{\frac{\varepsilon-1}{D-\varepsilon}} (\ln n)^{\beta(D-1)} - 2\ln n \to -\infty. \end{aligned}$$

Also, (2.13) is satisfied since D is a constant. Thus, by Lemma 2.10, $H(n,D)$ has diameter D almost surely.

Now check (2.16) and (2.17),

$$\frac{pn}{\ln n} = \frac{\alpha n^{\frac{1}{D-\varepsilon}} (\ln n)^{\beta}}{\ln n} = \alpha n^{\frac{1}{D-\varepsilon}} (\ln n)^{\beta-1} \to \infty,$$

$$(1-p)n\ln n \sim n\ln n \to \infty.$$

By Lemma 2.11, all the degrees of $H(n, D)$ are equal to $(1 + o(1))np$ almost surely. Noticing the relationship between the Laplacian matrix L and the adjacency matrix A, $L = \Delta - A$, where Δ is the diagonal matrix whose diagonal entries are degrees, we see that $\lambda_i = (1 + o(1))np - \bar{\lambda}_i$ for all i almost surely.

Next, we note that

$$\begin{aligned} p = \alpha n^{\frac{1}{D-\varepsilon}-1}(\ln n)^{\beta} &= \alpha \left(\frac{\ln n}{n}\right)^{1-\frac{1}{D-\varepsilon}} (\ln n)^{\beta - \left(1-\frac{1}{D-\varepsilon}\right)} \\ &\geq \alpha \left(\frac{\ln n}{n}\right)^{1-\frac{1}{D-\varepsilon}} \geq \alpha \left(\frac{\ln n}{n}\right). \end{aligned}$$

Applying Lemma 2.12 yields $\max_{2\leq i\leq n} |\bar{\lambda}_i| = O(\sqrt{np})$. Thus, we obtain

$$\lambda_2 = (1 + o(1))np - O(\sqrt{np}) \sim np = \alpha n^{\frac{1}{D-\varepsilon}} (\ln n)^{\beta} \to \infty, \tag{2.18}$$

$$\frac{\lambda_2}{\lambda_n} = \frac{(1 + o(1))np - O(\sqrt{np})}{(1 + o(1))np + O(\sqrt{np})} \to 1. \qquad \square$$

Remark. We can compare the growth rate of λ_2/λ_n of $H(n, D)$ with the upper bound on λ_2/λ_n given in Theorem 2.4. Since both of them tend to 1 as $n \to \infty$, it makes more sense to compare $(1 + \lambda_2/\lambda_n)/(1 - \lambda_2/\lambda_n)$ with the upper bound given in (2.7). For $H(n, D)$, we have

$$\begin{aligned} \frac{1 + \lambda_2/\lambda_n}{1 - \lambda_2/\lambda_n} &= \frac{\lambda_n + \lambda_2}{\lambda_n - \lambda_2} \sim \frac{2np}{2O(\sqrt{np})} \geq \frac{np}{c\sqrt{np}} = \frac{\sqrt{np}}{c} \\ &= \frac{1}{c}\alpha^{\frac{1}{2}} n^{\frac{1}{2(D-\varepsilon)}} (\ln n)^{\frac{\beta}{2}}, \end{aligned} \tag{2.19}$$

where $c > 0$ is a constant. Therefore, the growth-rate ratio

$$\lim_{n\to\infty} \frac{\ln \left[\frac{1}{c}\alpha^{\frac{1}{2}} n^{\frac{1}{2(D-\varepsilon)}} (\ln n)^{\frac{\beta}{2}}\right]}{\ln \frac{(2n)^{\frac{1}{D-1}}}{2}} = \frac{\frac{1}{2(D-\varepsilon)}}{\frac{1}{D-1}} = \frac{D-1}{2(D-\varepsilon)},$$

which is close to 1/2.

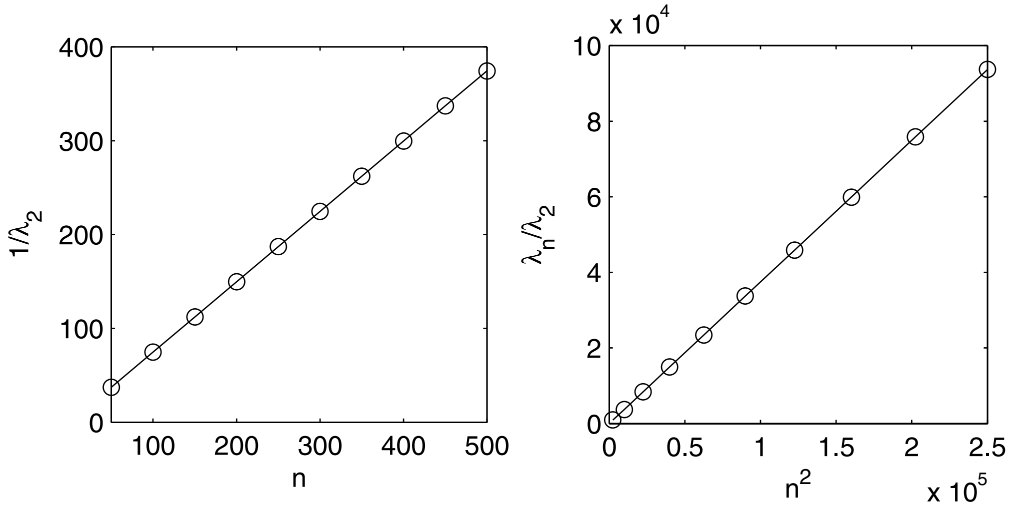

Figure 2.2. The effects of the network size n on λ_2 and λ_2/λ_n in the polarized network $P(n, D)$. The diameter D is fixed at 5.

2.3 Numerical Results

In this section, we design and carry out numerical experiments to verify the properties of the two classes of networks we have constructed.

Since Construction 2.5 is an explicit construction, the topology of a polarized network is determinate for each pair of values (n, D). To examine the effects of the two parameters on λ_2 and λ_2/λ_n, we fix one parameter and vary the other. The numerical results are shown in Figures 2.2 and 2.3. In the former, the diameter is fixed at 5, and it is seen that λ_2 and λ_2/λ_n are in inverse proportion to n and n^2, respectively. In the latter, the network size n is fixed at 200, and it is seen that λ_2 and λ_2/λ_n are both in inverse proportion to the diameter D. These results are in agreement with the upper bounds given in Theorem 2.8.

Next we turn to the diameter-preserved random networks in Construction 2.9. Since $H(n, D)$ is a random network with a distribution depending on the set of values $(n, D, \varepsilon, \alpha, \beta)$, we generate a sample of $H(n, D)$ and compute the sample means of λ_2 and λ_2/λ_n. Note that the values of ε, α, and β do not affect the

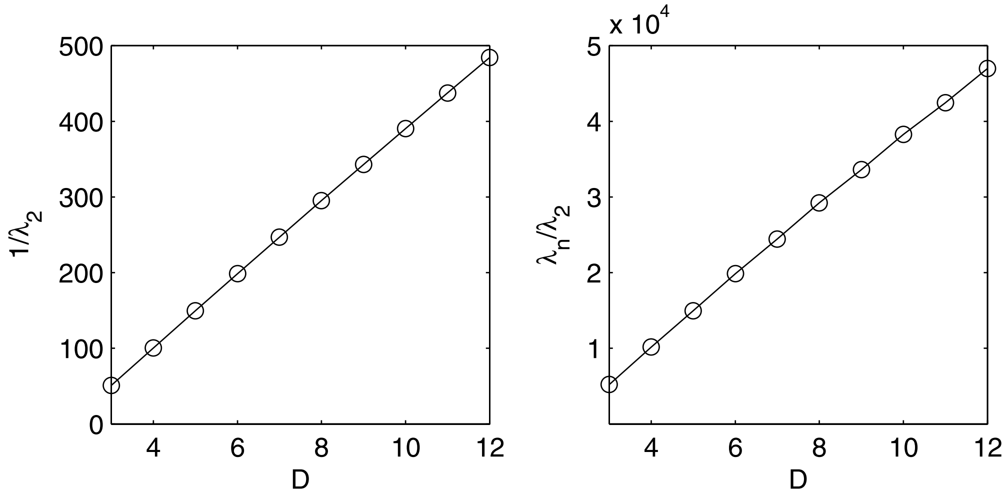

Figure 2.3. The effects of the diameter D on λ_2 and λ_2/λ_n in the polarized network $P(n, D)$. The network size n is fixed at 200.

convergence of the diameter to the specified value D, provided that they satisfy the assumptions in Construction 2.9. However, they do affect the convergence rate. To accelerate the convergence so that we can achieve the diameter D for smaller n's, it is important to choose the values of ε, α, and β appropriately. In our experiments, we fix $\varepsilon = 0.5$ and $\alpha = 0.8$, and let $\beta = 1 - \frac{1}{D-\varepsilon}$.

The numerical results for $D = 4$ are shown in Figure 2.4. For each value of n, we generate 10 random networks and compute the sample means of λ_2 and λ_2/λ_n, which are shown in the top two panels; the number of networks achieving the specified diameter D in each sample is shown in the bottom panel. It is clearly seen that the diameter converges quickly to D, and λ_2 and λ_2/λ_n grow slowly. These tendencies are in agreement with Theorem 2.13, although, due to high computational complexity, it is difficult to verify that λ_2 and λ_2/λ_n really tend to ∞ and 1, respectively.

To explore the effects of the diameter D on λ_2 and λ_2/λ_n in $H(n, D)$, we fix n at several values respectively, and vary D. For each pair of values (n, D), we generate

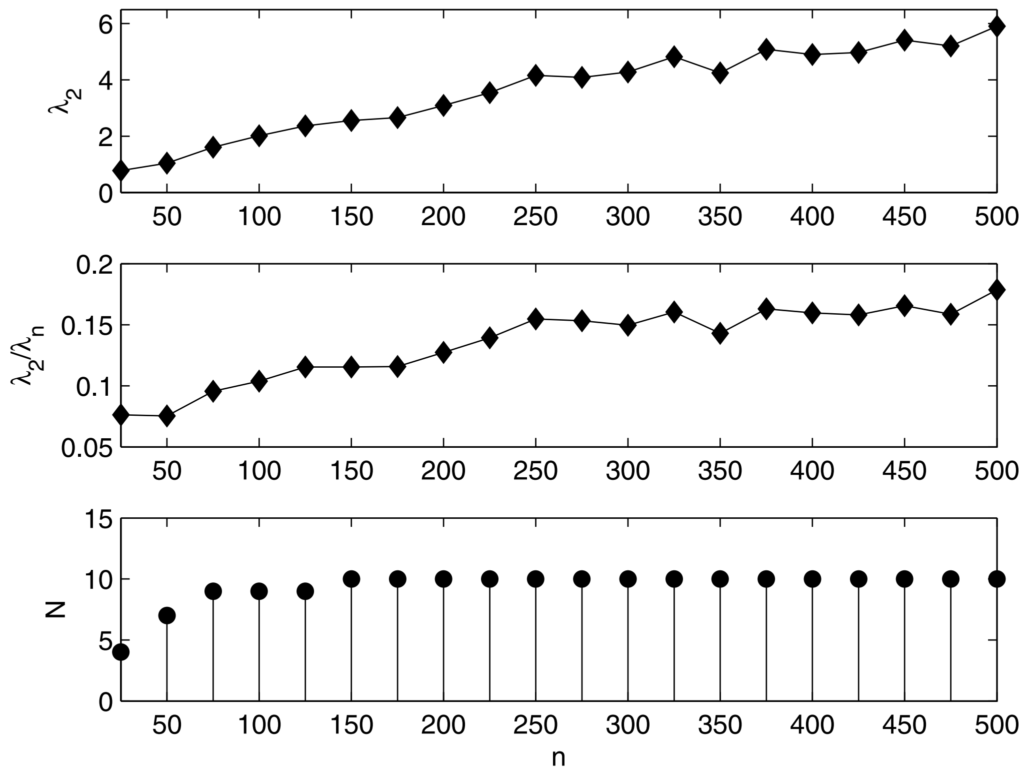

Figure 2.4. The effects of the network size n on λ_2 and λ_2/λ_n in the diameter-preserved random network $H(n, D)$. Here, $D = 4$, $\varepsilon = 0.5$, $\alpha = 0.8$, $\beta = 1 - 1/(4 - 0.5)$. For each n, 10 random networks are generated and the sample means of λ_2 and λ_2/λ_n are shown in the top two panels; the number of networks achieving the diameter D, denoted by N, is shown in the bottom panel.

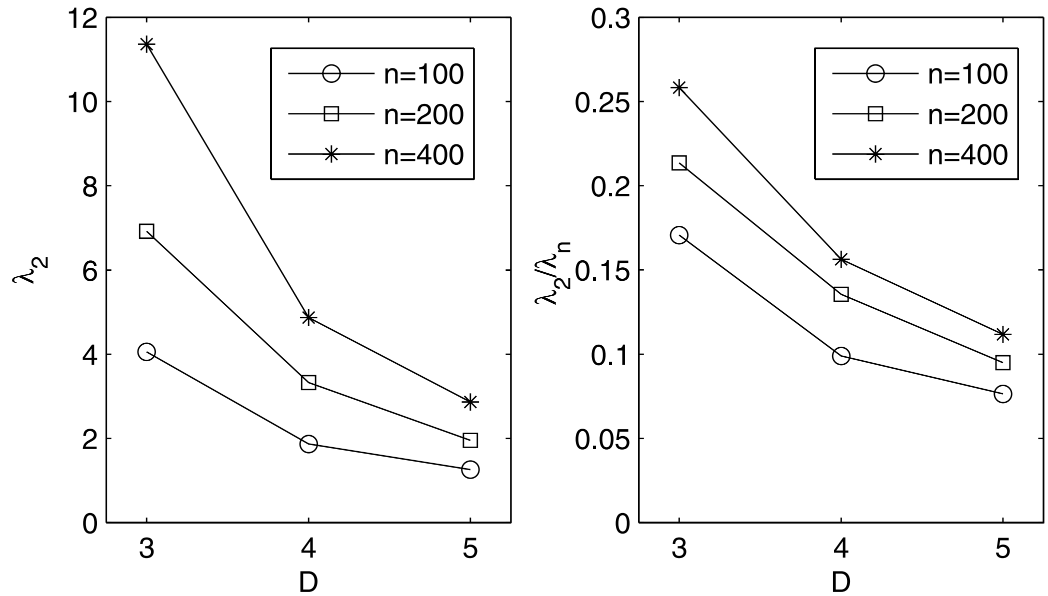

Figure 2.5. The effects of the diameter D on λ_2 and λ_2/λ_n in the diameter-preserved random network $H(n, D)$. Here, $\varepsilon = 0.5$, $\alpha = 0.8$, $\beta = 1 - 1/(D - 0.5)$. For each (n, D), 50 random networks are generated and sample means of λ_2 and λ_2/λ_n are computed.

50 random networks and compute the sample means of λ_2 and λ_2/λ_n. The results are shown in Figure 2.5. It is clearly seen that λ_2 and λ_2/λ_n both decrease as D increases, which is consistent with the asymptotic estimates in (2.18) and (2.19). Also, by comparing the lines for different n's in Figure 2.5, we note that the effects of D on λ_2 and λ_2/λ_n weaken with increasing n.

2.4 Weak Indicators of Synchronizability

In the previous sections, we have seen that the synchronizability of networks with a specified diameter may be very different. However, under certain assumptions, a small diameter could still be considered favorable. For instance, in Construction 2.9, it is seen from (2.18) and (2.19) that the growth rates of λ_2 and λ_2/λ_n depend on $1/(D-\varepsilon)$. Thus, a small diameter D has a positive effect on the synchronizability of $H(n, D)$, which is also confirmed by numerical results.

There are some other graph invariants with connection to the synchronizability of networks, such as the degree sequence, mean distance, girth, betweenness, etc. Many of these graph invariants have been found to have a similar property to the diameter; they have effects on synchronizability under certain assumptions, but in general, they are not a good indicator of synchronizability. As an example, it has been shown that the synchronizability of networks with a prescribed degree sequence may be quite different [Atay *et al.*, 2006; Wu, 2005]. In fact, the following result has been proved.

Theorem 2.14. *Given a graphical degree sequence* $0 < d_1 \leq d_2 \leq \cdots \leq d_n$ *(subject to certain conditions; see [Atay et al., 2006; Wu, 2005]), there exist two realizations of the degree sequence, of which one has* λ_2 *and* λ_2/λ_n *tending to 0, and the other has* λ_2 *and* λ_2/λ_n *bounded away from 0, as* $n \to \infty$.

Motivated by our results and related research on the connection between some graph invariants and synchronizability, we propose the following definition.

Definition 2.15. *A graph invariant* $\mathcal{I}$ *is called a weak indicator of synchronizability if there exist two sequences of networks* $\{\mathcal{G}_n\}$ *and* $\{\mathcal{H}_n\}$ *with the same value of* $\mathcal{I}$, *which satisfy that* $\lambda_2(\mathcal{G}_n)$ *and* $\frac{\lambda_2}{\lambda_n}(\mathcal{G}_n)$ *tend to 0, and* $\lambda_2(\mathcal{H}_n)$ *and* $\frac{\lambda_2}{\lambda_n}(\mathcal{H}_n)$ *are bounded away from 0, as* $n \to \infty$.

Hence, by Theorems 2.8, 2.13, and 2.14, we have the following result.

Theorem 2.16. *The diameter and degree sequence are both weak indicators of synchronizability.*

Note that Theorems 2.8 and 2.13 actually show that there exist two sequences of networks $\{\mathcal{G}_n\}$ and $\{\mathcal{H}_n\}$ with the same diameter D, which satisfy that $\lambda_2(\mathcal{G}_n) \to 0$,

$\frac{\lambda_2}{\lambda_n}(\mathcal{G}_n) \to 0$, $\lambda_2(\mathcal{H}_n) \to \infty$, and $\frac{\lambda_2}{\lambda_n}(\mathcal{H}_n) \to 1$, as $n \to \infty$, which is stronger than Definition 2.15.

Theorem 2.16 naturally leads to the question: Is there a graph invariant, except λ_2, λ_2/λ_n and their simple transformations, which is not a weak indicator of synchronizability? We guess there may exist one, but it is unlikely to be simple or common. This is an unanswered question in this thesis.

CHAPTER 3

OPTIMAL NETWORKS

3.1 Problem Formulation

In the previous chapter, we have already seen a network optimization problem, that is, to find the networks maximizing or minimizing synchronizability with the specified diameter. Now we switch to a more general optimization problem, that is, to find the networks maximizing synchronizability without restriction on the diameter. Also, we extend the type of networks in question to directed weighted networks, i.e., the connection weights w_{ij} can be any real nonnegative numbers, and it is not necessary to have that $w_{ij} = w_{ji}$. By defining measures of synchronizability and synchronization cost appropriately, Nishikawa and Motter [2006] proved the following result.

Theorem 3.1. *The following statements are equivalent:*

(i) A network has the maximum synchronizability.

(ii) A network has the minimum synchronization cost.

(iii) The Laplacian eigenvalues of a network satisfy $0 = \lambda_1 < \lambda_2 = \cdots = \lambda_n$.

Thus, the problem of maximizing synchronizability can be formulated into a matrix optimization problem by specifying the properties of the Laplacian matrices.

Problem 3.2. Characterize and find any matrix $L = (l_{ij}) \in \mathbb{R}^{n\times n}$ satisfying all of the following:

(i) $l_{ij} \leq 0$ for all $i \neq j$.

(ii) $\sum_{j=1}^{n} l_{ij} = 0$ for all i.

(iii) The eigenvalues of L satisfy $0 = \lambda_1 < \lambda_2 = \cdots = \lambda_n$.

Remark. Conditions (i) and (ii) guarantee that L is a Laplacian matrix of a directed weighted network.

Definition 3.3. *A matrix is optimal if it satisfies all the conditions in Problem 3.2. A network is optimal if its Laplacian matrix is optimal.*

Although Problem 3.2 is formulated in a simple form, its general case is not easy to answer. A class of triangular matrices has been found to be optimal [Nishikawa & Motter, 2006]; however, they are only a narrow class of possible solutions. On the other hand, the low-dimensional cases may be analytically accessible, which are of our interest. The case $n = 2$ is trivial, since condition (iii) becomes $0 = \lambda_1 < \lambda_2$, which is satisfied if and only if the network is connected. Hence, the lowest dimension of interest is three, which will be discussed in the next section.

3.2 Three-Oscillator Networks

We now discuss the three-dimensional case of Problem 3.2. A 3×3 matrix satisfying conditions (i) and (ii) is of the form

$$L = \begin{bmatrix} a+b & -a & -b \\ -d & c+d & -c \\ -e & -f & e+f \end{bmatrix}, \tag{3.1}$$

where a, b, c, d, e, $f \geq 0$. We have the following result.

Theorem 3.4. *The matrix L in (3.1) is optimal if and only if all of the following are satisfied:*

(i) $b - c,\ d - e,\ f - a \geq 0$, *or* $b - c,\ d - e,\ f - a \leq 0$.

(ii) $a + b + c + d + e + f > 0$.

(iii) $\sqrt{|f-a|} = \sqrt{|b-c|} \pm \sqrt{|d-e|}$.

The eigenvalues of such an optimal matrix are $0 = \lambda_1 < \lambda_2 = \lambda_3 = \frac{1}{2}(a+b+c+d+e+f)$.

Proof. The characteristic polynomial of L is

$$\det(\lambda I - L) = \begin{vmatrix} \lambda - a - b & a & b \\ d & \lambda - c - d & c \\ e & f & \lambda - e - f \end{vmatrix}$$
$$= \lambda\big[\lambda^2 - (a+b+c+d+e+f)\lambda + ac + ae + af + bc + bd + bf + ce + de + df\big].$$

To satisfy condition (iii), the quadratic equation

$$\lambda^2 - (a+b+c+d+e+f)\lambda + ac + ae + af + bc + bd + bf + ce + de + df = 0 \quad (3.2)$$

must have equal roots. Hence, we have

$$(a+b+c+d+e+f)^2 - 4(ac + ae + af + bc + bd + bf + ce + de + df) = 0,$$

or equivalently,

$$(f-a-b+c-d+e)^2 = 4(b-c)(d-e).$$

Taking square roots, we get

$$f-a-b+c-d+e = \pm 2\sqrt{(b-c)(d-e)}.$$

Then,

$$f-a = b-c+d-e \pm 2\sqrt{(b-c)(d-e)}$$
$$= \begin{cases} (\sqrt{|b-c|} \pm \sqrt{|d-e|})^2, & \text{if } b-c,\ d-e \geq 0, \\ -(\sqrt{|b-c|} \pm \sqrt{|d-e|})^2, & \text{if } b-c,\ d-e \leq 0. \end{cases}$$

Therefore, $b-c$, $d-e$, $f-a \geq 0$, or $b-c$, $d-e$, $f-a \leq 0$, and

$$\sqrt{|f-a|} = \sqrt{|b-c|} \pm \sqrt{|d-e|}.$$

Under these conditions, solving equation (3.2) yields $\lambda = \frac{1}{2}(a+b+c+d+e+f)$. The conclusion thus follows. □

For the symmetric case, we have the following corollary.

Corollary 3.5. *A symmetric matrix L in (3.1) is optimal if and only if $a = b = c > 0$. In other words, the optimal symmetric three-oscillator networks are the networks with all connection weights positive and equal. The eigenvalues of such an optimal matrix are $0 = \lambda_1 < \lambda_2 = \lambda_3 = 3a$.*

Proof. When $a = d$, $b = e$, $c = f$, the condition $b-c$, $d-e$, $f-a \geq 0$ becomes that $b-c$, $a-b$, $c-a \geq 0$, namely, $a \geq b \geq c \geq a$, resulting in $a = b = c$. The condition $b-c$, $d-e$, $f-a \leq 0$ also implies that $a = b = c$. Now condition (ii) in Theorem 3.4 becomes $a > 0$, and condition (iii) is clearly satisfied. Therefore, by Theorem 3.4, a symmetric matrix L in (3.1) is optimal if and only if $a = b = c > 0$. □

According to Theorem 3.4, we can design an algorithm to generate optimal three-oscillator networks.

Algorithm 3.6 (Generating Optimal Three-Oscillator Networks).
Step 1: Select b, c, d, $e \geq 0$ such that $b-c$, $d-e \geq 0$, or $b-c$, $d-e \leq 0$.
Step 2: If $b-c$, $d-e \geq 0$, select $a \geq 0$ and $f = a + (\sqrt{b-c} \pm \sqrt{d-e})^2$; otherwise, select $f \geq 0$ and $a = f + (\sqrt{c-b} \pm \sqrt{e-d})^2$.

As an interesting example, which shows that an optimal matrix may be neither diagonalizable nor triangular, we apply Theorem 3.4 to find all optimal networks which are directed cycles of length three.

Example 3.7 (Optimal Directed Cycles of Length Three). Letting $a = c = e = 0$ in (3.1), the Laplacian matrix of an optimal network which is a directed cycle of length three has the form

$$L = \begin{bmatrix} b & 0 & -b \\ -d & d & 0 \\ 0 & -f & f \end{bmatrix},$$

where $b,\ d,\ f \geq 0$. Then condition (i) in Theorem 3.4 is satisfied, and conditions (ii) and (iii) become $b + d + f > 0$ and

$$\sqrt{f} = \sqrt{b} \pm \sqrt{d}.$$

These conditions characterize all such optimal networks. The equal eigenvalues are

$$\lambda_2 = \lambda_3 = \frac{1}{2}(b + d + f) = \frac{1}{2}\left[b + d + \left(\sqrt{b} \pm \sqrt{d}\right)^2\right] = b + d \pm \sqrt{bd}.$$

When $b,\ d,\ f > 0$, L is nontriangular, and the eigenspace corresponding to eigenvalue $b + d \pm \sqrt{bd}$ has dimension 1, with eigenvector

$$\left[\mp\frac{b}{(\sqrt{b} \pm \sqrt{d})\sqrt{d}}, \pm\frac{\sqrt{bd}}{(\sqrt{b} \pm \sqrt{d})^2}, 1\right]^{\mathrm{T}},$$

whence L is nondiagonalizable.

3.3 Final Remarks

First note that Corollary 3.5 can be generalized to all dimensions. In fact, we have the following result.

Theorem 3.8. *A symmetric n-oscillator network is optimal if and only if all its connection weights are positive and equal. Denote this value by a, and the Laplacian eigenvalues of such an optimal network are $0 = \lambda_1 < \lambda_2 = \cdots = \lambda_n = na$.*

Proof. If $\lambda > 0$ is an eigenvalue of the Laplacian matrix L with multiplicity $n - 1$, then 0 is an eigenvalue of the matrix $\lambda I - L$ with multiplicity $n - 1$. Also, if L is symmetric, the corresponding eigenspace has dimension $n - 1$. Hence, the row space (as well as the column space) of $\lambda I - L$ has dimension 1. But $\lambda I - L$ has equal nonzero row sums λ, so all the rows must be the same. From the symmetry, it follows that all entries of $\lambda I - L$ are the same, denoted by a. Then L has the form

$$L = \begin{bmatrix} \lambda - a & -a & \cdots & -a \\ -a & \lambda - a & \cdots & -a \\ \vdots & \vdots & & \vdots \\ -a & -a & \cdots & \lambda - a \end{bmatrix}.$$

From zero row sums of L, we get $\lambda = na$. Since $\lambda > 0$, $a > 0$. The above argument can be reversed, and the conclusion follows. □

We conclude this thesis with several remarks:

(1) Nondiagonalizable vs. nontriangular. Nishikawa and Motter [2006] proved that all optimal matrices which are diagonalizable are of the form

$$L = \begin{bmatrix} \lambda - b_1 & -b_2 & \cdots & -b_n \\ -b_1 & \lambda - b_2 & \cdots & -b_n \\ \vdots & \vdots & & \vdots \\ -b_1 & -b_2 & \cdots & \lambda - b_n \end{bmatrix},$$

where $b_i \geq 0$ and $\lambda = \sum_{i=1}^{n} b_i > 0$. They further constructed a class of optimal networks whose Laplacian matrices are nondiagonalizable and concluded that most optimal matrices are nondiagonalizable. However, in their constructions, all the Laplacian matrices are triangular. In such an optimal matrix, the eigenvalues are exactly the diagonal elements which are all equal except the first one equal to zero. Note that the diagonal elements of the Laplacian matrix are the in-degrees of nodes. Thus, the triangular structure, which implies that no bi-direction connections exist and all nodes have equal in-degrees except one, is a rather restrictive condition. In fact, our study on three-dimensional case, especially Theorem 3.4, suggests that "most optimal matrices are nontriangular". Specifically, Example 3.7 shows a class of optimal matrices which is neither diagonalizable nor triangular. We guess this is true for all dimensions: Most optimal matrices are neither diagonalizable nor triangular.

(2) Symmetric vs. nonsymmetric. From Theorem 3.8, we see that the class of optimal symmetric matrices is so small that it has very little practical value.

However, undirected connections are very common in applications. Thus, it will be useful to find a larger class of networks which is close to optimal. In other words, for undirected networks, the optimization problem is meaningful only when specific connection constraints are taken into account and the goal is to maximize synchronizability under such constraints.

(3) Weighted vs. unweighted. Either from the remark following Lemma 2.2 or from Theorem 3.8, it is seen that the optimal undirected unweighted networks are the complete graphs. Therefore, like the symmetric vs. nonsymmetric case, a realistic optimization problem for undirected unweighted networks should be to maximize synchronizability under some connection constraints.

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